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Several New Methods for Solving Equations

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Several new methods for solving one nonlinear equation are developed. Most of the methods are of order three and they require the knowledge of f, f' and f''. The methods will be compared to others in the literature. An extensive bibliography is given.

KEY WORDS: Nonlinear equations, order of convergence, iteration, computational efficiency, efficiency index.

C.R. CATEGORIES: 5.1, 5.15.

1. INTRODUCTION

Many iterative procedures were developed to obtain a simple zero ξ of a nonlinear function $f(x)$. There are also methods for obtaining multiple zeros. The algorithms can be classified as bracketing techniques, fixed point methods and hybrid ones. The bracketing methods include the well-known bisection, Regula Falsi and modified Regula Falsi. Other algorithms of this class are: method F (King [66]), modified method F (Popovski [108]), Illinois algorithm (Snyder [116]), Pegasus (Dowell and Jarratt [24]), and improved Pegasus method (King [64]), algorithms A, M and R (Dekker [21]; Bus and Dekker [11]), algorithm B (Brent [9]), algorithm C

(Anderson and Bjorck [5]), Cox method (Cox [18]), and Stone's method (Stone [120]). In all these methods one assumes that an interval $[a, b]$ is given on which $f(x)$ changes its sign. The methods successively produce smaller and smaller intervals containing the zero. Thus one guarantees the convergence of the iterative process. On the other hand such methods *cannot* find zeros of even multiplicity. In order to overcome such difficulty one can use fixed point type method. The list of such methods is long and includes among others Cauchy, Chebyshev, Euler, Halley, Hansen and Patrick, Jarratt, King, Laguerre, Muller, Murakami, Neta, Newton, Ostrowski, Popovski, Steffensen, secant, Traub, Wegstein and Werner.

The last class uses a combination of two methods, one from each class to guarantee the convergence of the iterative process, see for example Nesdore [79] and Popovski [86–108].

In this article, we develop several new fixed-point type methods based on the idea of Popovski [107]. All such methods will require the evaluation of f, f' and f'' at each step. These methods are all of order three and thus the informational efficiency (see e.g., Ostrowski) is 1. The efficiency index is 1.442. A fourth-order method based on Nourein's algorithm [83] will be given. A special case of a fifth-order method developed by Murakami [77] will also be discussed. These methods will be compared numerically. Tables comparing the informational efficiency and the efficiency index of all methods (known to the author) will be given.

2. THIRD-ORDER METHODS

In 1982 Popovski has suggested the construction of third-order methods by what he called the method of replacement. Let

$$h = x_{n+1} - x_n, \quad f_n = f(x_n), \quad u_n = f_n / f'_n, \quad A_i = \frac{f^{(i)}(x_n)}{i!\, f'(x_n)},$$

then

$$0 = u + h + A_2 h^2. \tag{1}$$

This equation can be solved for h directly, which yields Cauchy's method [13]. If (1) is written as

$$0=u+h+A_2(h)\cdot\{h\} \tag{2}$$

one can replace (h) and $\{h\}$ by various iteration functions and then solve for h. This is the method of replacement. Popovski obtained the following 9 algorithms this way.

1. $$h=-\frac{u(uA_2-1)}{2uA_2-1}, \tag{3}$$

2. $$h=\frac{u}{(uA_2+1)uA_2-1}, \tag{4}$$

3. $$h=\frac{u(2uA_2-1)}{(uA_2-3)uA_2+1}, \tag{5}$$

4. $$h=-\frac{u[(uA_2-3)uA_2+1]}{3(uA_2-4)uA_2+1}, \tag{6}$$

5. $$h=-\frac{u[(uA_2+1)uA_2-1]}{2uA_2-1}, \tag{7}$$

6. $$h=u\left(\frac{uA_2}{(uA_2+1)uA_2-1}-1\right), \tag{8}$$

7. $$h=\frac{u[(uA_2+2)uA_2-1]}{(uA_2-3)uA_2+1]}, \tag{9}$$

8. $$h=-u\frac{uA_2}{(uA_2-1)^2+1}, \tag{10}$$

9. $$h=u\frac{(uA_2)^2+1}{uA_2-1}. \tag{11}$$

These methods based on combining $(h)=h, -u$ or one of the following methods

$$h=\frac{u}{uA_2-1} \qquad \text{(Halley)}, \tag{12}$$

$$h=-u(uA_2+1) \quad \text{(Euler)}, \tag{13}$$

$$h=-u[(2uA_2+1)uA_2+1], \tag{14}$$

and (3)–(5).

