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A sixth-order family of three-point modified Newton-like multiple-root finders and the dynamics behind their extraneous fixed points

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ABSTRACT

A class of three-point sixth-order multiple-root finders and the dynamics behind their extraneous fixed points are investigated by extending modified Newton-like methods with the introduction of the multivariate weight functions in the intermediate steps. The multivariate weight functions dependent on function-to-function ratios play a key role in constructing higher-order iterative methods. Extensive investigation of extraneous fixed points of the proposed iterative methods is carried out for the study of the dynamics associated with corresponding basins of attraction. Numerical experiments applied to a number of test equations strongly support the underlying theory pursued in this paper. Relevant dynamics of the proposed methods is well presented with a variety of illustrative basins of attraction applied to various test polynomials.

1. Introduction

Newton's method locates a numerical root of a nonlinear equation without difficulty under normal circumstances, provided that a proper initial guess is selected close to the true solution. Unfortunately, it has only linear convergence when locating repeated roots. For repeated roots of a nonlinear equation of the form $f(x) = 0$, given the multiplicity $m \geq 1$ a priori, modified Newton's method [36,37] in the following form

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

efficiently locates the desired multiple-root with quadratic-order convergence. It is known that numerical scheme (1.1) is a second-order one-point optimal [23] method on the basis of Kung-Traub's conjecture [23] that any multipoint method [35] without memory can reach its convergence order of at most 2^{r-1} for r functional evaluations. We can find other higher-order multiple-zero finders in a number of literatures [16–18,21,24,25,31,32,40,45].

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Assuming a known multiplicity of $m \geq 1$, we propose in this paper a family of new three-point sixth-order multiple-root finders of modified Newton type in the form of:

$$\begin{cases} y_n = x_n - m \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - m \cdot Q_f(x_n) \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - m \cdot K_f(x_n) \cdot \dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{1.2}$$

where the desired forms of weight functions Q_f and K_f will be extensively studied for sixth-order of convergence in Section 3. As a consequence, one can regard the last equation in (1.2) as a family of modified Newton-like methods.

The remaining portion of this paper is organized as follows. Section 2 shortly surveys existing studies on multiple-root finders. Fully described in Section 3 is methodology and convergence analysis for newly proposed multiple-root finders. A main theorem on the properties of the family of proposed methods (1.2) is drawn to discover convergence order of six as well as to induce asymptotic error constants and error equations by use of a family of weight functions Q_f and K_f dependent on two principal roots of function-to-function ratios. In Section 4, special cases of weight functions are considered based on polynomials and low-order rational functions. Section 5 extensively investigates the extraneous fixed points and related dynamics underlying the basins of attraction. Tabulated in Section 6 are computational results for a variety of numerical examples. Table 5 compares the magnitudes of $e_n = x_n - \alpha$ of the proposed methods with those of a member of an existing sixth-order family of methods. Dynamical characteristics of the proposed methods along with their illustrative basins of attraction are depicted at great length with detailed analyses, comparisons and comments. Briefly stated at the end is overall conclusion together with a possible development of future work.

2. Review of existing sixth-order multiple-root finders

The orders of convergence of existing multiple-root finders are mostly found to be less than or equal to 4, and more higher-order multiple-root finders are rarely to be found. Very recently Geum–Kim–Neta [19] have developed a class of two-point sixth-order multiple-root finders by extending the classical modified double-Newton method with extensive analysis of their relevant dynamics behind the basins of attraction from the viewpoint of the extraneous fixed points. One member of the class is introduced as follows shown by (2.1):

Let a function $f : \mathbb{C} \to \mathbb{C}$ have a repeated zero α with integer multiplicity $m > 1$ and be analytic [1] in a small neighborhood of α.

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{m + a_1 u}{1 + b_1 u + b_2 u^2} \times \dfrac{1}{1 + 2(m-1)t} \cdot \dfrac{f(y_n)}{f'(y_n)}, u = \left[\dfrac{f(y_n)}{f(x_n)}\right]^{\frac{1}{m}}, t = \left[\dfrac{f'(y_n)}{f'(x_n)}\right]^{\frac{1}{m-1}}, \end{cases} \tag{2.1}$$

where $a_1 = \frac{2m(4m^4-16m^3+31m^2-30m+13)}{(m-1)(4m^2-8m+7)}$, $b_1 = \frac{4(2m^2-4m+3)}{(m-1)(4m^2-8m+7)}$ and $b_2 = -\frac{4m^2-8m+3}{4m^2-8m+7}$. This member will be compared with another family of sixth-order multiple-root finders to be developed in the next section of this paper.

3. Methodology and convergence analysis

We assume that a function $f : \mathbb{C} \to \mathbb{C}$ has a repeated zero α with integer multiplicity $m \geq 1$ and is analytic in a small neighborhood of α. Given an initial guess x_0 sufficiently close to α, new three-point iterative methods proposed in (1.2) to find an approximate zero α of multiplicity m will take the specific form of:

$$\begin{cases} y_n = x_n - m \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - m \cdot Q_f(s) \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - m \cdot K_f(s, v) \cdot \dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{3.1}$$

where

$$s = \left[\frac{f(y_n)}{f(x_n)}\right]^{\frac{1}{m}}, \tag{3.2}$$

$$v = \left[\frac{f(w_n)}{f(x_n)}\right]^{\frac{1}{m}}, \tag{3.3}$$

and where $Q_f : \mathbb{C} \to \mathbb{C}$ is analytic in a neighborhood of 0 and $K_f : \mathbb{C}^2 \to \mathbb{C}$ is holomorphic [20,39] in a neighborhood of (0, 0). Since s and ν are respectively one-to-m multiple-valued functions, we consider their principal analytic branches [1]. Hence, it is convenient to treat s as a principal root given by $s = \exp[\frac{1}{m}\text{Log}(\frac{f(y_n)}{f(x_n)})]$, with $\text{Log}(\frac{f(y_n)}{f(x_n)}) = \text{Log}\left|\frac{f(y_n)}{f(x_n)}\right| + i\,\text{Arg}(\frac{f(y_n)}{f(x_n)})$ for $-\pi < \text{Arg}(\frac{f(y_n)}{f(x_n)}) \le \pi$; this convention of $\text{Arg}(z)$ for $z \in \mathbb{C}$ agrees with that of Log[z] command of Mathematica [44] to be employed later in numerical experiments of Section 6. By means of further inspection of s, we find that $s = \left|\frac{f(y_n)}{f(x_n)}\right|^{\frac{1}{m}} \cdot \exp[\frac{i}{m}\,\text{Arg}(\frac{f(y_n)}{f(x_n)})] = O(e_n)$. Similarly we treat $\nu = \left|\frac{f(w_n)}{f(x_n)}\right|^{\frac{1}{m}} \cdot \exp[\frac{i}{m}\,\text{Arg}(\frac{f(w_n)}{f(x_n)})] = O(e_n^3)$. In addition, we find that $O(\frac{f(x_n)}{f'(x_n)}) = O(e_n)$.

Definition 1 (Error equation, asymptotic error constant, order of convergence)**.** Let $x_0, x_1, \ldots, x_n, \ldots$ be a sequence converging to α and $e_n = x_n - \alpha$ be the nth iterate error. If there exist real numbers $p \in \mathbb{R}$ and $b \in \mathbb{R} - \{0\}$ such that the following error equation holds

$$e_{n+1} = b{e_n}^p + O(e_n^{p+1}), \tag{3.4}$$

then b or $|b|$ is called the asymptotic error constant and p is called the order of convergence [42].

In this paper, we investigate the maximal convergence order of proposed methods (3.1). We here establish a main theorem describing the convergence analysis regarding proposed methods (3.1) and find out how to construct weight functions Q_f and K_f for sixth-order convergence. It suffices to consider both weight functions Q_f and K_f up to the fifth-order terms in e_n due to the fact that $O(\frac{f(x_n)}{f'(x_n)}) = O(e_n)$.

Applying the Taylor's series expansion of f about α, we get the following relations:

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!}{e_n}^m\left[1 + \theta_2 e_n + \theta_3 e_n^2 + \theta_4 e_n^3 + \theta_5 e_n^4 + \theta_6 e_n^5 + \theta_7 e_n^6 + O(e_n^7)\right], \tag{3.5}$$

$$f'(x_n) = \frac{f^{(m)}(\alpha)}{(m-1)!}{e_n}^{m-1}\left[1 + \frac{m+1}{m}\theta_2 e_n + \frac{m+2}{m}\theta_3 e_n^2 + \frac{m+3}{m}\theta_4 e_n^3 + \frac{m+4}{m}\theta_5 e_n^4 + \frac{m+5}{m}\theta_6 e_n^5 + O(e_n^6)\right], \tag{3.6}$$

where $\theta_k = \frac{m!}{(m-1+k)!}\frac{f^{(m-1+k)}(\alpha)}{f^{(m)}(\alpha)}$ for $k \in \mathbb{N} - \{1\}$. For convenience, we denote e_n by e without subscript n whenever required to do so.

Dividing (3.5) by (3.6), we have

$$\frac{f(x_n)}{f'(x_n)} = \frac{e}{m} - \frac{\theta_2 e^2}{m^2} + \frac{Y_3 e^3}{m^3} + \frac{Y_4 e^4}{m^4} + \frac{Y_5 e^5}{m^5} + \frac{Y_6 e^6}{m^6} + O(e^7), \tag{3.7}$$

where $Y_3 = (1+m)\theta_2^2 - 2m\theta_3, Y_4 = -(1+m)^2\theta_2^3 + m(4+3m)\theta_2\theta_3 - 3m^2\theta_4$, $Y_5 = (1+m)^3\theta_2^4 - 2m(1+m)(3+2m)\theta_2^2\theta_3 + 2m^2(3+2m)\theta_2\theta_4 + 2m^2((2+m)\theta_3^2 - 2m\theta_5)$ and $Y_6 = -(1+m)^4\theta_2^5 + m(1+m)^2(8+5m)\theta_2^3\theta_3 - m^2(1+m)(9+5m)\theta_2^2\theta_4 + m^2\theta_2(-(2+m)(6+5m)\theta_3^2 + m(8+5m)\theta_5) + m^3((12+5m)\theta_3\theta_4 - 5m\theta_6)$.

Thus, from relation (3.7), we obtain

$$y_n = \alpha + \frac{\theta_2 e^2}{m} - \frac{Y_3 e^3}{m^2} - \frac{Y_4 e^4}{m^3} - \frac{Y_5 e^5}{m^4} - \frac{Y_6 e^6}{m^5} + O(e^7). \tag{3.8}$$

$$\begin{aligned} f(y_n) = \frac{f^{(m)}(\alpha)}{m!}\left(\frac{\theta_2}{m}\right)^m e^{2m}\Bigg\{ & 1 - \frac{Y_3}{\theta_2}e + \frac{(m-1)Y_3^2 - 2Y_4\theta_2 + 2\theta_2^4}{2m\theta_2^2}e^2 \\ & - \frac{(m-1)(m-2)Y_3^3 + 6Y_5\theta_2^2 + 6Y_3\theta_2(Y_4 - mY_4 + (m+1)\theta_2^3)}{6m^2\theta_2^3}e^3 \\ & + \frac{1}{24m^3\theta_2^4}[(m-1)(m-2)(m-3)(m-3)Y_3^4 + 24(m-1)Y_3Y_5\theta_2^2 \\ & + 12Y_3^2\theta_2(-(m-1)(m-2)Y_4 + m(m+1)\theta_2^3) \\ & + 12\theta_2^2((m-1)Y_4^2 - 2Y_6\theta_2 - 2(m+1)Y_4\theta_2^3 + 2m\theta_2^4\theta_3)]e^4 + O(e^5)\Bigg\}. \end{aligned} \tag{3.9}$$

By Taylor's expansion or multinomial expansion, we get an expression s in (3.2) as follows:

$$s = \frac{\theta_2}{m}e - \frac{Y_3 + \theta_2^2}{m^2}e^2 + \frac{-2Y_4 + \theta_2(2Y_3 + (m+3)\theta_2^2 - 2m\theta_3)}{2m^3}e^3 + \frac{W_4}{6m^4}e^4 + \frac{W_5}{24m^5}e^5 + O(e^6), \tag{3.10}$$

where $W_4 = (2m^2 + 3m + 7)\theta_2^4 + 3\theta_2^2((m+5)Y_3 - 2m(m+1)\theta_3) + 6(Y_5 - mY_3\theta_3) + 6\theta_2(-Y_4 + m^2\theta_4)$, and $W_5 = (6m^3 + 11m^2 + 6m + 25)\theta_2^5 + 4\theta_2^3((2m^2 + 3m + 13)Y_3 - 3m(m+1)(2m+1)\theta_3) + 24(-Y_6 + mY_4\theta_3) + 24m^2Y_3\theta_4 + \theta_2^2(-12(m+5)Y_4 + 24m^2(m+1)\theta_4) + \theta_2[12(2Y_3^2 + 2Y_5 - 2m(m+1)Y_3\theta_3 + m^2(m+1)\theta_3^2) - 24m^3\theta_5]$.

