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On a family of Halley-like methods to find simple roots of nonlinear equations

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ABSTRACT

There are many methods for solving a nonlinear algebraic equation. Here we introduce a family of Halley-like methods and show that Euler–Chebyshev and BSC are just members of the family. We discuss the conjugacy maps and the effect of the extraneous roots on the basins of attraction.

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1. Introduction

In 1694 Halley [1] developed the third order method given by

$$x_{n+1} = x_n - \frac{f_n}{f'_n - \frac{f_n f''_n}{2f'_n}}. \tag{1}$$

Here, and in the following, we denote $f_n = f(x_n)$ and similarly for the derivatives.

Since the method requires the evaluation of the function and its first and second derivatives, then we can say that the efficiency index (see Traub [2]) is $E = p^{1/d} = 3^{(1/3)} = 1.442$, which is higher than Newton's efficiency index of 1.4142. This is assuming that the cost of the derivatives is the same as the function.

Remark. Wynn [3] noted that methods using second derivatives are very useful for evaluating zeros of functions satisfying a second order ordinary differential equation (e.g., Bessel's functions). In such cases the evaluation of second derivatives is trivial and thus the increase in efficiency.

See also Candela and Marquina [4], Hernandez [5], Melman [6] and Scavo [7].

The method can also be obtained as a special case of Hansen and Patrick's family of methods [8]

$$x_{n+1} = x_n - \frac{(\alpha+1)f_n}{\alpha f'_n \pm \sqrt{(f'_n)^2 - (\alpha+1)f_n f''_n}}, \tag{2}$$

where $\alpha = 1$ and the square root is approximated linearly. It can also be obtained as a member of the family

$$x_{n+1} = x_n + (e-1)\frac{f'_n}{f''_n}\left\{\left[1 - \frac{e}{e-1}\frac{f_n f''_n}{(f'_n)^2}\right]^{1/e} - 1\right\}, \tag{3}$$

developed by Popovski [9] upon taking $e = -1$.

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This process was rediscovered by others (e.g., Frame [10], Hartree [11], Hamilton [12], Richmond [13], Salehov [14], Schröder [15] and Wall [16]). See also Neta [17] for a collection of algorithms for the solution of nonlinear equations and a comparison of their efficiency indices. Recently Petković et al. [18] have published a book on multipoint methods that unified many of the methods appearing in the literature.

In order to see the similarity to other methods, we use a different form of Halley's method, that is

$$x_{n+1} = x_n - \frac{f_n}{f'_n} - \left[\frac{f_n}{f'_n - \frac{f_n f''_n}{2f'_n}} - \frac{f_n}{f'_n}\right].$$

Upon simplifying the term in brackets, we get

$$x_{n+1} = x_n - \frac{f_n}{f'_n} - \frac{f_n^2 f''_n}{2(f'_n)^3 - f_n f'_n f''_n}. \tag{4}$$

In this paper we consider the family of methods based on this form of Halley's method (4). Halley's method was originally developed in 1694 and rediscovered by many. There have been many attempts to improve on the method. We will show that some other methods are just special cases of this and find the best member of the family in terms of simpler boundaries of the basin of attraction. Therefore, we can conclude that Halley's method (1) is the best third order available as was concluded by Neta et al. [19] based on several numerical experiments.

2. Halley-like family of methods

The one parameter family of methods we consider is

$$x_{n+1} = x_n - \frac{f_n}{f'_n} - \frac{f_n^2 f''_n}{2(f'_n)^3 - A f_n f'_n f''_n}. \tag{5}$$

Notice that this is just (4) with an additional parameter. Upon choosing $A = 1$ we have Halley's method (4). The choice $A = 0$ yields the well known Euler–Chebyshev method [20]. This latter method is also a special case of Hansen and Patrick's family (2) with $\alpha = 1$ or Popovski's family (3) with $e = \frac{1}{2}$. The choice $A = 2$ gives the BSC method [21].

Theorem 1. *Let $\eta \in I$ be a simple root of a sufficiently differentiable function $f : I \to \mathbf{R}$ for an open interval I. If x_0 is sufficiently close to η, then the method defined by (5) has third-order convergence, and satisfies the error equation*

$$e_{n+1} = \left(-B_3 + (2 - A)B_2^2\right) e_n^3 + O(e_n^4), \tag{6}$$

where $e_n = x_n - \eta$ and $B_i = \frac{f^{(i)}(\eta)}{i! f'(\eta)}$.