In a similar fashion, one can obtain the following 21 new methods.

$$h = -\frac{u}{1-uA_2(1+uA_2(1+2uA_2))}, \qquad (h), \quad (14) \quad (15)$$

$$h = -\frac{u}{1+uA_2/[(1+uA_2)uA_2-1]}, \qquad (h), \quad (4) \quad (16)$$

$$h = -u - u^2A_2[(2uA_2+1)uA_2+1], \qquad (-u), \quad (12) \quad (17)$$

$$h = -u - u^2A_2(uA_2+1), \qquad (-u), \quad (13) \quad (18)$$

$$h = -u - u^2A_2[(2uA_2+1)uA_2+1], \qquad (-u), \quad (14) \quad (19)$$

$$h = -u + u^2A_2\frac{(2uA_2+1)uA_2+1}{uA_2-1}, \qquad (12), \quad (14) \quad (20)$$

$$h = -u - \frac{u^2A_2}{(uA_2-1)[(uA_2+1)uA_2-1]}, \qquad (12), \quad (4) \quad (21)$$

$$h = -u - \frac{u^2A_2}{uA_2-1}\,\frac{2uA_2-1}{(uA_2-3)uA_2+1}, \qquad (12), \quad (5) \quad (22)$$

$$h = -u - u^2A_2(uA_2+1)^2, \qquad (13), \quad (13) \quad (23)$$

$$h = -u - A_2u^2(uA_2+1)[(2uA_2+1)uA_2+1], \qquad (13), \quad (14) \quad (24)$$

$$h = -u - A_2\frac{u^2(uA_2+1)(uA_2-1)}{2uA_2-1}, \qquad (13), \quad (3) \quad (25)$$

$$h = -u + A_2\frac{u^2(uA_2+1)(2uA_2-1)}{(uA_2-3)uA_2+1}, \qquad (13), \quad (5) \quad (26)$$

$$h = -u - A_2u^2[(2uA_2+1)uA_2+1]^2, \qquad (14), \quad (14) \quad (27)$$

$$h = -u - A_2\frac{u^2(uA_2-1)[(2uA_2+1)uA_2+1]}{2uA_2-1}, \qquad (14), \quad (3) \quad (28)$$

$$h=-u+A_2\frac{u^2[(2uA_2+1)uA_2+1]}{(uA_2+1)uA_2-1}, \qquad (14), \quad (4) \quad (29)$$

$$h=-u+A_2\frac{u^2(2uA_2-1)[(2uA_2+1)uA_2+1]}{(uA_2-3)uA_2+1}, \qquad (14), \quad (5) \quad (30)$$

$$h=-u-A_2\left[\frac{u(uA_2-1)}{(2uA_2-1)}\right]^2, \qquad (3), \quad (3) \quad (31)$$

$$h=-u+A_2\frac{u^2(uA_2-1)}{(2uA_2-1)[(uA_2+1)uA_2-1]}, \qquad (3), \quad (4) \quad (32)$$

$$h=-u-A_2\left[\frac{u}{(uA_2+1)uA_2-1}\right]^2, \qquad (4), \quad (4) \quad (33)$$

$$h=-u-A_2\frac{u^2(2uA_2-1)}{[(uA_2+1)uA_2-1][(uA_2-3)uA_2+1]}, \qquad (4), \quad (5) \quad (34)$$

$$h=-u-A_2\left[\frac{u(2uA_2-1)}{(uA_2-3)uA_2+1}\right]^2. \qquad (5), \quad (5) \quad (35)$$

Remark The two quantities in parentheses to the left of the equation number indicate how the new method was developed.

Since Popovski [107] recommended the use of method (3) we compare that method with the newly developed ones in Section 5.

3. FOURTH-ORDER METHOD

This method is based on Nourein's algorithm [83]. The method of neglecting discussed by Popovski [107] is used here. Setting $f^{(4)}(x_i)=0$, in Nourein's method one obtains

$$h=\frac{(2uA_2-1-6u^2A_3)u}{(uA_2-3)uA_2+1+6u^2A_3}. \qquad (36)$$

The method is of order four and requires four new pieces of information. Therefore the informational efficiency is 1 as for the previous methods. On the other hand the efficiency index is only 1.414.

4. FIFTH-ORDER METHOD

Murakami [77] has developed the following family of methods of order five,

$$h = -a_1 u_n - a_2 w_2(x_n) - a_3 w_3(x_n) - \psi(x_n), \tag{37}$$

where

$$w_2(x_n) = \frac{f_n}{f'(x_n - u_n)}, \tag{38}$$

$$w_3(x_n) = \frac{f_n}{f'(x_n + \beta u_n + \gamma w_2(x_n))}, \tag{39}$$

$$\psi(x_n) = \frac{f_n}{b_1 f'_n + b_2 f'_n(x_n - u_n)}. \tag{40}$$

The asymptotic error constant is

$$C = -\left[\tfrac{32}{3}\gamma^2 + \tfrac{8}{3}\gamma - \tfrac{2}{3} + \frac{1}{6(4\gamma+1)}\right] A_2^4 + (8\gamma^2 + 4\gamma) A_2^2 A_3 + \tfrac{128}{3}\gamma A_2 A_4 - \tfrac{3}{8} A_3^2 + \tfrac{1}{24} A_5. \tag{41}$$

