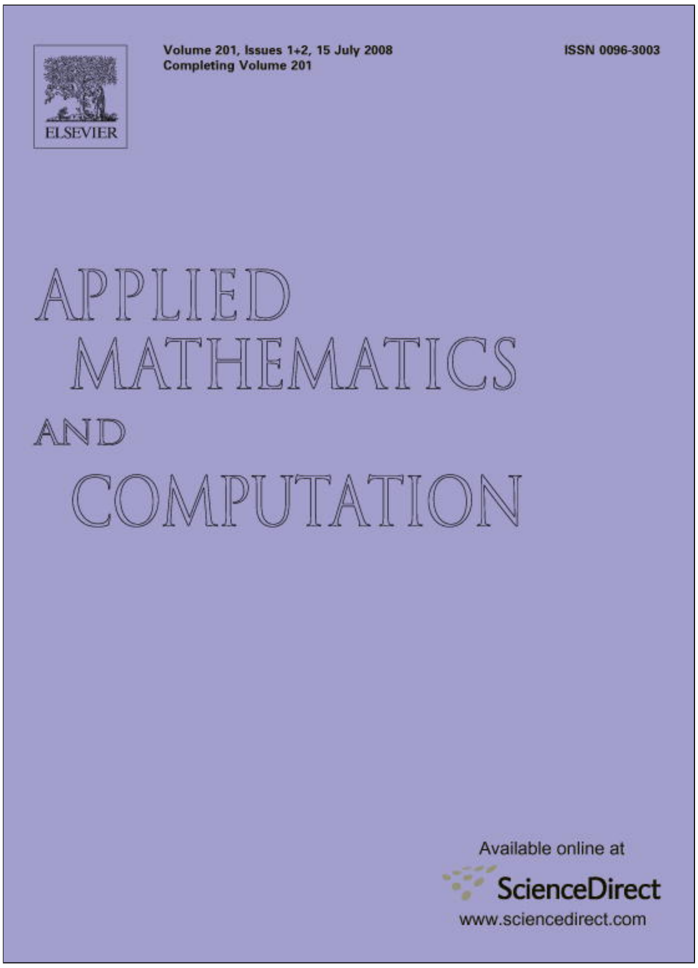
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On Popovski's method for nonlinear equations

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Abstract

Two different modifications of Popovski's method are developed, both are free of second derivatives. In the first modified scheme we traded the second derivative by an additional function evaluation. In the second method we replaced the second derivative by a finite difference and thus reducing the order slightly and reducing the number of evaluations per step by one. Therefore the second modification is more efficient.
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Keywords: Nonlinear equations; Simple roots; Popovski's method; Chebyshev's method

1. Introduction

There is a vast literature on the solution of nonlinear equations and nonlinear systems, see for example Ostrowski [1], Traub [2], Neta [3] and references there. In general, methods for the solution of polynomial equations are treated differently and will not be discussed here. The methods can be classified as bracketting or fixed point methods. The first class include methods that at every step produce an interval containing a root, whereas the other class produces a point which is hopefully closer to the root than the previous one. Here we develop two third-order fixed point type methods based on Popovski's family of methods [4]. In the first modified method we traded the second derivative by an additional function evaluation. The informational efficiency and efficiency index (see [2]) are the same as Popovski's. In the second modified scheme we replaced the second derivative by a finite difference and thus reducing the order slightly and reducing the number of function evaluations. This method is more efficient than Popovski's.

2. Popovski's third order family of methods

Popovski's family of methods to obtain a simple root of the nonlinear equation

$$f(x) = 0, \tag{1}$$

is given by the iteration

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$$x_{n+1} = x_n - (1-e)\frac{f'_n}{f''_n}\left\{\left[1 - \frac{e}{e-1}u_n\frac{f''_n}{f'_n}\right]^{1/e} - 1\right\}, \tag{2}$$

where

$$\begin{aligned} f_n^{(i)} &= f^{(i)}(x_n), \quad i = 0,1,2, \\ u_n &= \tfrac{f_n}{f'_n}. \end{aligned} \tag{3}$$

Popovski [4] has shown that this method is of order 3 with an asymptotic error constant

$$C = \frac{e-2}{6(e-1)}\left(\frac{f'''}{f'}\right)^2 - \frac{f''}{6f'}. \tag{4}$$