Proof. Using Taylor expansion of $f(x_n)$ about η, we have

$$f(x_n) = e_n(1 + B_2 e_n + B_3 e_n^2 + B_4 e_n^3 + B_5 e_n^4 + B_6 e_n^5 + O(e_n^6)), \tag{7}$$

$$f'(x_n) = 1 + 2B_2 e_n + 3B_3 e_n^2 + 4B_4 e_n^3 + 5B_5 e_n^4 + 6B_6 e_n^5 + O(e_n^6), \tag{8}$$

$$f''(x_n) = 2B_2 + 6B_3 e_n + 12B_4 e_n^2 + 20B_5 e_n^3 + 30B_6 e_n^4 + O(e_n^5), \tag{9}$$

where $B_i = \frac{f^{(i)}(\eta)}{i! f'(\eta)}$ and $e_n = x_n - \eta$.

Dividing (7) by (8) gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - B_2 e_n^2 + L_3 e_n^3 + L_4 e_n^4 + O(e_n^5), \tag{10}$$

where

$$L_3 = -2B_3 + 2B_2^2,$$

$$L_4 = -3B_4 - 4B_2^3 + 7B_2 B_3$$

and similarly upon dividing (9) by (8) we have

$$\frac{f''(x_n)}{f'(x_n)} = 2B_2 + K_1 e_n + K_2 e_n^2 + K_3 e_n^3 + K_4 e_n^4 + O(e_n^5), \tag{11}$$

where

$$K_1 = 6B_3 - 4B_2^2,$$
$$K_2 = 12B_4 + 8B_2^3 - 18B_2B_3,$$
$$K_3 = 20B_5 - 18B_3^2 - 32B_2B_4 + 48B_3B_2^2 - 16B_2^4,$$
$$K_4 = 90B_2B_3^2 + 32B_2^5 + 80B_2^2B_4 - 50B_2B_5 - 120B_2^3B_3 + 30B_6 - 60B_3B_4.$$

We now use Maple to collect all these expansions into (5) to have the denominator $1 - A/2f(x_n)/f'(x_n)f''(x_n)/f'(x_n)$ given by

$$1 - A/2(e_n - B_2e_n^2 + L_3e_n^3 + L_4e_n^4)(2B_2 + K_1e_n + K_2e_n^2 + K_3e_n^3 + K_4e_n^4) + O(e_n^5) \tag{12}$$

and the numerator $(1/2)\,[f(x_n)/f'(x_n)]^2 f''(x_n)/f'(x_n)$ given by

$$(e_n + B_2e_n^2 + L_3e_n^3 + L_4e_n^4)^2(2B_2 + K_1e_n + K_2e_n^2 + K_3e_n^3 + K_4e_n^4)/2 + O(e_n^5). \tag{13}$$

Therefore when we collect terms we have

$$e_{n+1} = N_3\,e_n^3 + N_4\,e_n^4 + O(e_n^5) \tag{14}$$

where

$$N_3 = -AB_2^2 - B_3 + 2B_2^2,$$
$$N_4 = 12B_2B_3 - A^2B_2^3 - 6AB_2B_3 - 9B_2^3 + 7AB_2^3 - 3B_4,$$

which indicates that the order of convergence of the methods defined by (5) is at least three. The error constant is the coefficient N_3. This completes the proof. □

Remark. As a special case, we get the constant for Halley's method (see also e.g. Traub [2])

$$N_3 = -B_3 + B_2^2,$$

for Euler–Chebyshev method ($A = 0$)

$$N_3 = -B_3 + 2B_2^2$$

and for BSC method ($A = 2$)

$$N_3 = -B_3.$$

One may conclude that the BSC method is superior to the others. We will see later that the asymptotic error constant is not the best indicator.

3. Corresponding conjugacy maps for quadratic polynomials

Given two maps f and g from the Riemann sphere into itself, an analytic conjugacy between the two maps is a diffeomorphism h from the Riemann sphere onto itself such that $h \circ f = g \circ h$. Here we consider only quadratic polynomials.