With the use of s in (3.10), expanding Taylor series of $Q_f(s)$ about 0 up to fifth-order terms we find:

$$Q_f(s) = A_0 + A_1 s + A_2 s^2 + A_3 s^3 + A_4 s^4 + A_5 s^5 + O(e^6), \tag{3.11}$$

where $A_j = \frac{Q_f^{(j)}(0)}{j!}$ for $0 \le j \le 5$.

Hence by substituting (3.5)–(3.11) into w_n in (3.1) with explicit use of $Y_j (3 \le j \le 6)$ from relation (3.7), we find:

$$w_n = \alpha + (1 - A_0)e + \frac{(A_0 - A_1)}{m}\theta_2 e^2 + \frac{2m(A_0 - A_1)\theta_3 - [(m+1)A_0 - (m+3)A_1 + A_2]\theta_2^2}{m^2} e^3 + Z_4 e^4 + Z_5 e^5 + Z_6 e^6 + O(e^7), \tag{3.12}$$

where $Z_i = Z_i(\theta_2, \theta_3 \ldots, \theta_7, A_0, A_1, \ldots, A_3)$ for $4 \le i \le 6$. By selecting $A_0 = 1, A_1 = 1, A_2 = 2,$ we have

$$w_n = \alpha + \frac{(m + 9 - 2A_3)\theta_2^3 - 2m\theta_2\theta_3}{2m^3} e^4 + Z_5 e^5 + Z_6 e^6 + O(e^7), \tag{3.13}$$

Hence, we obtain $f(w_n)$ as follows:

$$f(w_n) = \frac{f^{(m)}(\alpha)}{m!}\left(\frac{\theta_2}{2m^3}\right)^m e^{4m}\Bigg[((9 + m - 2A_3)\theta_2^2 - 2m\theta_3)^m + \frac{2m^4 Z_5((9 + m - 2A_3)\theta_2^2 - 2m\theta_3)^{m-1}}{\theta_2} e + \frac{2m^4((m-1)m^3 Z_5^2 + (9 + m - 2A_3)Z_6\theta_2^3 - 2mZ_6\theta_2\theta_3)((9 + m - 2A_3)\theta_2^2 - 2m\theta_3)^{m-2}}{\theta_2^2} e^2 + O(e)^3\Bigg]. \tag{3.14}$$

With the use of (3.9) and (3.14), we get an expression ν in (3.3) after Taylor's expansion or multinomial expansion as follows:

$$\nu = \frac{\theta_2((9 + m - 2A_3)\theta_2^2 - 2m\theta_3)e^3}{2m^3} + \left(Z_5 + \frac{-(9 + m - 2A_3)\theta_2^4 + 2m\theta_2^2\theta_3}{2m^4}\right)e^4 + \frac{4m^4(mZ_6 - Z_5\theta_2) + (9 - 2A_3)\theta_2^5 - 2m(A_3 - 5)\theta_2^3(\theta_2^2 - 2\theta_3) + m^2\theta_2(\theta_2^2 - 2\theta_3)^2}{4m^5} e^5 + O(e^6). \tag{3.15}$$

Using s in (3.10) and ν in (3.15) and expanding Taylor series of $K_f(s, \nu)$ about (0, 0) up to fifth-order terms we find:

$$K_f(s, \nu) = K_{00} + K_{10}s + K_{20}s^2 + K_{30}s^3 + K_{40}s^4 + K_{50}s^5 + (K_{01} + K_{11}s + K_{21}s^2)\nu + O(e^6), \tag{3.16}$$

where $K_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j}}{\partial s^i \partial \nu^j} K_f(s, \nu)|_{(s=0,\nu=0)}$ for $0 \le i < 5$ and $0 < j \le 1$.

Hence by substituting (3.5)–(3.16) into the proposed method (3.1) with explicit uses of $Y_j (3 \le j \le 6)$, Z_5, Z_6, we obtain the error equation as

$$x_{n+1} - \alpha = x_n - \alpha - K_f(s, \nu) \cdot \frac{f(x_n)}{f'(x_n)} = L_1 e + L_2 e^2 + L_3 e^3 + L_4 e^4 + L_5 e^5 + L_6 e^6 + O(e^7), \tag{3.17}$$

where $L_1 = (1 - K_{00})$ and the coefficients $L_i (2 \le i \le 6)$ generally depend on m, the parameters $Q_j (j = 0, 1, \cdots, 5)$ and $\theta_i (i = 1, 2, \cdots,)$. Solving $L_1 = 0$ for K_{00}, we get

$$K_{00} = 1. \tag{3.18}$$

Substituting $K_{00} = 1$ into $L_2 = 0$ and simplifying, we obtain $\frac{(1-K_{10})}{m}\theta_2 = 0$, from which

$$K_{10} = 1 \tag{3.19}$$

follows independently of θ_2. Substituting $K_{00} = 1, K_{10} = 1$ into $L_3 = 0$ and simplifying yields:

$$-\frac{(K_{20} - 2)}{m^2}\theta_2^2 = 0, \tag{3.20}$$

from which we find

$$K_{20} = 2. \tag{3.21}$$

Substituting $K_{00} = 1, K_{10} = 1, K_{20} = 2$ into $L_4 = 0$ and simplifying yields:

$$L_4 = \frac{9 - 2K_{30} + m - K_{01}(9 + m - 2A_3)}{2m^3}\theta_2^3 + \frac{(K_{01} - 1)}{m^2}\theta_2\theta_3 = 0, \tag{3.22}$$

from which

$$K_{01} = 1, K_{30} = A_3 \tag{3.23}$$

follows independently of θ_2 and θ_3.

Substituting $K_{00} = 1, K_{10} = 1, K_{20} = 2, K_{01} = 1, K_{30} = A_3$ into $L_5 = 0$ and simplifying yields:

$$L_5 = \theta_2^2 \frac{(-K_{11}(9 + m - 2A_3) + 2(9 - K_{40} + m - 2A_3 + A_4))\theta_2^2 + 2(K_{11} - 2)m\theta_3}{2m^4} = 0, \tag{3.24}$$

from which we obtain independently of θ_2 and θ_3:

$$K_{40} = A_4, K_{11} = 2. \tag{3.25}$$

Substituting $K_{00} = 1, K_{10} = 1, K_{20} = 2, K_{01} = 1, K_{30} = A_3, K_{40} = A_4, K_{11} = 2$ into L_6, we obtain

$$L_6 = \frac{\theta_2}{4m^5}\left[\phi\theta_2^4 + 4m(K_{21} - 10 - m + A_3)\theta_2^2\theta_3 + 4m^2\theta_3^2\right], \tag{3.26}$$

where

$$\phi = 99 - 4K_{50} + 20m + m^2 - 2K_{21}(9 + m - 2A_3) - 2(11 + m)A_3 + 4A_5. \tag{3.27}$$

The consequence of the analysis carried out thus far immediately leads us to the following theorem.

Theorem 3.1. *Let $m \in \mathbb{N}$ be given. Let $f : \mathbb{C} \to \mathbb{C}$ have a zero α of multiplicity m and be analytic in a small neighborhood of α. Let $k \in \mathbb{N}$ be given. Let $\theta_j = \frac{m!}{(m-1+j)!} \cdot \frac{f^{(m-1+j)}(\alpha)}{f^{(m)}(\alpha)}$ for $j \in \mathbb{N} - \{1\}$. Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α. Let $A_j (0 \le j \le 5)$ and $K_{ij} (0 \le i \le 5, 0 \le j \le 1)$ be respectively defined in (3.11) and (3.16). Suppose that $A_0 = A_1 = 1, A_2 = 2, |A_3| < \infty, |A_4| < \infty, |A_5| < \infty$, and $K_{00} = K_{10} = K_{01} = 1, K_{11} = K_{20} = 2, K_{30} = A_3, K_{40} = A_4, |K_{50}| < \infty, |K_{21}| < \infty$ hold. Then iterative methods (3.1) are of sixth-order and possess the following error equation:*

$$e_{n+1} = \frac{\theta_2}{4m^5}\left[\phi\theta_2^4 + 4m(K_{21} - 10 - m + A_3)\theta_2^2\theta_3 + 4m^2\theta_3^2\right]e_n^6 + O(e_n^7), \tag{3.28}$$

where ϕ is given in (3.27).

4. Special cases of weight functions

As a result of Theorem 3.1, Taylor-polynomial forms of $Q_f(s)$ and $K_f(s, \upsilon)$ are easily given by

$$\begin{cases} Q_f(s) = A_0 + A_1 s + A_2 s^2 + A_3 s^3 + A_4 s^4 + A_5 s^5, \\ K_f(s, \upsilon) = K_{00} + K_{10}s + K_{20}s^2 + K_{30}s^3 + K_{40}s^4 + K_{50}s^5 + (K_{01} + K_{11}s + K_{21}s^2)\upsilon, \end{cases} \tag{4.1}$$

where $A_0 = A_1 = 1, A_2 = 2, A_3, A_4, A_5$ (maybe free) and $K_{00} = K_{01} = K_{10} = 1, K_{20} = K_{11} = 2, K_{30} = A_3, K_{40} = A_4$.

Although a variety forms of weight functions $Q_f(s)$ and $K_f(s, \upsilon)$ are available, we will limit ourselves to considering several forms of low-order polynomials or simple rational functions.

Case 1: Polynomial weight functions: $A_0 = A_1 = 1, A_2 = 2, A_3 = A_4 = A_5 = 0$ and $K_{00} = K_{10} = K_{01} = 1, K_{20} = K_{11} = 2, K_{30} = A_3 = 0, K_{40} = A_4 = 0$, $K_{50}, K_{21} =$ free.

$$\begin{cases} Q_f(s) = 1 + s + 2s^2, \\ K_f(s, \upsilon) = 1 + s + 2s^2 + K_{50}s^5 + (1 + 2s + K_{21}s^2)\upsilon, \end{cases} \tag{4.2}$$

Case 1A: When $K_{50} = K_{21} = 0$

$$\begin{cases} Q_f(s) = 1 + s + 2s^2, \\ K_f(s, \upsilon) = 1 + s + 2s^2 + (1 + 2s)\upsilon, \end{cases} \tag{4.3}$$

Case 1B: When $K_{50} = 0$

$$\begin{cases} Q_f(s) = 1 + s + 2s^2, \\ K_f(s, \upsilon) = 1 + s + 2s^2 + (1 + 2s + K_{21}s^2)\upsilon, \end{cases} \tag{4.4}$$

Case 1C: When $K_{21} = 0$

$$\begin{cases} Q_f(s) = 1 + s + 2s^2, \\ K_f(s, \upsilon) = 1 + s + 2s^2 + K_{50}s^5 + (1 + 2s)\upsilon, \end{cases} \tag{4.5}$$

Case 2: Rational weight functions

$$\begin{cases} Q_f(s) = \dfrac{1 + (b-1)s + bs^2}{1 + (b-2)s}, b \in \mathbb{R} - \{2\}, \\ K_f(s, \upsilon) = \dfrac{q_0 + q_1 s + q_2 s^2 + (q_3 + q_4 s)\upsilon}{1 + r_1 s + r_2 s^2 + (r_3 + r_4 s)\upsilon}, \end{cases} \tag{4.6}$$

with $A_3 = 2(2-b), A_4 = 2(2-b)^2, A_5 = 2(2-b)^3$ and $q_0 = 1, q_1 = -1 + b, q_2 = b, q_3 = 1 - b + q_4 - r_4, r_1 = -2 + b, r_2 = 0, r_3 = -b + q_4 - r_4$. If $b = 2$, then Q_f becomes a polynomial being equivalent to **Case 1**. One should note that four parameters q_3, q_4, r_3, r_4 define a linear system of rank 2, if b is given. Hence, any two of them can be solved in terms of remaining two free parameters for a given b. The following sub-cases are of interest with a choice of $b = 1, q_1 = 0, q_2 = 1$ and $r_1 = -1$.