The method requires one function- and two derivative-evaluation per step. Thus the informational efficiency is 1, and the efficiency index is 1.442. The following are four well known special cases. For $e = 1$, the method reduces to Newton's second order method which does not contain second derivative. Therefore this case will not be considered here. For $e = -1$, the method is due to Halley [6]

$$x_{n+1} = x_n - \frac{u_n}{1 - \frac{1}{2}u_n\frac{f''_n}{f'_n}}. \tag{5}$$

For $e = 2$, the method is due to Cauchy [5]

$$x_{n+1} = x_n - \frac{f'_n - \sqrt{(f'_n)^2 - 2f_n f''_n}}{f''_n}. \tag{6}$$

For $e = 1/2$, the method is due to Chebyshev (see [4])

$$x_{n+1} = x_n - u_n\left[1 + \frac{1}{2}u_n\frac{f''_n}{f'_n}\right]. \tag{7}$$

Popovski [7] has also developed an extension of Chebyshev's method

$$x_{n+1} = x_n - u_n\left[1 + \frac{1}{2}u_n\frac{f''_n}{f'_n}\left(1 + u_n\frac{f''_n}{f'_n}\right)\right]. \tag{8}$$

This method have the same order and number of function evaluation, but with asymptotic error constant

$$C = -\frac{f'''}{6f'}. \tag{9}$$

3. New third order schemes free of second derivatives

Kou et al. [9] have modified Halley's method to have several third order schemes free of second derivative. Their family of methods is as follows

$$x_{n+1} = x_n - u_n\frac{\theta^2 f_n}{(\theta^2 - \theta + 1)f_n - f(y_n)}, \tag{10}$$

where θ is a nonzero real number, u_n is given by (3) and

$$y_n = x_n - \theta u_n. \tag{11}$$

Three particular cases are given, one of them ($\theta = 1$) is the Newton–Steffensen scheme (see [10]). Kou and Li [8] modified Chebyshev's method (7) by removing the second derivative, i.e.

$$x_{n+1} = x_n - u_n\left[\frac{\theta^2 + \theta - 1}{\theta^2} + \frac{f(y_n)}{\theta^2 f_n}\right]. \tag{12}$$

Here we use this idea to modify Popovski's method (2), for $e \neq 1$. First we expand $f(y_n)$ in Taylor series

$$f(y_n) = f_n + f_n'(y_n - x_n) + \frac{1}{2} f_n''(y_n - x_n)^2 + \cdots \tag{13}$$

Now substitute for $y_n - x_n$ from (11) and drop the terms of higher than second order

$$f(y_n) = f_n - \theta f_n + \frac{1}{2} f_n'' \theta^2 \left(\frac{f_n}{f_n'}\right)^2. \tag{14}$$

Now solve this for the second derivative and substitute in (2)

$$x_{n+1} = x_n - (1-e)\frac{\theta^2 f_n^2}{2f_n'[f(y_n) - (1-\theta)f_n]}\left\{\left[1 - \frac{2e}{e-1}\frac{f(y_n) - (1-\theta)f_n}{\theta^2 f_n}\right]^{1/e} - 1\right\}. \tag{15}$$

It is easy to see that if we let $e = 1/2$ in (15) we get (12). If we let $e = -1$ in (15) we get the family of methods given in Kou et al. [9].

Theorem. *Assume that the function $f : \mathscr{D} \subset \Re \to \Re$ for an open interval $\mathscr{D}$ has a simple root $\xi \in \mathscr{D}$. Let θ be a nonzero real number and $f(x)$ be sufficiently smooth in the neighborhood of ξ, then the order of convergence of the method defined by (15) is three.*

Proof. Let

$$e_n = x_n - \xi \tag{16}$$

and

$$\hat{e}_n = y_n - \xi \tag{17}$$

and expand f_n, f_n', u_n and $f(y_n)$ in Taylor series about the root ξ, we have (recall that $f(\xi) = 0$)

$$f_n = f'(\xi)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)\right], \tag{18}$$

where

$$c_k = \frac{f^{(k)}(\xi)}{k! f'(\xi)}. \tag{19}$$

Furthermore

$$f_n' = f'(\xi)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + O(e_n^3)\right]. \tag{20}$$

Thus upon dividing, we have

$$u_n = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4) \tag{21}$$

and

$$\hat{e}_n = e_n - \theta u_n = (1-\theta)e_n + \theta c_2 e_n^2 - 2\theta(c_2^2 - c_3)e_n^3 + O(e_n^4). \tag{22}$$

Using $\hat{e}_n$ in the expansion of $f(y_n)$ we have (neglecting terms of order higher than three)

$$f(y_n) = f'(\xi)\left\{(1-\theta)e_n + (\theta^2 - \theta + 1)c_2 e_n^2 - \left[2\theta^2 c_2^2 + (\theta^3 - 3\theta^2 + \theta - 1)c_3\right]e_n^3\right\}. \tag{23}$$

We now substitute all these expansions in (2), using the symbolic manipulator MAPLE [11], and neglect all terms of order higher than three.

$$e_{n+1} = \frac{1}{3}\left\{\frac{2(e-2)}{e-1}c_2^2 + 3(\theta - 1)c_3\right\}e_n^3. \tag{24}$$

Therefore the order of convergence is three. The difference between this method (15) and Popovski's is the fact that the second derivatives are not used. Clearly this is useful when the second derivative is more expensive than the function evaluation. The informational efficiency is 1.