Theorem 2 (Halley's family of methods(5)). *For a rational map $R_p(z)$ arising from Halley's method applied to $p(z) = (z-a)(z-b),\ a \neq b,\ R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = \frac{-z-2+A}{-2z+Az-1}z^3.$$

Proof. Let $p(z) = (z-a)(z-b),\ a \neq b$ and let M be the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \frac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p\left(\frac{ub-a}{u-1}\right) = \frac{-u-2+A}{-2u+Au-1}u^3.$$

As a special case we see that for Halley's method ($A = 1$) we have $S(z) = z^3$. For BSC method ($A = 2$), we have $S(z) = z^4$ and for Euler–Chebyshev ($A = 0$), we have $S(z) = \frac{z+2}{2z+1}z^3$. □

4. Extraneous fixed points

In solving a nonlinear equation iteratively we are looking for fixed points which are zeros of the given nonlinear function. Many iterative methods have fixed points that are not zeros of the function of interest. Those points are called extraneous fixed points (see Vrcsay and Gilbert [22]). Those points could be attractive which will trap an iteration sequence and give erroneous results. Even if those extraneous fixed points are repulsive or indifferent they can complicate the situation by converging to a root not close to the initial guess.

The Halley family of methods can be written as

$$x_{n+1} = x_n - u_n H_f(x_n).$$

Clearly the root η of $f(x)$ is a fixed point of the method, since u_n vanishes at η. The points $\xi \neq \eta$ at which $H_f(\xi) = 0$ are also fixed points of the family, since the second term on the right vanishes.

It is easy to see that for the family of Halley-like methods we have

$$H_f(\xi) = 1 + \frac{f(\xi)f''(\xi)}{2f'(\xi)^2 - Af(\xi)f''(\xi)}$$

or

$$H_f(\xi) = \frac{2f'(\xi)^2 - (A-1)f(\xi)f''(\xi)}{2f'(\xi)^2 - Af(\xi)f''(\xi)}. \tag{15}$$

Theorem 3. *There are no extraneous fixed points for Halley's method.*

Proof. For Halley's method (1) we have

$$H_f = \frac{1}{1 - \frac{1}{2}\frac{f(\xi)}{f'(\xi)}\frac{f''(\xi)}{f'(\xi)}}.$$

This function does not vanish and therefore there are no extraneous fixed points. □

Theorem 4. *There are two extraneous fixed points for Halley-like family of methods. They are the roots of*

$$2f'(\xi)^2 - (A-1)f(\xi)f''(\xi) = 0. \tag{16}$$

Proof. The extraneous fixed point can be found by solving (15). For the quadratic polynomial $z^2 - 1$ this leads to the equation

$$\frac{(A-5)z^2 - A + 1}{(A-4)z^2 - A} = 0$$

for which the roots are $\pm\sqrt{\frac{(A-1)}{(5-A)}}i$. These fixed points are repulsive if $6 - A > 1$, i.e. $A < 5$. Vrcsay and Gilbert [22] show that if the points are attractive then the method will give erroneous results. If the points are repulsive then the method may not converge to a root near the initial guess.

The poles are at $z = \pm\sqrt{\frac{A}{4-A}}i$. For the three members mentioned above the poles are on the imaginary axis. One should **not** choose $A \geqslant 4$ since in those cases we have poles on the real axis. □

Remark. Since we are discussing the quadratic polynomial $z^2 - 1$, then theoretically the imaginary axis is the boundary between the two basins, see Kneisl [23]. Any extraneous root on the imaginary axis will either give erroneous results or complicate the situation as discussed earlier. Any pole will cause the method to diverge and thus it should be on the boundary. In order for the poles and the extraneous roots to be on the imaginary axis, we must have $1 \leqslant A \leqslant 4$. In the case $A = 1$ there are no extraneous roots.

Recently Basto et al. [24] have experimented with Halley, Euler–Chebyshev and BSC methods. They have constructed basins of attraction for these methods for nine nonlinear equations. They concluded that Halley's method shows the simplest boundaries and confirms the best performance already suggested by the studies made by Scott et al. [25] and Neta et al. [19,26]. Such numerical studies were initiated by Stewart [27] and followed by the works of Amat et al. [28–31] and Chun et al. [32].

Remark. For the BSC and Euler–Chebyshev the extraneous fixed points are repulsive. For Halley's method there are **no** extraneous fixed points. This is why Halley's method perform better than Euler–Chebyshev and BSC.

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