Case 2A: $b = 1, r_3 = -1, q_3 = 0, r_4 = q_4 = 0.$

$$\begin{cases} Q_f(s) = \dfrac{1+s^2}{1-s}, \\ K_f(s,v) = \dfrac{1+s^2}{1-s-v}. \end{cases} \tag{4.7}$$

Case 2B: $b = 1, r_3 = -2, q_3 = -1, r_4 = 1, q_4 = 0.$

$$\begin{cases} Q_f(s) = \dfrac{1+s^2}{1-s}, \\ K_f(s,v) = \dfrac{1+s^2-v}{1-s+(s-2)v}. \end{cases} \tag{4.8}$$

Case 2C: $b = 1, r_3 = -1, q_3 = 0, r_4 = 1, q_4 = 1.$

$$\begin{cases} Q_f(s) = \dfrac{1+s^2}{1-s}, \\ K_f(s,v) = \dfrac{1+s^2+sv}{1-s+(s-1)v}. \end{cases} \tag{4.9}$$

Case 3: Mixture of rational and polynomial weight functions

$$\begin{cases} Q_f(s) = \dfrac{1+(b-1)s+bs^2}{1+(b-2)s}, b \in \mathbb{R} - \{2\}, \\ K_f(s,v) = 1+s+2s^2+2(2-b)s^3+2(2-b)^2s^4+K_{50}s^5+(1+2s+K_{21}s^2)v. \end{cases} \tag{4.10}$$

Case 3A: $b = 1, K_{50} = K_{21} = 0.$

$$\begin{cases} Q_f(s) = \dfrac{1+s^2}{1-s}, \\ K_f(s,v) = 1+s+2s^2+2s^3+2s^4+(2s+1)v, \end{cases} \tag{4.11}$$

Case 3B: $b = 1, K_{50} = 0, K_{21} = 1.$

$$\begin{cases} Q_f(s) = \dfrac{1+s^2}{1-s}, \\ K_f(s,v) = 1+s+2s^2+2s^3+2s^4+(s+1)^2v, \end{cases} \tag{4.12}$$

Case 4: Mixture of polynomial and rational weight functions

$$\begin{cases} Q_f(s) = 1+s+2s^2, \\ K_f(s,v) = \dfrac{1+s+2s^2+(q_3+q_4s)v}{1+(r_3+r_4s)v}, \end{cases} \tag{4.13}$$

where $q_3 = 1 + r_3, r_4 = -2 + q_4 - r_3$. One should note that four parameters q_3, q_4, r_3, r_4 define a linear system of rank 2. Hence, any two of them can be solved in terms of remaining two free parameters. The following sub-cases are of interest.

Case 4A: $r_4 = 0, q_4 = 0.$

$$K_f(s,v) = \frac{1+s+2s^2-v}{1-2v}, \tag{4.14}$$

Case 4B: $r_4 = 0, q_3 = 0.$

$$K_f(s,v) = \frac{1+s+2s^2+sv}{1-v}, \tag{4.15}$$

Case 4C: $q_3 = 0, q_4 = 0.$

$$K_f(s,v) = \frac{1+s+2s^2}{1-(1+s)v}, \tag{4.16}$$

Case 4D: $q_4 = 0, r_3 = 0.$

$$K_f(s,v) = \frac{1+s+2s^2+v}{1-2sv}, \tag{4.17}$$

Case 5: Low-order weight functions for purely imaginary extraneous fixed points

$$\begin{cases} Q_f(s) = \dfrac{1+(b-c-1)s+bs^2}{1+(b-c-2)s+cs^2},\ b,c\in\mathbb{R}, \\ K_f(s,v) = \dfrac{1+(b-c-1)s+bs^2}{1+(b-c-2)s+cs^2+[(1-b+c)s-1]v}, \end{cases} \tag{4.18}$$

where $b, c \in \mathbb{R}$ are free parameters excluding $b = 2, c = 0$. Both weight functions Q_f and K_f clearly satisfy the required conditions for their coefficients stated in (4.1). The detailed analysis for a possible combination of (b, c)-parameters leading to purely imaginary extraneous fixed points is described in the latter part of Section 5. The nature of $F_1(\zeta)$ in (5.5) and $F_2(\zeta)$ in (5.6) enables us to consider two cases **5X** and **5Y**, respectively. The following sub-cases are our interest.

Case 5X: Selection of parameters (b, c) leading to the negative roots of $F_1(\zeta)$ given by (5.5).

Case 5XA: $b = 0, c = \frac{4(2+b)}{3} = \frac{8}{3}$.

$$\begin{cases} Q_f(s) = \dfrac{3-11s}{(2s-3)(4s-1)}, \\ K_f(s,v) = \dfrac{3-11s}{3-14s+8s^2-(3-11s)v}. \end{cases} \tag{4.19}$$

Case 5XB: $b = 1, c = \frac{4(2+b)}{3} = 4$.

$$\begin{cases} Q_f(s) = \dfrac{1-4s+s^2}{(s-1)(4s-1)}, \\ K_f(s,v) = \dfrac{1-4s+s^2}{(4s-1)(s+v-1)}. \end{cases} \tag{4.20}$$

Case 5XC: $b = 2, c = \frac{4(2+b)}{3} = \frac{16}{3}$.

$$\begin{cases} Q_f(s) = \dfrac{3-13s+6s^2}{(4s-3)(4s-1)}, \\ K_f(s,v) = \dfrac{3-13s+6s^2}{3-16s+16s^2-(3-13s)v}. \end{cases} \tag{4.21}$$

Case 5XD: $b = 4, c = \frac{4(2+b)}{3} = 8$.

$$\begin{cases} Q_f(s) = \dfrac{s-1}{2s-1}, \\ K_f(s,v) = \dfrac{(s-1)(4s-1)}{1-6s+8s^2-(1-5s)v}. \end{cases} \tag{4.22}$$

Case 5XE: $b = \frac{1}{2}, c = \frac{1}{2}$.

$$\begin{cases} Q_f(s) = \dfrac{2-2s+s^2}{2-4s+s^2}, \\ K_f(s,v) = \dfrac{2-2s+s^2}{2-4s+s^2-2(1-s)v}. \end{cases} \tag{4.23}$$

Case 5XF: $b = \frac{6}{5}, c = 3$.

$$\begin{cases} Q_f(s) = \dfrac{5-14s+6s^2}{5-19s+15s^2}, \\ K_f(s,v) = \dfrac{5-14s+6s^2}{5-19s+15s^2-(5-14s)v}. \end{cases} \tag{4.24}$$

Case 5XG: $b = 1, c = 2$.

$$\begin{cases} Q_f(s) = \dfrac{s-1}{2s-1}, \\ K_f(s,v) = \dfrac{(s-1)^2}{(2s-1)(s-1+v)}. \end{cases} \tag{4.25}$$

Case 5XH: $b = 5, c = 9$.

$$\begin{cases} Q_f(s) = \dfrac{1-5s+5s^2}{(3s-1)^2}, \\ K_f(s,v) = \dfrac{1-5s+5s^2}{1-6s+9s^2-(1-5s)v}. \end{cases} \tag{4.26}$$

Note that sub-cases **5XA, 5XB, 5XC, 5XD** and **5XE, 5XF, 5XG, 5XH** yield uniparametric and biparametric negative roots of $F_1(\zeta)$, respectively.

Case 5Y: Selection of parameters (b, c) leading to the negative roots of $F_2(\zeta)$ given by (5.6).
Case 5YA: $b = 12, c = \frac{(12+5b)}{4} = 18$.

$$\begin{cases} Q_f(s) = \dfrac{(3s-1)(4s-1)}{1-8s+18s^2}, \\ K_f(s,v) = \dfrac{(3s-1)(4s-1)}{1-8s+18s^2-(1-7s)v}. \end{cases} \tag{4.27}$$

Case 5YB: $b = 6, c = \frac{(12+5b)}{4} = \frac{21}{2}$.

$$\begin{cases} Q_f(s) = \dfrac{(3s-2)(4s-1)}{(3s-1)(7s-2)}, \\ K_f(s,v) = \dfrac{(3s-2)(4s-1)}{2-13s+21s^2-(2-11s)v}. \end{cases} \tag{4.28}$$

Case 5YC: $b = 8, c = \frac{(12+5b)}{4} = 13$.

$$\begin{cases} Q_f(s) = \dfrac{(2s-1)(4s-1)}{1-7s+13s^2}, \\ K_f(s,v) = \dfrac{(2s-1)(4s-1)}{1-7s+13s^2-(1-6s)v}. \end{cases} \tag{4.29}$$

Case 5YD: $b = 1, c = \frac{5}{2}$.

$$\begin{cases} Q_f(s) = \dfrac{(s-2)(2s-1)}{(s-1)(5s-2)}, \\ K_f(s,v) = \dfrac{(s-2)(2s-1)}{(5s-2)(s+v-1)}. \end{cases} \tag{4.30}$$

Case 5YE: $b = 4, c = 7$.

$$\begin{cases} Q_f(s) = \dfrac{(2s-1)^2}{1-5s+7s^2}, \\ K_f(s,v) = \dfrac{(2s-1)^2}{1-5s+7s^2+(4s-1)v}. \end{cases} \tag{4.31}$$

Case 5YF: $b = \frac{1}{6}, c = 0$.

$$\begin{cases} Q_f(s) = \dfrac{(s-3)(s-2)}{6-11s}, \\ K_f(s,v) = \dfrac{(s-3)(s-2)}{6-11s+(5s-6)v}. \end{cases} \tag{4.32}$$

Note that sub-cases **5YA, 5YB, 5YC** and **5YD, 5YE, 5YF** yield uniparametric and biparametric negative roots of $F_2(\zeta)$, respectively.

For selected cases **5XA, 5XH, 5YA, 5YF**, Table 2 lists the corresponding purely imaginary extraneous fixed points.

5. Extraneous fixed points

In this section, we will investigate the extraneous fixed points [22,43] of the iterative map (3.1) and relevant dynamics associated with their basins of attraction. The dynamics underlying basins of attraction was initiated by Stewart [41] and followed by works of Amat et al. [2–5], Scott et al. [38], Chun et al. [10], Chun-Neta [11], Chicharro et al. [8], Cordero et al. [15], Neta et al. [28,33], Argyros-Magreñan [7], Magreñan [27], Magreñan et al. [26], Andreu et al. [6] and Chun et al. [12]. The only papers comparing basins of attraction for methods to obtain multiple roots are due to Neta et al. [29], Neta-Chun [30,34], Chun-Neta [13,14] and Geum-Kim-Neta [19].

A zero α of a nonlinear equation $f(x) = 0$ can be located by a fixed point ξ of iterative methods of the form

$$x_{n+1} = R_f(x_n), n = 0, 1, \ldots, \tag{5.1}$$

where R_f is the iteration function under consideration. In general, R_f might possess other fixed points $\xi \neq \alpha$. Such fixed points are called the *extraneous fixed points* of the iteration function R_f. Extraneous fixed points may result in attractive, indifferent or repulsive cycles as well as other periodic orbits influencing the dynamics behind the basins of attraction. Exploration of such dynamics is clearly another goal of our current analysis, which leads us to a more specific form of iterative maps (5.1) as follows:

$$x_{n+1} = R_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n), \tag{5.2}$$

where $H_f(x_n) = m \cdot K_f(s, \upsilon)$ can be regarded as a weight function of the classical Newton's method. It is obvious that α is a fixed point of R_f. The points $\xi \neq \alpha$ for which $H_f(\xi) = 0$ are extraneous fixed points of R_f.

For an analysis of the relevant dynamics, we limit ourselves to considering only combinations of weight functions $Q_f(s)$ and $K_f(s, \upsilon)$ in the form of quadratic rational functions as shown in **Case 5** of Section 4. Other types of combinations have empirically shown poor convergence in the existing studies by [13,19,29,34]. A special attention will be paid to some selected cases **1A, 2A, 2B, 2C, 3A, 4A** as well as all **5X** and **5Y** in order to pursue further properties of extraneous fixed points and relevant dynamics associated with their basins of attraction. The existence of such extraneous fixed points would affect the global iteration dynamics, which was demonstrated for simple zeros via König functions and Schröder functions applied to a family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$ by Vrscay and Gilbert [43]. Especially the presence of attractive cycles induced by the extraneous fixed points of R_f may alter the basins of attraction due to the trapped sequence $\{x_n\}$. Even in the case of repulsive or indifferent fixed points, an initial value x_0 chosen near a desired root may converge to another unwanted remote root. Indeed, these aspects of the Schröder functions [43] were observed in an application to the same family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$.