Murakami has suggested that $\gamma = 0$ or $\gamma = -\frac{1}{2}$. These two choices annihilate one term of the asymptotic error constant. Another possibility is $\gamma = 17795/131072$ which annihilates the first term. This choice leads to the values

$$\beta = -\tfrac{1}{2} - \gamma, \quad a_1 = 0.3879870, \quad a_2 = -1.420700, \quad a_3 = \tfrac{2}{3}, \quad b_1 = -0.1186015, \quad b_2 = 0.8506410, \tag{42}$$

and the asymptotic error constant is then

$$C = 0.6905251 A_2^3 A_3 + 5.792695 A_2 A_4 - \tfrac{3}{8} A_3^2 + \tfrac{1}{24} A_5. \tag{43}$$

5. NUMERICAL EXPERIMENTS

We have used the following six examples to compare the performance of the methods (3), (15)–(37).

$$f_1(x) = \sin x - \tfrac{1}{2}x, \qquad x_0 = 2$$

$$f_2(x) = x^5 + x - 10\,000, \quad x_0 = 4$$

$$f_3(x) = x^{1/2} - \frac{1}{x} - 3, \qquad x_0 = 1$$

$$f_4(x) = e^x + x - 20, \qquad x_0 = 0$$

$$f_5(x) = \ln x + x^{1/2} - 5, \quad x_0 = 1$$

$$f_6(x) = x^3 - x^2 - 1, \qquad x_0 = 0.5.$$

The first example is simple and it's taken from Gerald and Wheatley [35]. All methods performed very well in this case. The other examples are taken from Popovski [86] and thus we added the method of that article to our comparison. In our notations, the method is

$$h = -\frac{e^{2uA_2} - 1}{2A_2}. \tag{44}$$

The following table (Table 1) gives the number of iterations required to obtain the zero with tolerance of 10^{-14}. All computations were performed in double precision on an IBM 3033. The letter D stands for divergence (within 30 iterations) and * denotes computational difficulties (overflow, etc.). Note that the method recommended in Popovski [107] didn't perform as well as the new

Table 1

	Function					
Method	*1*	*2*	*3*	*4*	*5*	*6*
(3)	3	6	D	*	D	6
(15)	3	10	D	8	D	D
(16)	3	D	*	D	*	13
(17)	3	29	4	14	4	16
(18)	3	12	4	26	4	17
(19)	3	29	4	14	4	16
(20)	3	16	*	10	*	11
(21)	3	6	5	6	5	12
(22)	3	7	*	D	*	7
(23)	3	14	5	13	5	14
(24)	3	16	7	*	6	19
(25)	3	8	5	*	5	23
(26)	3	D	4	D	4	16
(27)	3	20	*	11	5	19
(28)	3	D	4	*	4	11
(29)	3	9	*	25	*	13
(30)	3	D	5	D	5	8
(31)	3	11	6	5	5	5
(32)	3	15	5	*	5	11
(33)	3	6	7	9	4	7
(34)	3	D	D	D	D	D
(35)	3	6	4	7	4	6
(36)	3	6	5	8	5	15
(37)	2	5	4	4	3	7
(44)	3	6	4	5	4	D

methods developed here. The method taken from Popovski [86] did not converge for one of the examples. The following new methods converge for all the examples (17)–(19), (21), (23), (31), (33), (35)–(37). Counting the total number of iterations one can say that methods (21), (31), (33) (35) are the best third-order ones of those considered. Next comes (23), and then (17)–(19). Certainly method (37) which is fifth-order performed better than the others. Surprisingly though the fourth-order method (36) didn't perform better than the third-order methods.

Remark More experiments were performed with functions suggested by Nerinckx and Haegemans [78] and the results were consistently better than those reported there.

6. EFFICIENCY COMPARISON

In this section we collected information concerning the order, informational usage, informational efficiency and efficiency index (for definitions, see e.g. Neta [81]) of all methods known to the author. The first table (Table 2) consists of the information for bracketing methods and the others will give fixed point methods (Tables 3–7). For the fixed point methods we gave separate tables for derivative free methods, for methods using f' at only one point, for those using f' at more than one point, for those using f' and f'' and those using derivatives of order 3 or higher.

