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Numerical Solution of a Nonlinear Integro-differential Equation

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Galerkin's method with appropriate discretization in time is considered for approximating the solution of the nonlinear integro-differential equation

$$u_t(x, t) = \int_0^t a(t-\tau) \frac{\partial}{\partial x} \sigma(u_x(x, \tau))\, d\tau + f(x, t), \qquad 0 < x < 1, \quad 0 < t < T.$$

An error estimate in a suitable norm will be derived for the difference $u - u^h$ between the exact solution u and the approximant u^h. It turns out that the rate of convergence of u^h to u as $h \to 0$ is optimal. This result was confirmed by the numerical experiments.

1. INTRODUCTION

The nonlinear problem

$$u_t(x, t) = \int_0^t a(t-\tau) \frac{\partial}{\partial x} \sigma(u_x(x, \tau))\, d\tau + f(x, t), \qquad 0 < x < 1, \quad 0 < t < T, \tag{1}$$

$$u(0, t) = u(1, t) = 0, \tag{2}$$

$$u(x, 0) = u_0(x), \tag{3}$$

serves as a very special model for one-dimensional heat flow in materials with memory [7]. The problem also arises in the theory of one-dimensional viscoelasticity [3, 11–13]. It is also an example in the general theory of equations of the form

$$\dot{u}(t) = -\int_0^t a(t-\tau)\, g(u(\tau))\, d\tau + f(t) \tag{4}$$

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$$u(0) = u_0 \tag{5}$$

on a Hilbert space H, with g a nonlinear-unbounded operator [1].

As a special case of (1)–(3) one obtains the damped nonlinear-wave equation. Let

$$a(t) = e^{-\alpha t}, \qquad \alpha > 0 \tag{6}$$

$$g(x, t) = f_t(x, t) + \alpha f(x, t), \tag{7}$$

then (1) can be written in the form

$$u_{tt}(x, t) + \alpha u_t(x, t) = \frac{\partial}{\partial x} \sigma(u_x(x, t)) + g(x, t), \qquad 0 < x < 1, \quad 0 < t < T, \tag{8}$$

with the initial conditions,

$$u(x, 0) = u_0(x), \tag{9}$$

$$u_t(x, 0) = f(x, 0), \tag{10}$$

and the same boundary conditions (2).

In [18], Nohel studied the global existence, uniqueness, and continuous dependence on data of smooth solutions to (8)–(10), and (2). For the special case $g \equiv 0$, Nishida [17] established the existence and uniqueness of global-smooth solutions for smooth and sufficiently small data (9)–(10).

This result was generalized to several space dimensions (still with $g \equiv 0$) by Matsumara [14, 15] and Klainerman [8]. The case $\alpha = 0$ and $g \equiv 0$ is treated in [5, 6, 9, 10].

For our special model (1)–(3), MacCamy [12] and recently Staffans [19] discussed the existence, uniqueness, boundedness, and asymptotic behavior of solutions.

The purpose of this paper is to describe a Galerkin-type method for approximating the solution of (1)–(3). We shall follow the assumptions in [12] to ensure the existence of a unique solution. For completeness we shall give those conditions:

$$a \in C^2[0, \infty), \qquad a(0) > 0, \qquad \dot{a}(0) < 0, \tag{11}$$

$$t^j a^{(k)} \in L_1(0, \infty), \qquad k = 0, 1, 2, \quad j \leqslant 3 + N \quad \text{for some} \quad N \geqslant 0, \tag{12}$$

$$\operatorname{Re} \hat{a}(i\eta) > 0 \qquad \text{for all} \quad \eta, \tag{13}$$

where $\hat{a}(s)$ is the Laplace transform of $a(t)$.

$$\sigma \in C^2(-\infty, \infty), \qquad \sigma(0) = 0, \tag{14}$$

$$0 \leqslant \mu \leqslant \sigma'(\xi) \leqslant \gamma \qquad \text{for all} \quad \xi, \tag{15}$$

$$f \in C^3([0, 1] \times [0, \infty)), \tag{16}$$

$$\bar{f}(t) \in L_1(0, \infty) \cap L_2(0, \infty) \cap L_\infty(0, \infty), \tag{17}$$

where

$$\bar{f}(t) = \sup_{x \in [0,1]} (|f(x, t)|, |f_t(x, t)|). \tag{18}$$

As a special case one obtains the approximate solution of the nonlinear hyperbolic equation (8)–(10), and (2). Dendy [4] discussed Galerkin's method for a special case of this problem ($\alpha = 0$).

An error estimate in the norm

$$|||E(x, t)|||_1 = \left\{ \int_0^T \int_0^1 [E^2(x, t) + E_x^2(x, t)]\, dx\, dt \right\}^{1/2} \tag{19}$$

will be derived. It turned out that the rate of convergence of the approximant u^h to the exact solution u as $h \to 0$ is optimal.

The next section will be devoted to the variational formulation of the problem. The error estimate will be derived in Section 3. In Section 4 the numerical solution will be discussed. Some of the numerical experiments performed will be described in the last section.