For simplified analysis of such dynamics related to the extraneous fixed points underlying the basins of attraction for iterative maps (5.2), we first choose a quadratic polynomial from the family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$ employed by Vrscay and Gilbert [43] . By closely following the works of Chun et al. [9,13] and Neta et al. [28,33,34], we then construct $H_f(x_n) = m \cdot K_f(s, \upsilon)$ in (5.2). We now take the multiplicity m of the zeros α into consideration and apply a polynomial $f(z) = (z^2 - 1)^m$ to $H_f(x_n)$ and construct $H(z)$, with a change of a variable $\zeta = z^2$, in the form of

$$H(z) = \mathcal{A}(\zeta) \cdot F(\zeta), \tag{5.3}$$

where $\mathcal{A}(\zeta)$ may represent a term of a repeated zero root ζ of integer multiplicity with a constant factor which may be dependent on m; $F(\zeta)$ may indeed contain the extraneous fixed points H. Thus the extraneous fixed points ξ of H can be found from the roots $\zeta \neq 0$ (other wise s in (3.2) is not defined) of $F(\zeta)$ via relation $\xi = \zeta^{\frac{1}{2}}$. Note that $F(\zeta)$ contains rational terms with fractional powers. It must be emphasized that any general algebraic ways of zero-finding of $F(\zeta)$ seem to be infeasible. By a suitable change of variables for the terms with fractional powers as well as through a finite number of algebraic operations, $F(\zeta)$ can be transformed into a multivariate rational function, which then can be solved with known polynomial root-finding methods. In fact, $F(\zeta)$ for the selected cases **1A, 2A, 2B, 2C, 3A, 4A** as well as all **5X** and **5Y** fortunately form rational equations in ζ, whose numbers of roots $\zeta \neq 0$ are respectively given by 6, 4, 5, 6, 6, 5, 3, 4, 4, 3, 6, 6, 6, 6, 5, 5, 5, 5, 6, 4, after a close inspection of their numerators. From Remark 5.1, we find that the desired extraneous fixed points are determined regardlessly of m. Functions $\mathcal{A}(\zeta)$, $F(\zeta)$ and the number of ζ are explicitly displayed for the selected cases **1A, 2A, 2B, 2C, 3A, 4A** as well as all **5X** and **5Y** in Table 2.

Remark 5.1. With $Q_f(s) = \frac{1+(b-c-1)s+bs^2}{1+(b-c-2)s+cs^2}$ for $f(z) = (z^2 - 1)^m$, we find that s and υ are independent of m below: $s = \left[\frac{f(y)}{f(x)}\right]^{\frac{1}{m}} = \frac{1}{4}(1 - \frac{1}{z^2})$ and $\upsilon = \left[\frac{f(w)}{f(x)}\right]^{\frac{1}{m}} = \frac{(z^2-1)^3[b^2+2(-5b^2-2c^2+b(4+6c))z^2+(4+3b-2c)^2z^4]}{4z^2[c+2(4-2b+c)z^2+(8+4b-3c)z^4]^2}$, where $y = z - m \cdot \frac{f(z)}{f'(z)}$ and $w = y - m \cdot Q_f(s)\frac{f(z)}{f'(z)}$. As a result, $K_f(s, \upsilon)$ in (4.18) is independent of m. Hence, the roots of $H(z) = m \cdot K_f(s, \upsilon) = 0$, i.e., the roots of $K_f(s, \upsilon) = 0$ other than zeros of f are the desired extraneous fixed points, being independent of m.

It is interesting to find a combination of Q_f and K_f leading to purely imaginary extraneous fixed points, whose investigation was done by Chun et al. [9]. We first describe the following lemma on the negative real roots of a quadratic equation for later use.

Lemma 5.1. *Let $q(x) = ax^2 + bx + c$ be a quadratic equation with real coefficients $a \neq 0$, b, c satisfying $b^2 - 4ac \geq 0$. Let r_1 and r_2 be the two roots of $q(x) = 0$. Then both roots $r_1 < 0$ and $r_2 < 0$ hold if and only if all three coefficients a, b, c have the same sign.*

Proof. The hypothesis $b^2 - 4ac \geq 0$ guarantees that all the roots of $q(x) = 0$ are real. One should note that $r_1 < 0$ and $r_2 < 0$ hold if and only if $-\frac{b}{a} = r_1 + r_2 < 0$ and $\frac{c}{a} = r_1 r_2 > 0$. We easily get $ab > 0$ and $ac > 0$ from relations $-\frac{b}{a} < 0$ and $\frac{c}{a} > 0$. Hence, $r_1 < 0$ and $r_2 < 0$ if and only if all three coefficients a, b, c have the same sign. □

We now consider **Case 5** described by (4.18) to discuss purely imaginary extraneous fixed points. After applying $f(z) = (z^2 - 1)^m$ to compute s and v, we get K_f with $\zeta = z^{1/2}$ below:

$$K_f(s, v) = \frac{F_1(\zeta)^2 \cdot F_2(\zeta)}{c^3 + b^2(1 + c - b) + \sum_{j=1}^{6} \rho_j \zeta^j}, \tag{5.4}$$

where

$$F_1(\zeta) = c + 2(4 - 2b + c)\zeta + (8 + 4b - 3c)\zeta^2, \tag{5.5}$$

$$F_2(\zeta) = b - 2(-2 + 3b - 2c)\zeta + (12 + 5b - 4c)\zeta^2, \tag{5.6}$$

and ρ_j $(1 \leq j \leq 6)$ is a bivariate polynomial in b and c.

As a result, we can obtain the extraneous fixed points $\xi = \zeta^{1/2}$ by finding the zeros ζ of F_1 or F_2. The corresponding repeated real zeros ζ of F_1 are easily found to be:

$$\zeta = \begin{cases} \dfrac{2b - c - 4 \pm 2\sqrt{b^2 + c^2 - 2b(2 + c) + 4}}{4b - 3c + 8}, & \text{if } 4b - 3c + 8 \neq 0 \text{ and } b^2 + c^2 - 2b(2 + c) + 4 \geq 0, \\ \dfrac{b + 2}{b - 10}, & \text{if } 4b - 3c + 8 = 0 \text{ and } b \neq 10. \end{cases} \tag{5.7}$$

Similarly, the corresponding real zeros ζ of F_2 are found to be:

$$\zeta = \begin{cases} \dfrac{3b - 2c - 2 \pm 2\sqrt{b^2 + (1 + c)^2 - 2b(3 + c)}}{5b - 4c + 12}, & \text{if } 5b - 4c + 12 \neq 0 \text{ and } b^2 + (1 + c)^2 - 2b(3 + c) \geq 0, \\ \dfrac{b}{b - 16}, & \text{if } 5b - 4c + 12 = 0 \text{ and } b \neq 16. \end{cases} \tag{5.8}$$

One should be aware that the one-parametric second solutions in (5.7) and (5.8) are found from the degenerated linear cases of $F_1(\zeta)$ and $F_2(\zeta)$ with vanishing coefficients in their quadratic-order terms. We are now ready to begin an analysis leading to purely imaginary extraneous fixed points from the roots of F_1 and F_2. We start with F_1 for its one-parametric solution followed by its two-parametric solution. In view of relation $\xi = \zeta^{1/2}$ between extraneous fixed points ξ and the zero ζ, values of one-parametric zeros ζ should be negative for purely imaginary extraneous fixed points ξ. Hence

$$\zeta = \frac{b + 2}{b - 10} < 0, \tag{5.9}$$

from which the value of b must satisfy the inequality

$$-2 < b < 10, \tag{5.10}$$

and the corresponding purely imaginary extraneous fixed points ξ are given by:

$$\xi = \left(\frac{b + 2}{b - 10}\right)^{1/2} \text{ for } -2 < b < 10. \tag{5.11}$$

Typical values of $b \in \{0, 1, 2, 4\}$ with $c = \frac{4(2+b)}{3}$ are considered in sub-cases **5XA, 5XB, 5XC, 5XD**.

For all values of two-parametric zeros ζ of F_1 to be negative, all the coefficients should have the same sign according to Lemma 5.1. After a lengthy algebra to have the coefficients of the same sign with the help of Mathematica symbolic capability, we find that (b, c) satisfies the relation for desired negative values of ζ:

$$\begin{cases} 0 < c < \dfrac{4(2 + b)}{3}, & \text{if } -2 < b \leq 1, \\ 0 < c \leq -2\sqrt{b - 1} + b \text{ or } 2\sqrt{b - 1} + b \leq c < \dfrac{4(2 + b)}{3}, & \text{if } 1 < b \leq 2, \\ 2\sqrt{b - 1} + b \leq c < \dfrac{4(2 + b)}{3}, & \text{if } 2 < b < 10, \end{cases} \tag{5.12}$$

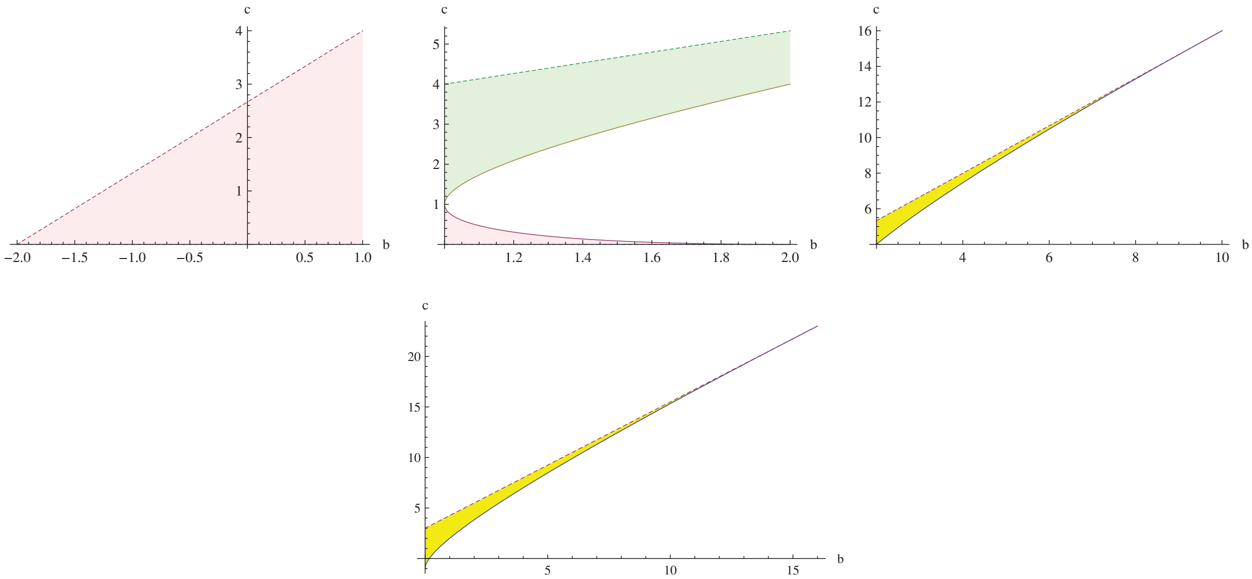

Fig. 1. Regions of selectable (b, c)-parameters for negative roots of $F_1(\zeta)$ and $F_2(\zeta)$, respectively from top to bottom.

and obtain the desired purely imaginary extraneous fixed points ξ given by:

$$\xi = \left(\frac{2b - c - 4 \pm 2\sqrt{b^2 + c^2 - 2b(2+c) + 4}}{4b - 3c + 8}\right)^{1/2}. \tag{5.13}$$

Typical values of $(b, c) \in \{(\frac{1}{2}, \frac{1}{2}), (\frac{6}{5}, 3), (1, 2), (5, 9)\}$ are considered in sub-cases **5XE, 5XF, 5XG, 5XH**.

Similar treatment for F_2 leads us to obtaining purely imaginary extraneous fixed points ξ given by:

$$\xi = \left(\frac{b}{b - 16}\right)^{1/2} \quad \text{for } 0 < b < 16. \tag{5.14}$$

as well as

$$\xi = \left(\frac{3b - 2c - 2 \pm 2\sqrt{b^2 + (1+c)^2 - 2b(3+c)}}{5b - 4c + 12}\right)^{1/2}, \tag{5.15}$$

for (b, c) satisfying the relation below:

$$-1 + 2\sqrt{b} + b \le c < \tfrac{1}{4}(12 + 5b), \qquad \text{for } 0 < b < 16. \tag{5.16}$$

Typical values of $b \in \{4, 6, 8\}$ and $c = \frac{12+5b}{4}$ are considered in sub-cases **5YA, 5YB, 5YC** as well as $(b, c) \in \{(1, \frac{5}{2}), (4, 7), (\frac{1}{6}, 0)\}$ in sub-cases **5YD, 5YE, 5YF**. Indeed, Fig. 1 illustrates appropriate shaded (b, c)-parameter regions for a bi-parametric family of negative roots of F_1 and F_2. Consequently, combinations of parameters (b, c) can be selected from these shaded regions for purely imaginary extraneous fixed points, and some of them are shown in sub-cases of **Case 5**, which give the desired purely imaginary extraneous fixed points listed in Table 1.