Table 2

	P	d	E	I	*Total no.*	f'
Bisection	1	1	1	1	$N = \log_2 \frac{\lvert b-a \rvert}{t}$	
Regula Falsi	1	1	1	1		
Modified R.F.	1.618	1	1.618	1.618		
Algorithm A	1.618	1	1.618	1.618	2^N	
Algorithm B	1.618	1	1.618	1.618	$(N+1)^2 - 2$	
Algorithm M	1.618	1	1.618	1.618	$4N$	
Algorithm R	1.839	1	1.839	1.839	$5N$	
Method F	1.839	1	1.839	1.839		
Modified Method F	1.839	1	1.839	1.839		
Illinois	3	3	1	1.442		
Pegasus	7.275	4	1.818	1.642		
Improved Pegasus	5	3	1.667	1.710		
	3	2	1.5	1.732		
Algorithm C	5	3	1.667	1.710		
	8	4	2	1.682		
Cox	2	2	1	1.414		1
Stone	3	2	1.5	1.732		1

In the last two algorithms we assumed equal complexity in evaluating f, f'. The efficiency may be higher in other cases.

Table 3

	P	d	E	I
Fixed point (Picard)	1	1	1	1
Wegstein	1.618	1	1.618	1.618
Secant	1.618	1	1.618	1.618
Popovski [98]	1.839	1	1.839	1.839
Muller	1.839	1	1.839	1.839
Jarratt and Nudds	1.839	1	1.839	1.839
Popovski [100]	1.839	1	1.839	1.839
Traub (3 methods)	1.839	1	1.839	1.839
Steffensen	2	2	1	1.414
Chambers	2	2	1	1.414
Chambers	2.732	2	1.366	1.653

Only f values are required.

Table 4

	P	d	E	I
Newton	2	2	1	1.414
Dordjevic	2	2	1	1.414
Ostrowski	2.414	2	1.207	1.554
Popovski [99]	2.414	2	1.207	1.554
Popovski [105] 3 methods	2.414	2	1.207	1.554
Werner [131]	2.414	2	1.207	1.554
Chambers [15]	2.414	2	1.207	1.554
Jarratt [53]	2.732	2	1.366	1.653
Jain (implicit)	3	–	–	–
Werner [134]	3	3	1	1.442
King [65]	4	3	1.333	1.587
Popovski [105] 10 methods	4.562	3	1.520	1.66
Murakami	5	4	1.25	1.495
Neta (this article)	5	4	1.25	1.495
Neta [80]	6	4	1.5	1.565
Popovski [104]	7	4	1.75	1.626
Neta [82]	10.815	4	2.704	1.813
Neta [81]	14	5	2.8	1.695
Neta [81]	16	5	3.2	1.741
Werner [133]	$2k$	$k+2$	–	
Werner [133]	$k+\sqrt{k^2+1}$	$k+2$	–	
Werner [133]	$2k+1$	$k+2$		
Werner [133]	$2k+2$	$k+2$	–	

f' is required at one point only.

Table 5

	P	d	E	I	f' at
Jarratt [52]	3	3	1	1.442	2
Jarratt [55]	4	3	1.333	1.587	2
Jarratt [51] 4 methods	4	3	1.333	1.587	2
Jarratt [52] 2 methods	4	4	1	1.414	3
Jain (semi-explicit)	4	–	–	–	2
Jarratt [52]	5	4	1.25	1.495	3
Jain (implicit)	5	–	–	–	2
King [62]	5	4	1.25	1.495	2
Popovski [101]	7.464	4	1.866	1.653	2
Werner	$\frac{m}{2}+\sqrt{\frac{m^2}{4}+1}$	m	–	–	$m-1$

Methods require f and f' at more than one point.

Table 6

	P	d	E	I
Hansen and Patrick	3	3	1	1.442
Popovski [93]	3	3	1	1.442
Halley	3	3	1	1.442
Laguerre	3	3	1	1.442
Chebyshev	3	3	1	1.442
Cauchy	3	3	1	1.442
Euler	3	3	1	1.442
Ostrowski	3	3	1	1.442
Popovski [89]	3	3	1	1.442
Milovanovic *et al.*	3	3	1	1.442
Popovski [90]	3	3	1	1.442
Popovski [94]	3	3	1	1.442
Neta (this article) 21 methods	3	3	1	1.442
Popovski [107] 9 methods	3	3	1	1.442
Werner [132]	4	3	1.333	1.587
Werner [133] 4 methods	4	3	1.333	1.587
Popovski [106]	6	4	1.5	1.565

Methods require f and f''.

Table 7

	P	d	E	I	f'	f''	f'''	$f^{(4)} \ldots f^{(k)}$
Kiss/Lika	4	4	1	1.414	1	1	1	
Neta (this article)	4	4	1	1.414	1	1	1	
Nourein	5	5	1	1.380	1	1	1	1
Werner	$k+2$	$k+1$	–	–	1	1	1	1 ... 1
Varyukhin and Kas'yanyuk	$k+2$	$k+2$	–	–	1	1	1	1 ... 1

Methods requiring derivatives of f of order three or higher.

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