The choice $\theta = 1$ annihilates one term in the asymptotic error constant and the method is

$$x_{n+1} = x_n - \frac{1-e}{2} u_n \frac{f_n}{f(y_n)} \left\{ \left[1 - \frac{2e}{e-1} \frac{f(y_n)}{f_n} \right]^{1/e} - 1 \right\}. \tag{25}$$

Another possibility is to choose θ so that

$$(e-1)\theta^2 + 2e(1-\theta) = 0,$$

i.e.

$$\theta = \frac{2e \pm \sqrt{8e - 4e^2}}{2(e-1)}. \tag{26}$$

Clearly that requires e to satisfy $4e(2-e) \geqslant 0$ which excludes Halley's method ($e = -1$). If $e \neq 1$, the method is

$$x_{n+1} = x_n - e \frac{u_n}{v_n - 1} \left(v_n^{1/e} - 1 \right), \tag{27}$$

where

$$v_n = \frac{f(y_n)}{(1-\theta) f_n}. \tag{28}$$

If $e = -1$ then $\theta^2 - \theta + 1 = 0$ and we have no real value for θ, thus this second possibility is not realistic for $e = -1$. □

4. New more efficient methods

The idea in the previous section allowed us to get the same order and the same number of function evaluations. Therefore the efficiency is the same. In this section, we will use a different idea of removing the second derivative. The method will be of lower than third order but more efficient.

Let us replace the second derivative by the second order differencing

$$f''_n = \frac{6}{h^2}(f_{n-1} - f_n) + \frac{2}{h} f'_{n-1} + \frac{4}{h} f'_n, \tag{29}$$

where $h = x_n - x_{n-1}$. This approximation of the second derivative can be obtained by using the method of undetermined coefficients. Let

$$f''_n = A f_n + B f_{n-1} + C f'_n + D f'_{n-1}. \tag{30}$$

Expand all the terms on the right about the point x_n and collect terms. Upon comparing the coefficients of the derivatives of f at x_n, we have the following system of equations for the unknowns $A, \ldots, D$

$$\begin{aligned} &A + B = 0, \\ &-Bh + C + D = 0, \\ &B\frac{h^2}{2} - Dh = 1, \\ &-B\frac{h^3}{6} + D\frac{h^2}{2} = 0. \end{aligned} \tag{31}$$

Solving the last two equations, we get

$$B = \frac{6}{h^2}, \quad D = \frac{2}{h}. \tag{32}$$

Substituting in the other two equation we get

$$A = -\frac{6}{h^2}, \quad C = \frac{4}{h}. \tag{33}$$

The method is now

$$x_{n+1} = x_n - \frac{(1-e)}{w(x_n)}\left\{\left[1 - \frac{e}{e-1}u_n w(x_n)\right]^{1/e} - 1\right\}, \tag{34}$$

where

$$w(x_n) = \frac{6(f_{n-1} - f_n) + 2hf'_{n-1} + 4hf'_n}{h^2 f'_n}. \tag{35}$$

This modified method requires one function- and one derivative-evaluation per step. It also requires an additional starting value which we can obtain using Newton's method (first derivative is required anyway). Let us now show that the order of the method is 2.732 and thus the informational efficiency is 1.366 and the efficiency index is 1.6529. Both of these efficiency measures are higher than Popovski's method and higher than the first modification.