Our next goal is to extensively investigate the complex dynamics of the iterative map R_p of the form

$$z_{n+1} = R_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)} H_p(z_n), \tag{5.17}$$

in connection with the basins of attraction for a variety of polynomials $p(z_n)$ and a weight function $H_p(z_n)$. Indeed, $R_p(z)$ represents the classical Newton's method with weight function $H_p(z)$ and may possess its fixed points as zeros of $p(z)$ or extraneous fixed points associated with $H_p(z)$. As a result, basins of attraction for the fixed points or the extraneous fixed points as well as their attracting periodic orbits would reflect complex dynamics whose illustrative description will be made for various polynomials in the latter part of Section 6.

We now continue to describe the dynamical behavior of (5.17) when $p(z) = (z^2 - 1)^m$ with selected values of $m \in \{2, 3, 4, 5\}$. Table 2 lists corresponding extraneous fixed points ξ of H for any value of m. By direct computation of multipliers $R'_p(\xi)$, we find that the parabolic fixed points are given by $\xi = \zeta^{1/2}$ satisfying repeated roots arising from cases **2A, 5XA, 5XH, 5YA, 5YF**, which are highlighted in bold face in Table 2. Attractive extraneous fixed points are indicated by framed

Table 1
$\mathcal{A}(\zeta), F(\zeta)$ and number of nonzero roots ζ for the selected cases.

Case	$\mathcal{A}(\zeta)$	$F(\zeta)$	No. of ζ
1A	$\frac{m}{512}$	$\frac{1-12\zeta+73\zeta^2-232\zeta^3+427\zeta^4-524\zeta^5+779\zeta^6}{\zeta^6}$	6
2A	$4m\zeta$	$\frac{(1+3\zeta)^2(1-2\zeta+17\zeta^2)}{1-5\zeta+74\zeta^2-10\zeta^3+581\zeta^4+383\zeta^5}$	4
2B	$4m\zeta$	$\frac{1-\zeta+74\zeta^2-98\zeta^3+485\zeta^4+563\zeta^5}{1+2\zeta+23\zeta^2+316\zeta^3-353\zeta^4+2722\zeta^5+1385\zeta^6}$	5
2C	m	$\frac{1-6\zeta+79\zeta^2-148\zeta^3+527\zeta^4+1146\zeta^5+2497\zeta^6}{1-2\zeta+43\zeta^2+84\zeta^3+263\zeta^4+2126\zeta^5+1581\zeta^6}$	6
3A	$\frac{m}{64}$	$\frac{1-5\zeta+34\zeta^2-138\zeta^3+309\zeta^4-65\zeta^5+888\zeta^6}{\zeta^4(1+3\zeta)^2}$	6
4A	$\frac{m}{2}$	$\frac{1-9\zeta+46\zeta^2-62\zeta^3-47\zeta^4+327\zeta^5}{1-9\zeta+46\zeta^2-94\zeta^3+81\zeta^4+103\zeta^5}$	5
5XA	-16ζ	$\frac{(11+\zeta)(1+5\zeta)^2}{-32-304\zeta-2923\zeta^2-3488\zeta^3-158\zeta^4-8\zeta^5+\zeta^6}$	3
5XB	-4	$\frac{(1+3\zeta)^2(1+14\zeta+\zeta^2)}{-17-123\zeta-490\zeta^2-374\zeta^3-21\zeta^4+\zeta^5}$	4
5XC	128	$\frac{(1+2\zeta)^2(3+20\zeta+\zeta^2)}{1141+5498\zeta+13559\zeta^2+7084\zeta^3+403\zeta^4-38\zeta^5+\zeta^6}$	4
5XD	16	$\frac{(1+\zeta)^2(1+3\zeta)}{37+80\zeta+114\zeta^2+24\zeta^3+\zeta^4}$	3
5XE	1	$\frac{(1+6\zeta+25\zeta^2)(1+14\zeta+17\zeta^2)^2}{(3+10\zeta+19\zeta^2)(1+20\zeta+190\zeta^2+580\zeta^3+233\zeta^4)}$	6
5XF	2	$\frac{(3+22\zeta+15\zeta^2)(15+46\zeta+19\zeta^2)^2}{(3+\zeta)(1293+9767\zeta+32626\zeta^2+47550\zeta^3+30289\zeta^4+6475\zeta^5)}$	6
5XG	1	$\frac{(7+18\zeta+7\zeta^2)^2(3+18\zeta+11\zeta^2)}{397+2586\zeta+7771\zeta^2+11148\zeta^3+7923\zeta^4+2618\zeta^5+325\zeta^6}$	6
5XH	$\frac{1}{2}$	$\frac{(3+\zeta)^4(5+10\zeta+\zeta^2)}{427+544\zeta+787\zeta^2+216\zeta^3+65\zeta^4+8\zeta^5+\zeta^6}$	6
5YA	2	$\frac{(3+\zeta)(9-2\zeta+\zeta^2)^2}{855-960\zeta+1049\zeta^2-632\zeta^3+249\zeta^4-56\zeta^5+7\zeta^6}$	5
5YB	4	$\frac{(3+\zeta)^2(7+\zeta)^2(3+5\zeta)}{10845+8178\zeta+13283\zeta^2-260\zeta^3+723\zeta^4-14\zeta^5+13\zeta^6}$	5
5YC	8	$\frac{(1+\zeta)(13+2\zeta+\zeta^2)^2}{2581-290\zeta+2303\zeta^2-780\zeta^3+323\zeta^4-50\zeta^5+9\zeta^6}$	5
5YD	2	$\frac{(1+\zeta)(1+3\zeta)^2(5+3\zeta)(1+7\zeta)}{29+255\zeta+946\zeta^2+1550\zeta^3+1089\zeta^4+227\zeta^5}$	5
5YE	4	$\frac{(1+\zeta)^2(7+6\zeta+3\zeta^2)^2}{407+738\zeta+1261\zeta^2+1004\zeta^3+513\zeta^4+146\zeta^5+27\zeta^6}$	6
5YF	$16\zeta^2$	$\frac{(1+7\zeta)(1+11\zeta)(11+13\zeta)^2}{5+194\zeta+3755\zeta^2+86556\zeta^3+273355\zeta^4+394114\zeta^5+126757\zeta^6}$	4

Table 2
Extraneous fixed points $\xi = \zeta^{1/2}$ for selected cases for any $m \geq 1$.

Case	ξ	No. of ξ
1A	± 0.526337 ± 0.570728*i*, ±0.523321 ± 0.138562*i*, ±0.40816 ± 0.190345*i*	12
2A	± **0.57735***i*, ±**0.57735***i*, ±0.388175 ± 0.303078*i*	8
2B	± 1.05974*i*, ±0.241527 ± 0.250925*i*, ±0.481292 ± 0.310196*i*	10
2C	± 0.331444 ± 0.712687*i*, ±0.281664 ± 0.219642*i*, ±0.442388 ± 0.24126*i*	12
3A	± 0.47636 ± 0.639187*i*, ±0.307256 ± 0.320966*i*, ±0.497158 ± 0.142524*i*	12
4A	± 0.795894*i*, ±0.384237 ± 0.186681*i*, ±0.572952 ± 0.22907*i*	10
5XA	± 3.31662*i*, ±**0.447214***i*, ±**0.447214***i*	6
5XB	± 3.73205*i*, ±**0.57735***i*, ±**0.57735***i*, ±0.267949*i*	8
5XC	± $\boxed{4.45521\ i}$, ±**0.707107***i*, ±**0.707107***i*, ±0.38877*i*	8
5XD	± **1.0***i*, ±**1.0***i*, ±0.57735*i*	6
5XE	± **0.862856***i*, ±**0.862856***i*, ±0.2 ± 0.4*i*, ±**0.281085***i*, ±**0.281085***i*	12
5XF	± **1.42571***i*, ±**1.42571***i*, ±1.14653*i*, ±**0.623213***i*, ±**0.623213***i*, ±0.39006*i*	12
5XG	± **1.44701***i*, ±**1.44701***i*, ±1.20334*i*, ±**0.69108i**, ±**0.69108i**, ±0.433988*i*	12
5XH	± $\boxed{3.07768\ i}$, ±**1.73205***i*, ±**1.73205***i*, ±**1.73205***i*, ±**1.73205***i*, ±0.726543*i*	12
5YA	± 1.73205*i*, ±**1.41421 ± 1.0***i*, ±**1.41421 ± 1.0***i*	10
5YB	± **2.64575***i*, ±**2.64575***i*, ±**1.73205***i*, ±**1.73205***i*, ±0.774597*i*	10
5YC	± 1.0*i*, ±**1.14139 ± 1.51749***i*, ±**1.14139 ± 1.51749***i*	10
5YD	± 1.29099*i*, ±1.0*i*, ±**0.57735***i*, ±**0.57735***i*, ±0.377964*i*	10
5YE	± **1.0***i*, ±**1.0***i*, ±**0.513578 ± 1.12417***i*, ±**0.513578 ± 1.12417***i*	12
5YF	±**0.919866***i*, ±**0.919866***i*, ±0.377964*i*, ± $\boxed{0.301511\ i}$	8

values in Table 2 for three cases **5XC, 5XH** and **5YF**. All other extraneous fixed points ξ of H in each case are found to be repulsive.

Before closing this section, we denote 20 iterative maps in Table 1 corresponding to cases **1A, 2A, 2B, 2C, 3A, 4A** as well as all **5X** and **5Y** respectively by **GKN1A, GKN2A, GKN2B, GKN2C, GKN3A, GKN4A** and **GKN5XA** through **GKN5YF** for convenience and later use. In addition, the map for iterative method (2.1) is denoted by **GKNPA**.

Table 3
Additional test functions $f_i(x)$ with zeros α, multiplicity m and initial guesses x_0.

i	$f_i(x)$	α	m	x_0
1	$(4+3\sin x-2x^2)^3$	1.85471014256339	3	1.90
2	$[2x-\pi+\cos^2 xe^{1-x^2}]^7$	$\frac{\pi}{2}$	7	1.6
3	$[2x^2+3e^{-x}+4\sin(x^3)-5]^6$	$\approx 0.846491745344542$	6	0.86
4	$[x\cos(\frac{\pi x}{6})+\frac{1}{x^3+1}-\frac{1}{28}](x-3)^3$	3	4	3.05
5	$(x-1)^2+\frac{1}{12}-\log(\frac{25}{12}-2x+x^2)$	$\frac{1-i\sqrt{3}}{6}$	2	$1.05-0.28i$
6	$(x\log x-\sqrt{x}+x^3)^3$	1	3	1.05

Here $\log z (z\in\mathbb{C})$ represents a principal analytic branch with $-\pi \le Im(\log z) < \pi$.

6. Numerical experiments and complex dynamics

This section is basically composed of two parts. The first part deals with computational aspects of proposed methods (3.1) for a variety of test functions in comparison with other existing methods. Selected cases **1A, 2A, 2B, 2C, 3A, 4A** as well as all **5X** and **5Y** have been implemented to verify the convergence developed in this paper. Later on in the second part of this section, the complex dynamics will be explored together with basins of attraction of selected rational iterative maps **GKN1A, GKN2A, GKN2B, GKN2C, GKN3A, GKN4A** and **GKN5XA** through **GKN5YF**.

A number of numerical experiments have been implemented with Mathematica programming to confirm the developed theory. Throughout these experiments, we have maintained 160 digits of minimum number of precision, via Mathematica command $\$MinPrecision = 160$, to achieve the specified accuracy. In case that α is not exact, it is replaced by a more accurate value which has more number of significant digits than the preassigned number $\$MinPrecision = 160$.

Definition 2 (Computational Convergence Order)**.** Assume that theoretical asymptotic error constant $\eta = \lim_{n\to\infty}\frac{|e_n|}{|e_{n-1}|^p}$ and convergence order $p \ge 1$ are known. Define $p_n = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$ as the computational convergence order. Note that $\lim_{n\to\infty} p_n = p$.

Remark 6.1. Note that p_n requires knowledge at two points x_n, x_{n-1}, while the usual COC(computational order of convergence) $\frac{\log(|x_n-x_{n-1}|/|x_{n-1}-x_{n-2}|)}{\log(|x_{n-1}-x_{n-2}|/|x_{n-2}-x_{n-3}|)}$does require knowledge at four points $x_n, x_{n-1}, x_{n-2}, x_{n-3}$. Hence p_n can be handled with a less number of working precision digits than the usual COC whose number of working precision digits is at least p times as large as that of p_n.

Computed values of x_n are accurate up to $\$MinPrecision$ significant digits. If α has the same accuracy of $\$MinPrecision$ as that of x_n, then $e_n = x_n - \alpha$ would be nearly zero and hence computing $|e_{n+1}|/e_n^p|$ would unfavorably break down. To clearly observe the convergence behavior, we desire α to have more significant digits that are Φ digits higher than \$MinPrecision. To supply such α, a set of following Mathematica commands are used:

$$sol = FindRoot[f(x), \{x, x_0\}, PrecisionGoal \to \Phi + \$MinPrecision,$$
$$WorkingPrecision \to 2 * \$MinPrecision];$$
$$\alpha = sol[[1, 2]]$$

In this experiment, we assign $\Phi = 16$. As a result, the numbers of significant digits of x_n and α are found to be 160 and 176, respectively. Nonetheless, the limited paper space allows us to list both of them only up to 15 significant digits. We set the error bound ϵ to $\frac{1}{2}\times 10^{-112}$ satisfying $|x_n - \alpha| < \epsilon$.

Iterative methods (3.1) associated with case numbers are identified by W-prefixed names. Typical methods with cases **1A, 2A, 3A, 4A**are respectively identified by **W1A, W2A, W3A, W4A**. These four typical methods have been successfully applied to the test functions $F_1 - F_4$ below:

$$\begin{cases} \mathbf{W1A}: F_1(x) = \left[\cos\left(\frac{\pi x}{2}\right) + e^{1-x^2} - x - 2\right]^4, m = 4, \alpha = -1 \\ \mathbf{W2A}: F_2(x) = [\cos(x^2+1) - x\log(x^2-\pi+2)+1]^2(x^2+1-\pi), m = 3, \alpha = \sqrt{\pi-1}, \\ \mathbf{W3A}: F_3(x) = [\sin^{-1}(x^2-1) + e^{2-x^2} - 5x - 3]^2, m = 2, \alpha \approx 1.46341814037882, \\ \mathbf{W4A}: F_4(x) = x^2[x^4 + \log(1+x^3)], m = 5, \alpha \approx 0.434401024257508, \\ \text{where } \log z (z\in\mathbb{C}) \text{ represents a principal analytic branch such that} -\pi < Im(\log z) \le \pi \end{cases}$$

As seen in Table 4, they clearly confirmed sextic-order convergence. The values of computational asymptotic error constant agree up to 10 significant digits with η. It appears that the computational convergence order well approaches 6.

Table 3 shows additional test functions to further confirm the convergence behavior of proposed scheme (3.1).

In Table 5, we compare numerical errors $|x_n - \alpha|$ of proposed methods **W1A, W2A, W2B, W2C, W3A, W4A, W5XA, W5XH, W5YA, W5YF** with those of method **WPA** which identifies method (2.1). The least errors within the prescribed error bound are highlighted in bold face. Although we are limited to the selected current experiments, within two iterations, a

Table 4
Convergence for test functions $F_1(x) - F_4(x)$ with typically selected methods **W1A, W2A, W3A, W4A.**

MT	F	n	x_n	$\|F(x_n)\|$	$\|x_n - \alpha\|$	$\|e_n/e_{n-1}^6\|$	η	p_n
		0	−0.92	0.00199559	0.0800000			
		1	−0.999999923690214	1.481×10^{-27}	7.630×10^{-8}	0.2910987337	0.699148242	6.34691
W1A	F_1	2	−1.00000000000000	1.586×10^{-170}	1.380×10^{-43}	0.6991476541		6.00000
		3	−1.00000000000000	0.0×10^{-638}	0.0×10^{-160}			
		0	1.425	0.00335045	0.0384181			
		1	1.46341813420998	1.260×10^{-23}	6.168×10^{-9}	1.918604750	2.837841985	6.12010
W2A	F_2	2	1.46341814037882	2.053×10^{-145}	1.563×10^{-49}	2.837841806		6.00000
		3	1.46341814037882	0.0×10^{-479}	0.0×10^{-159}			
		0	0.45	0.0192094	0.0155990			
		1	0.434401024265989	5.608×10^{-21}	8.481×10^{-12}	0.5887215858	0.7002340415	6.04169
w3A	F_3	2	0.434401024257508	5.299×10^{-132}	0.0×10^{-160}	0.7002340415		6.00000
		0	0.01	1.010×10^{-10}	0.01			
		1	8.679×10^{-15}	4.925×10^{-71}	8.679×10^{-15}	0.008679381883	0.00896	6.00691
W4A	F_4	2	3.830×10^{-87}	8.245×10^{-433}	3.830×10^{-87}	0.008960000000		6.00000
		3	0.0×10^{-246}	0.0×10^{-1230}	0.0×10^{-246}			

MT = method.

Table 5
Comparison of $|x_n - \alpha|$ for selected multiple-zero finders.

f, x_0; m	$\|x_n - \alpha\|$	W1A	W2A	W2B	W2C	W3A	W4A	W5XE	W5XF	W5YD	W5YF	WPA
f_1, 1.9; 3	$\|x_1 - \alpha\|$	2.66e-9*	5.96e-10	3.20e-10	8.65e-10	1.82e-9	9.49e-10	2.43e-10	1.11e-10	1.22e-10	1.75e-10	2.69e-9
	$\|x_2 - \alpha\|$	1.40e-52	3.64e-57	4.65e-59	5.04e-56	9.53e-54	9.78e-56	6.63e-60	**2.74e-62**	5.20e-62	6.59e-61	1.61e-52
f_2, 1.6; 5	$\|x_1 - \alpha\|$	4.20e-12	7.64e-12	7.43e-12	7.83e-12	8.43e-12	7.99e-12	7.27e-12	7.19e-12	7.17e-12	7.17e-12	8.03e-12
	$\|x_2 - \alpha\|$	1.03e-68	3.59e-69	2.93e-69	4.30e-69	7.39e-69	5.01e-69	2.48e-69	**2.29e-69**	2.25e-69	2.26e-69	6.51e-69
f_3, 0.86; 6	$\|x_1 - \alpha\|$	4.20e-11	2.20e-11	1.88e-11	2.49e-11	3.46e-11	2.70e-11	1.68e-11	1.55e-11	1.537e-11	1.56e-11	6.08e-11
	$\|x_2 - \alpha\|$	4.85e-62	4.73e-64	1.57e-64	1.17e-63	1.23e-62	2.08e-63	6.84e-65	3.82e-65	**3.57e-65**	3.98e-65	7.45e-61
f_4, 3.05; 4	$\|x_1 - \alpha\|$	2.96e-12	1.28e-12	1.03e-12	1.52e-12	2.32e-12	1.68e-12	8.85e-13	7.79e-13	7.71e-13	7.94e-13	3.91e-12
	$\|x_2 - \alpha\|$	1.44e-73	3.98e-76	8.90e-77	1.34e-75	2.64e-74	2.70e-75	2.89e-77	1.18e-77	**1.10e-77**	1.35e-77	1.04e-72
f_5, 1.05	$\|x_1 - \alpha\|$	4.86e-4	2.79e-4	2.77e-4	2.80e-4	2.83e-4	4.78e-4	8.27e-7	2.06e-5	2.65e-7	5.38e-7	1.16e-5
$-0.28i$; 2	$\|x_2 - \alpha\|$	9.59e-18	6.29e-20	2.72e-20	1.01e-19	2.50e-19	2.69e-18	5.27e-12	**8.17e-28**	5.41e-13	2.23e-12	2.32e-14
f_6, 1.05; 3	$\|x_1 - \alpha\|$	3.12e-7	6.54e-8	3.12e-8	9.82e-8	2.11e-7	1.04e-7	2.37e-8	5.45e-9	7.67e-9	1.59e-8	2.75e-7
	$\|x_2 - \alpha\|$	3.03e-38	4.66e-43	2.46e-45	8.34e-42	1.94e-39	1.29e-41	3.58e-46	**1.21**e-50	1.28e-49	2.13e-47	1.49e-38

* 2.66e-9 denotes 2.66×10^{-9}.

strict comparison shows that Method **W5XF** displays slightly better convergence for test functions f_1, f_2, f_5, f_6, while Method **W5YF** for test functions f_3 and f_4.

By inspecting the asymptotic error constant $\eta(\theta_i, m, Q_f, K_f) = \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p}$ when p is known, we should be aware that the local convergence is dependent on the function $f(x)$, an initial value x_0, the multiplicity m, the zero α itself and the weight functions Q_f and K_f. Accordingly, for a given set of test functions, one method is hardly expected to always show better performance than the others.

We introduce the efficiency index [42] defined by $EI = p^{\frac{1}{d}}$ where p is the order of convergence and d is the number of distinct functional or derivative evaluations per iteration. The proposed methods (3.1) evidently show a reasonable EI of $6^{1/4} \approx 1.56508$ as compared with that of classical modified Newton's method. Weight functions Q_f and K_f dependent on two function-to function ratios $[\frac{f(y_n)}{f(x_n)}]^{\frac{1}{m}}$ and $[\frac{f(w_n)}{f(x_n)}]^{\frac{1}{m}}$ play a crucial role in obtaining sixth-order of convergence for proposed methods (3.1).

It is, in general, a matter of importance to properly select initial values influencing the convergence behavior of iterative methods. For ensured convergence of iterative map (5.17) with a weight function $H_p(z)$, it requires good initial values close to zero α. It is, however, not a simple task to determine how close the initial values are to zero α, since initial values are generally dependent upon computational precision, error bound and the given function $f(x)$ under consideration. One effective way of selecting stable initial values is to directly use visual basins of attraction. Since the area of convergence can be seen on the basins of attraction, it would be reasonable to say that a method having a larger area of convergence implies a more stable method. Clearly a quantitative analysis becomes an essential tool for measuring the size of area of convergence.

To this end, we provide Table 6 featuring a statistical data describing the average number of iterations per point. In the following 6 examples, we take a 6 by 6 square centered at the origin and containing all the zeros of the given functions. We assume that all zeros are of the same multiplicity m. We then take 360,000 equally spaced points in the square as initial points for the iterative methods. We color the point based on the root it converged to. This way we can find out if the method converged within the maximum number of iteration allowed and if it converged to the root closer to the initial point.

Table 6
Average number of iterations per point for each example (1–6).

Map	Example						
	1: $m=2$	2: $m=3$	3: $m=3$	4: $m=4$	5: $m=5$	6: $m=5$	Average
GKN5XA	8.8286	9.69	15.23	-	-	-	-
GKN5XB	6.1744	6.92	10.44	7.89	12.31	17.86	10.26
GKN5XC	8.9915	14.44	-	-	-	-	-
GKN5XD	11.2374	10.47	15.15	-	-	-	-
GKN5XE	4.1803	6.50	12.14	6.80	7.07	17.18	8.98
GKN5XF	3.824	51.19	8.87	6.20	5.93	12.10	7.02
GKN5XG	4.5768	7.67	10.86	8.89	9.34	17.07	9.73
GKN5XH	25.5374	-	-	-	-	-	-
GKN5YA	37.5688	-	-	-	-	-	-
GKN5YB	29.4858	-	-	-	-	-	-
GKN5YC	32.3239	-	-	-	-	-	-
GKN5YD	**3.8273**	**5.09**	**6.69**	**6.06**	**5.85**	**8.87**	**6.06**
GKN5YE	14.4975	-	-	-	-	-	-
GKN5YF	4.2731	7.43	14.80	-	-	-	-
GKNPA	8.1742	7.7825	9.9977	8.8558	13.1151	14.9314	10.4761

Table 7
CPU time (in seconds) required for each example (1–6) using a Dell Multiplex-990.

Map	Example						
	1: $m=2$	2: $m=3$	3: $m=3$	4: $m=4$	5: $m=5$	6: $m=5$	Average
GKN5XA	1400.53	4617. 817	6959.048	-	-	-	-
GKN5XB	1014.007	3362.055	4857.668	3506.106	5256.19	8340.91	4389.488
GKN5XC	1477.08	7001.918	-	-	-	-	-
GKN5XD	1801.453	5074.26	6987.862	-	-	-	-
GKN5XE	708.369	3136.463	5736.375	3079.117	3092.47	7964.1	3952.815
GKN5XF	665.796	2500.322	4273.007	2732.622	2691.984	5622.775	3081.084
GKN5XG	736.746	3652.311	4976.821	3777.564	3994.531	7708.649	4141.104
GKN5XH	4158.176	-	-	-	-	-	-
GKN5YA	6073.945	-	-	-	-	-	-
GKN5YB	4813.754	-	-	-	-	-	-
GKN5YC	5316.202	-	-	-	-	-	-
GKN5YD	**643.909**	**2410.387**	**3148.443**	**2713.325**	**2500.431**	**4113.84**	**2588.389**
GKN5YE	2309.829	-	-	-	-	-	-
GKN5YF	667.778	3471.927	6036.489	-	-	-	-
GKNPA	987.27	2914.50	3275.10	3520.85	5183.44	6586.66	3744.64

We now are ready to discuss the complex dynamics of selected iterative maps **GKN1A, GKN2A, GKN2B, GKN2C, GKN3A, GKN4A** and **GKN5XA** through **GKN5YF** applied to various polynomials $p_k(z), k \in \mathbb{N}$.