Table 1
Functions, zeros and initial guesses

Number	Function	Zero	Initial guesses
1	$x^3 + 4x^2 - 15$	1.6319808055660636	1, 2
2	$x^2 - e^x - 3x + 2$	0.25753028543986084	−1, 0
3	$xe^{x^2} - \sin^2 x + 3\cos x + 5$	−1.207647827130919	−3, −2, −1
4	$\sin x - \frac{1}{2}x$	1.8954942670339809	1.6, 2
5	$(x+2)e^x - 1$	−0.44285440100238854	−1, 1, 3
6	$10xe^{-x^2} - 1$	1.67963061042845	1.5, 2
7	$\sin^2 x - x^2 + 1$	1.4044916482153411	1, 3
8	$e^{x^2+7x-30} - 1$	3	3.25, 3.5

Table 2
Number of function evaluations and accuracy for Chebyshev's method and ours

Function number	Initial guess	Chebyshev		Our method (25)		Our method (34)	
		No. of functions	Accuracy $\lvert f(x_n)\rvert$	No. of functions	Accuracy $\lvert f(x_n)\rvert$	No. of functions	Accuracy $\lvert f(x_n)\rvert$
1	1	12	8(−21)	12	1(−16)	8	0
1	2	9	1(−19)	9	7(−19)	8	0
2	−1	9	2(−18)	9	3(−18)	8	0
2	0	9	1(−24)	9	1(−24)	6	1(−19)
3	−3	27	6(−24)	30	6(−24)	20	1(−17)
3	−2	18	1(−23)	18	2(−23)	12	4(−19)
3	−1	15	4(−20)	12	1(−22)	8	4(−19)
4	1.6	12	2(−25)	12	2(−25)	8	4(−20)
4	2	9	2(−25)	9	2(−25)	6	4(−20)
5	−1	12	1(−17)	18	2(−18)	8	9(−16)
5	1	15	0	15	0	10	0
5	3	18	6(−18)	21	0	14	0
6	1.5	9	3(−19)	9	8(−20)	8	9(−20)
6	2	12	3(−15)	15	3(−21)	8	7(−19)
7	1	15	1(−24)	48	1(−20)	8	1(−19)
7	3	12	3(−16)	12	3(−22)	8	1(−16)
8	3.25	18	0	18	2(−23)	12	0
8	3.5	24	0	24	2(−15)	18	0

The accuracy is given as $m(-n)$ which is a shorthand for $m \times 10^{-n}$.

To this end, we use a result from Traub [2], p. 105: "Interpolatory one point iteration with memory use s pieces of information at x_i and reuse s old pieces at $x_{i-1},\dots,x_{i-n}$. Thus the order is determined by the unique positive real root of $t^{n+1}-s\sum_{j=0}^{n}t^j=0$." In our case $s=2$ and $n=1$ and therefore the root is 2.732.

5. Numerical experiments

We have experimented with our method (using $\theta=1$ and $e=1/2$) and compared it to Chebyshev's method. We have used the following functions and initial guesses listed in Table 1.

In the next table we compare the number of function evaluations required to achieve $|f(x_n)|\leqslant 10^{-14}$ and the accuracy for Chebyshev's method (7) ($e=1/2$ in (2)) and our modified methods (25) and (34). The accuracy achieved in each case is given in the form of $m(-n)$ which stands for $m\times 10^{-n}$. It can be seen that the number of function evaluations to achieve the accuracy always smaller for the modified method (34). When comparing the modified Popovski method to Chebyshev's, we found that in 5 out of 18 cases the former requires more function evaluations. (see Table 2)

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References

[1] A.M. Ostrowski, Solutions of Equations and System of Equations, Academic Press, New York, 1960.
[2] J.F. Traub, Iterative Methods for the Solution of Equations, Prentice Hall, New Jersey, 1964.
[3] B. Neta, Numerical Methods for the Solution of Equations, Net-A-Sof, California, 1983.
[4] D.B. Popovski, A family of one point iteration formulae for finding roots, Int. J. Comput. Math. 8 (1980) 85–88.
[5] A. Cauchy, Sur la Determination Approximative des Racines d'une Équation Algebrique ou Transcendante, Oeuvres Complétes Serie 2, Tome 4, Gauthier Villars, Paris, 1899, pp. 573–609.
[6] E. Halley, A new, exact and easy method of finding the roots of equations generally and that without any previous reduction, Phil. Trans. Roy. Soc. London 18 (1694) 136–148.
[7] D.B. Popovski, An extension of Chebyshev's iteration, Informatica 4 (1980) 26–28.
[8] J. Kou, Y. Li, Modified Chebyshev's method free from second derivative for nonlinear equations, Appl. Math. Comput. 187 (2007) 1027–1032.
[9] J. Kou, Y. Li, X. Wang, Modified Halley's method free from second derivative, Appl. Math. Comput. 183 (2006) 704–708.
[10] J.R. Sharma, A composite third-order Newton–Steffensen method for solving nonlinear equations, Appl. Math. Comput. 169 (2005) 242–246.
[11] D. Redfern, The Maple Handbook, Springer-Verlag, New York, 1994.