Example 1. As a first example, we have taken a quadratic polynomial raised to the power of 2 with all real roots:

$$p_1(z) = (z^2 - 1)^2. \tag{6.1}$$

Clearly the roots are $\pm$ 1 with multiplicity 2. Basins of attraction for **GKN5XA** – **GKN5XH** are given in the top two rows of Fig. 2. The last two rows present the basins of attraction for **GKN5YA** – **GKN5YF**. It is clear that the best methods are **GKN5XF** and **GKN5YD** and the worst are **GKN5XH**, **GKN5YA** – **GKN5YC** and **GKN5YE**. Consulting Tables 6–8, we find the methods **GKN5XF** and **GKN5YD** use the least number of iterations per point on average, they also use the least amount of CPU time and have the least number of black points. The method **GKN5YF** is the next best. In the following examples we will not show the 5 worst methods.

Example 2. In our second example, we have taken a cubic polynomial raised to the power of 3:

$$p_2(z) = (z^3 + 4z^2 - 10)^3. \tag{6.2}$$

Basins of attraction are given in Fig. 3. In the top row the basins for **GKN5XA** – **GKN5XD**, center row for **GKN5XE** – **GKN5XG** and on the bottom row the basins for **GKN5YD** and **GKN5YF**. It is clear that the best methods are **GKN5XF** and **GKN5YD** and the worst are **GKN5XA** and **GKN5XC**. Based on Tables 6–8, we find that **GKN5YD** is fastest followed by **GKN5XF** and the slowest is **GKN5XC**. The average number of iterations per point is least for **GKN5YD** (5.09) followed by **GKN5XE** (6.50) and **GKN5XB** (6.92) and the highest is for **GKN5XF** (51.19). The least number of black points is for **GKN5YD** (652) and the highest for **GKN5XA** and **GKN5XC**. We will therefore eliminate **GKN5XC** from the rest of the experiments.

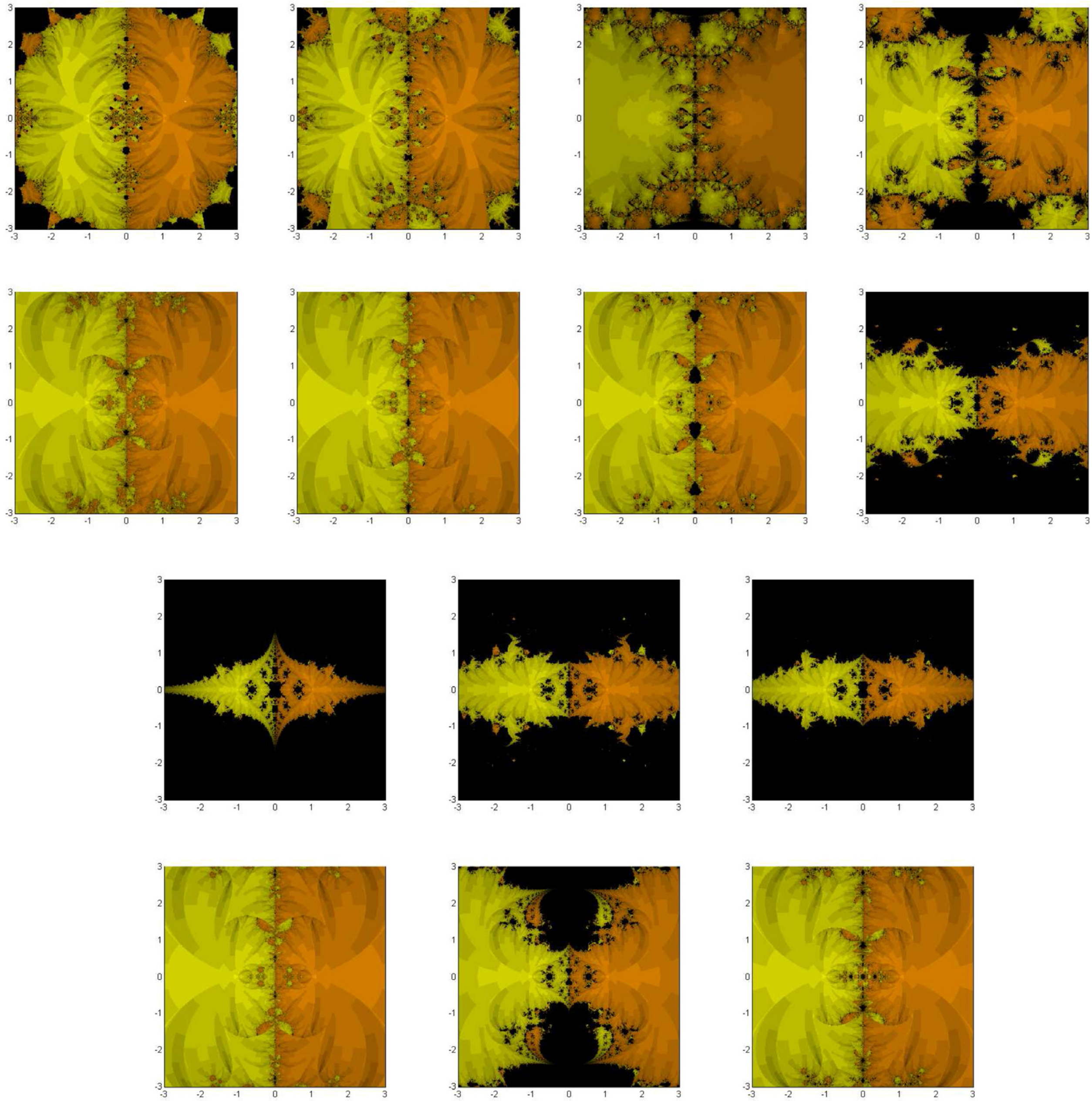

Fig. 2. The top row for **5XA**, **5XB**, **5XC**, **5XD** in order from left to right, the second row for **5XE**, **5XF**, **5XG** ,**5XH**, the third row for **5YA**, **5YB**, **5YC**, and the bottom row for **5YD**, **5YE**, **5YF**, for the roots of the polynomial $(z^2-1)^2$.

Example 3. As a third example, we have taken a quintic polynomial raised to the power of 3:

$$p_3(z) = (z^5 - 1)^3. \tag{6.3}$$

The basins for this example are plotted in Fig. 4. In the top row, we have the basins for **5XA**, **5XB** and **5XD**. Below that we have the basins for **5XE**, **5XF** and **5XG** and on the third row the basins for **5YD** and **5YF**. The best methods are **5XB**, **5XF** and **5YD**. The worst are **5XA** and **5YF**. Upon consulting Table 6, we find that **5YD** uses the least number of iterations per point (6.69) followed by **5XF** with 8.87 iterations. The methods **5XA**, **5XD** and **5YF** require between 14.80 and 15.23 iterations per point. Based on the CPU in Table 7, we arrive at the same conclusion. Based on the number of black points, we find that **5YD** is by far the best (5488 points) with the rest having at least 24843 points. The worst are **5YF** with 94342 points, **5XD** with 70466 points and **5XA** with 68063 points. These 3 methods will be excluded from the rest of the experiments.

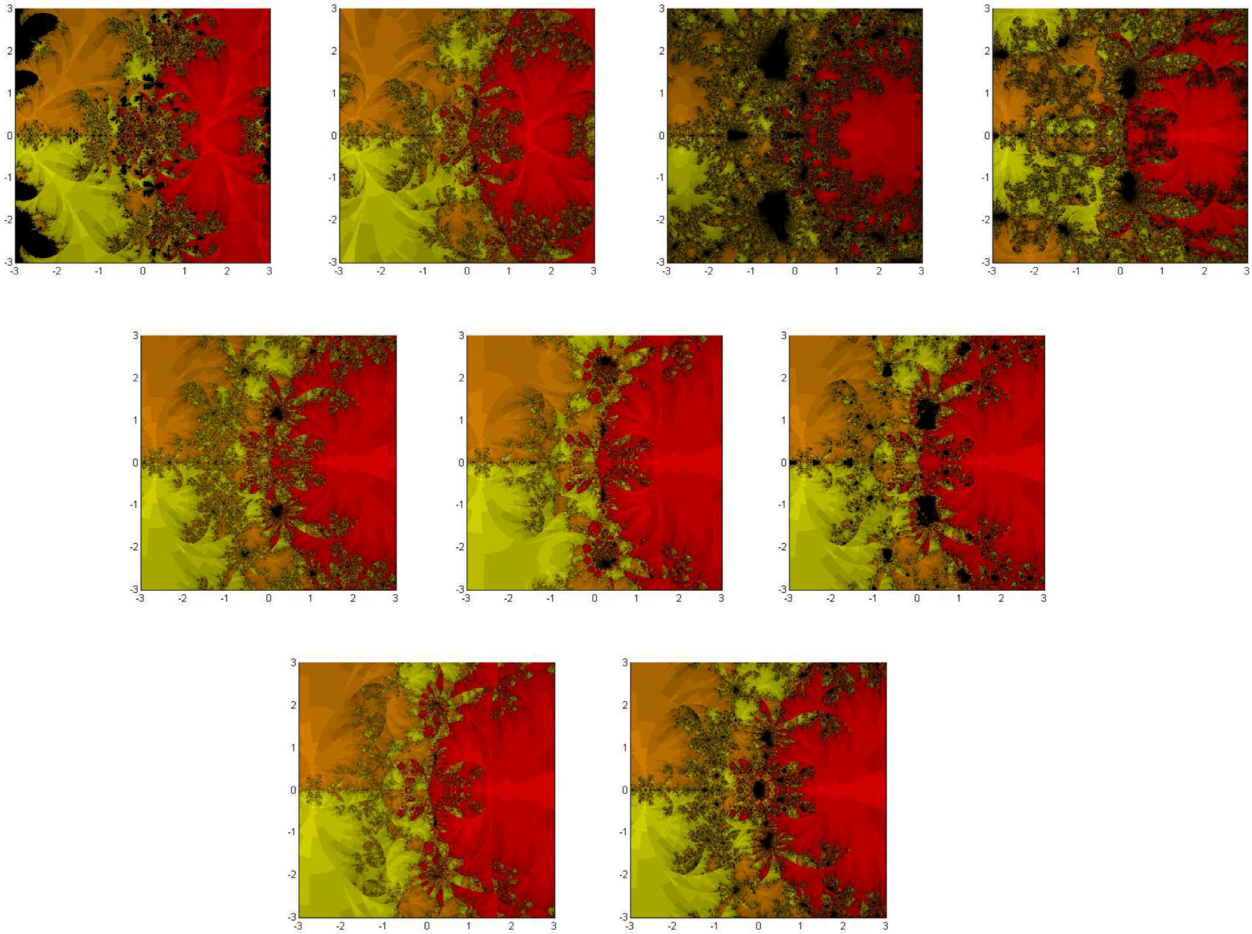

Fig. 3. The top row for **5XA**, **5XB**, **5XC**, **5XD** in order from left to right, the second row for **5XE**, **5XF**, and **5XG**, and the bottom row for **5YD**, and **5YF**, for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$.

Example 4. As a fourth example, we have taken a different cubic polynomial raised to the power of 4:

$$p_4(z) = (z^3 - z)^4. \tag{6.4}$$

Now all the roots are real. The basins are given in Fig. 5 in two rows. The top row have **5XB**, **5XE** and **5XF**. The bottom row shows the basins for **5XG** and **5YD**. The best are **5YD** and **5XF**. The worst methods are **5XB** and **5XG**. the number of iterations per point is now in the range of 6.06 (for **5YD**) to 8.89 (for **5XG**). The fastest methods are **5YD** (2713.325 s) followed by **5XF** (2732.622 s) and the slowest is **5XG** with 3777.564 s.The method **5YD** has the least number of black points (1642) and **5XG** has the most (30584) black points.

Example 5. As a fifth example, we have taken a quadratic polynomial raised to the power of 5:

$$p_5(z) = (z^2 - 1)^5. \tag{6.5}$$

The basins for the best 5 methods so far are plotted in Fig. 6. Based on the plots and the Tables, we conclude that **5YD** is the best performer followed closely by **5XF** and the worst is **5XB**.

Example 6. As a last example, we have taken a quartic polynomial raised to the power of 5:

$$p_6(z) = (z^4 - 1)^5. \tag{6.6}$$

The basins for the best 5 methods left are plotted in Fig. 7. The conclusions are the same as in the previous example based on the plots and the tables.

In summary, we find that **5YD** is best followed closely by **5XF**. The worst is **5XB**. To summarize the results of the 6 examples, we have averaged the results in Tables 6–8 across examples. Based on Table 6 we find that **5YD** uses the least number of iterations per point (6.06 on average) followed closely by **5XF** (7.02). The method requiring the highest number of iterations per point is **5XB** (10.26) which is slightly less than the best sixth order method **GKNPA** in our previous paper

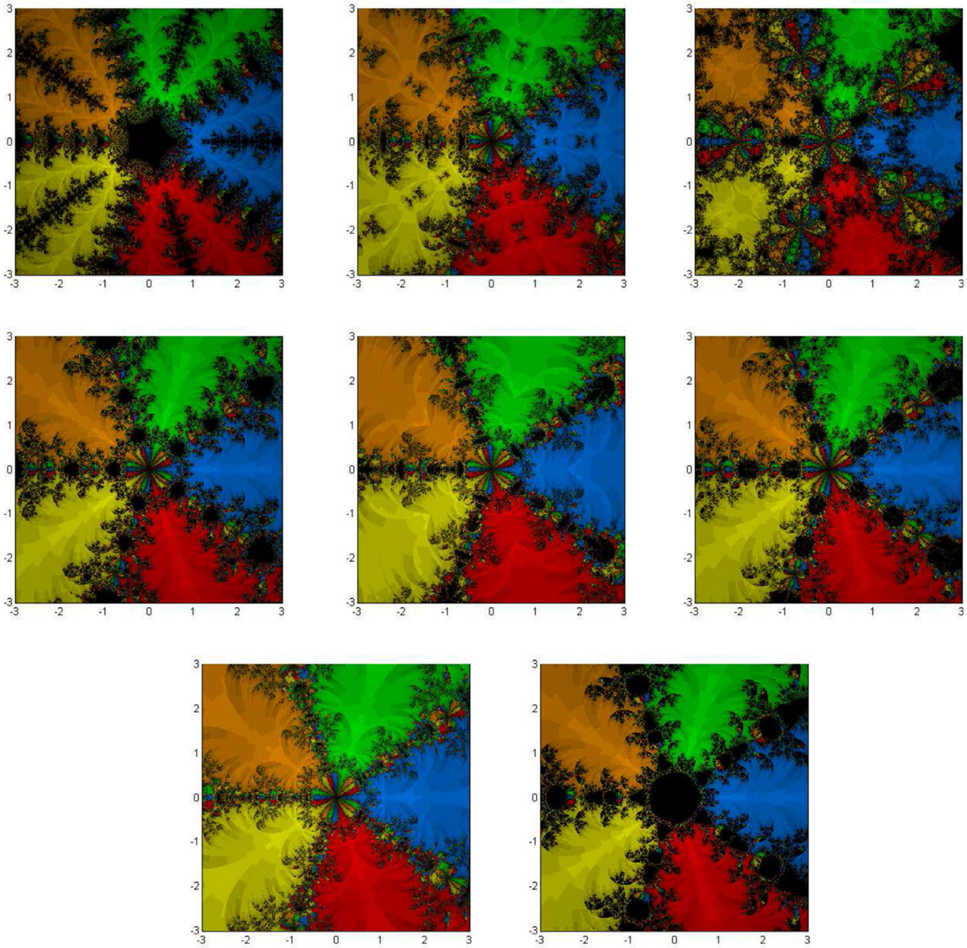

Fig. 4. The top row for **5XA**, **5XB**, **5XD** in order from left to right, the second row for **5XE**, **5XF**, and **5XG**, and the bottom row for **5YD**, and **5YF**, for the roots of the polynomial $(z^5 - 1)^3$.

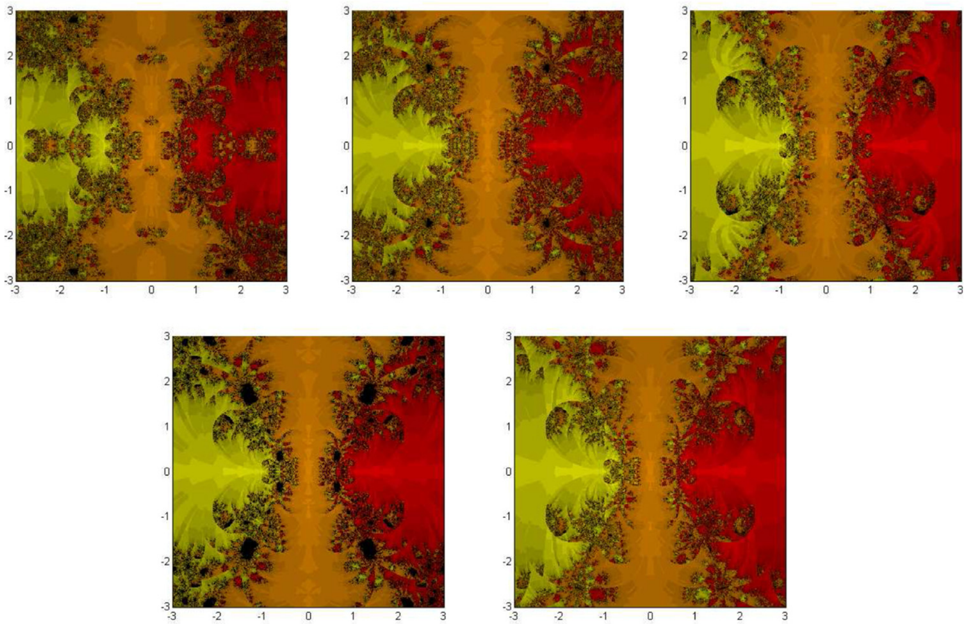

Fig. 5. The top row for **5XB**, **5XE**, **5XF**, in order from left to right, and the bottom row for **5XG**, and **5YD**, for the roots of the polynomial $(z^3 - z)^4$.

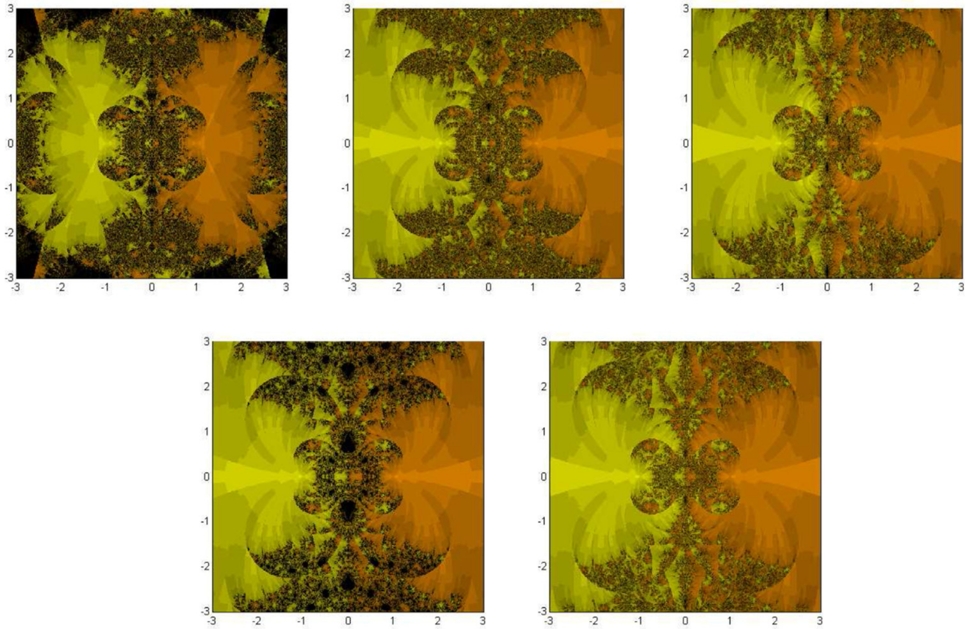

Fig. 6. The top row for **5XB**, **5XE**, **5XF**, in order from left to right, and the bottom row for **5XG**, and **5YD**, for the roots of the polynomial $(z^2 - 1)^5$.

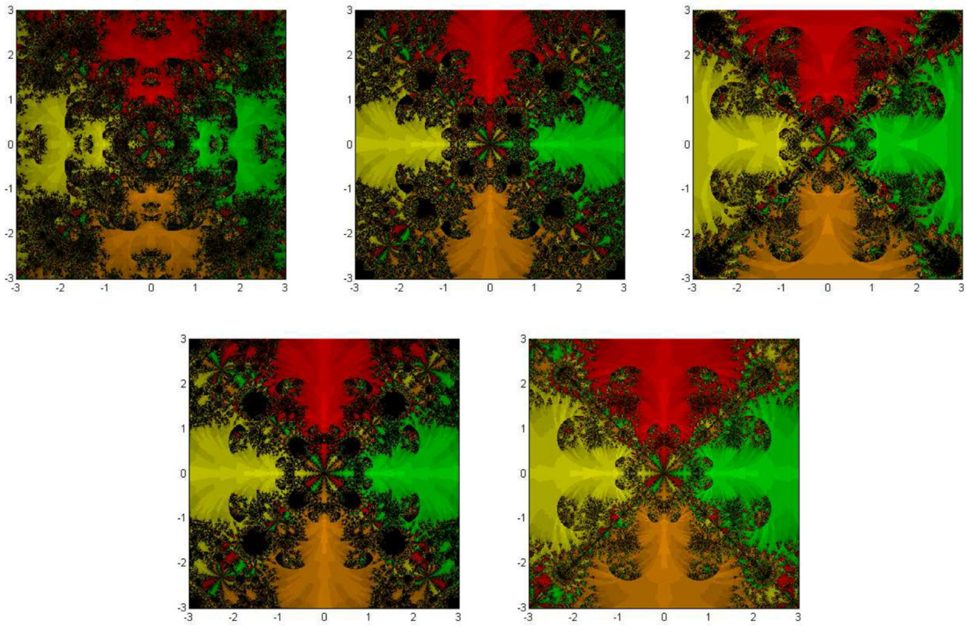

Fig. 7. The top row for **5XB**, **5XE**, **5XF**, in order from left to right, and the bottom row for **5XG**, and **5YD**, for the roots of the polynomial $(z^4 - 1)^5$.

[19]. The fastest method is **5YD** (2588.39 s) followed by **5XF** (3081.08 s). The slowest is **5XB** (4389.49 s), slower than **GKNPA** (3744 s). As for the number of black points (see Table 8) we find that **GKNPA** has the lowest number (426 points) followed by **5YD** (3351 points).

We conclude the current study as follows. Convergence order of proposed methods (3.1) has been improved with the introduction of weight functions expressed in terms of function-to-function ratios. Computational aspects through a variety of test equations in a number of selected cases well agree with the developed theory, verifying the convergence order and asymptotic error constants. To determine what type of initial values of the proposed methods chosen near the zero α must be given for their ensured convergence, we have not only carefully investigated the extraneous fixed points of the proposed maps applied to a polynomial $f(z) = (z^2 - 1)^m$ motivated by the earlier work of Vrscay and Gilbert [43], but also extensively illustrated relevant complex dynamics of a family of selected methods **5X** and **5Y** behind the basins of attraction for a wide variety of exemplary polynomials $p_k(z)$. We conclude that **5YD** is the best method overall. We have tried to find connection

Table 8
Number of points requiring 40 iterations for each example (1–6).

Map	Example						
	1: $m = 2$	2: $m = 3$	3: $m = 3$	4: $m = 4$	5: $m = 5$	6: $m = 5$	Average
GKN5XA	47311	41564	68063	-	-	-	-
GKN5XB	17797	2970	24843	5654	40125	71921	27218
GKN5XC	24161	41102	-	-	-	-	-
GKN5XD	67853	17300	70466	-	-	-	-
GKN5XE	1117	3038	58168	3256	1617	96321	27253
GKN5XF	1133	2210	30559	3810	1523	49141	14729
GKN5XG	5993	21922	46058	30584	31775	107633	40672
GKN5XH	215319	-	-	-	-	-	-
GKN5YA	332263	-	-	-	-	-	-
GKN5YB	254945	-	-	-	-	-	-
GKN5YC	283479	-	-	-	-	-	-
GKN5YD	**791**	**652**	**5488**	**1642**	**861**	**10673**	**3351**
GKN5YE	102871	-	-	-	-	-	-
GKN5YF	2319	11576	94342	-	-	-	-
GKNPA	601	2	1128	0	7	817	425.83

between location and multiplicity of the extraneous fixed points (see Table 2) and the performance of the methods. Most methods have purely imaginary extraneous fixed point except **5XE**, **5YA**, **5YC** and **5YE**. Of these 4 methods only **5XE** did reasonably well but not as well as **5YD**. We can conclude that if the extraneous fixed points are not on the imaginary axis, the method will not perform well. We conjecture that **5XH** did not perform well because one of the extraneous fixed points has a multiplicity 4 and the rest have only double roots.

As our future work developing a family of new higher-order multiple-zero finders, we essentially need to make the best use of principal analytic branches of function-to-function ratios in selecting free parameters of the weight functions that would enhance relevant basins of attraction under consideration.

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