2. Variational Formulation

Consider the problem (1.1)–(1.3). One of the ingredients of finite-element method is a variational formulation of the problem (see, e.g., [2, 20]). Let us denote by H the linear space of functions u satisfying

$$u(0, t) = u(1, t) = 0, \tag{1}$$

$$\|u(\cdot, t)\|_1 < \infty, \tag{2}$$

where

$$\|u(\cdot, t)\|_r = \left\{ \int_0^1 \left(|u(x, t)|^2 + \sum_{i=1}^r \left| \frac{\partial^i u(x, t)}{\partial x^i} \right|^2 \right) dx \right\}^{1/2}. \tag{3}$$

The variational formulation of the problem can be stated as follows:
Find a function $u(x, t) \in H$ for which

$$\langle v, u_t \rangle + \int_0^t a(t-\tau) < \sigma(u_x(x, \tau)), \\ v_x(x, t) > d\tau = \langle v, f \rangle, \qquad \forall v \in H, \tag{4}$$

and

$$\langle v, u(x, 0) \rangle = \langle v, u_0(x) \rangle, \qquad \forall v \in H, \tag{5}$$

where

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)\, q(x)\, dx \tag{6}$$

and

$$\| p \|_0 = \langle p(x), p(x) \rangle^{1/2}. \tag{7}$$

To approximate the solution of (4)–(5) we require that u and v lie in a finite-dimensional subspace S_h of H for each t. (See, e.g., [16].) The following property concerning approximability in S_h can be readily verified for finite-element spaces. (See, e.g., [20].)

Approximation Property

There is an integer $r \geqslant 2$ and numbers C_0, C_1 independent of h such that for any $v \in H$ there exists a $v^h \in S_h$ satisfying

$$\| v - v^h \|_l \leqslant C_l h^{t-l} \| v \|_t \quad \text{for } 0 \leqslant l \leqslant 1 \text{ and } l < t \leqslant r. \tag{8}$$

The approximation $u^h \in S_h$ to u is defined by the following variational analog of (4)–(5):
Find a $u^h \in S_h$ such that

$$\langle v^h, u_t^h \rangle + \int_0^t a(t-\tau) < \sigma(u_x^h(x, \tau)), \\ v_x^h(x, t) > d\tau = \langle v^h, f \rangle, \qquad \forall v^h \in S_h \tag{9}$$

and

$$\langle v^h, u^h(x, 0) \rangle = \langle v^h, u_0(x) \rangle, \qquad \forall v^h \in S_h. \tag{10}$$

Once a basis has been selected for S_h, (9)–(10) are equivalent to a set of N integro-differential equations. The solution of such a system will be discussed in Section 4.

3. Error Estimates

In this section we shall estimate the error in the finite-element approximation using the norm (1.19)

$$|||E|||_1 = \left\{ \int_0^T \int_0^1 [E^2(x,t) + E_x^2(x,t)]\, dx\, dt \right\}^{1/2}. \tag{1}$$

We will also use the following norms:

$$\square\xi\square_r = \left\{ \int_0^1 \sum_{i=0}^{r} \left[\sup_{0 \leqslant t \leqslant T} \left| \frac{\partial^i \xi(x,t)}{\partial x^i} \right| \right]^2 dx \right\}^{1/2}, \tag{2}$$

$$\blacksquare\xi\blacksquare_r = \left\{ \int_0^1 \int_0^T \sum_{i=0}^{r} \left| \frac{\partial^i \xi(x,t)}{\partial x^i} \right|^2 dt dx \right\}^{1/2}. \tag{3}$$

This error estimate depends upon the assumption that there exists $\beta > 0$ such that for any function $\omega(x,t)$

$$\int_0^T \int_0^t a(t-\tau) \int_0^1 \sigma'(\omega(x,\tau))\, \xi(x,\tau)\, \xi(x,t)\, dx d\tau dt$$
$$\geqslant \frac{\beta A^2}{a(0)} \int_0^1 \left\{ \int_0^T |\xi(x,\tau)| d\tau \right\}^2 dx, \tag{4}$$

where $A = \inf_{0 \leqslant t \leqslant T} a(t)$. This assumption analogous to the inequality for positive-definite functions proved by Staffans [19, Lemma 4.1]. It becomes the consequence of that lemma if $\sigma'(w(x,\tau)) \equiv \beta$.

Theorem. *The error in the finite-element approximation u^h generated by* (2.9)–(2.10) *satisfies the relation*

$$|||u - u^h|||_1 \leqslant h^{r-1} \{ C_2^2 h^2 \|u_0\|_r^2 + (C_0^2 h^2/\varepsilon) \square u_t \square_r^2$$
$$+ (\gamma^2 M^2 C_1^2/\sqrt{\beta}) \blacksquare u \blacksquare_r^2 \}^{1/2} + C_1 h^{r-1} |||u|||_r. \tag{5}$$

Proof. Subtracting (2.9) from (2.4) with v^h instead of v we obtain

$$\langle v^h(x,t), u_t^h(x,t) \rangle$$
$$+ \int_0^t a(t-\tau) < \sigma(u_x^h(x,\tau)), v_x^h(x,t) > d\tau$$
$$= \langle v^h(x,t), u_t(x,t) \rangle$$

$$+\int_0^t a(t-\tau)\langle\sigma(u_x(x,\tau)), v_x^h(x,t)\rangle\, d\tau$$

$$\text{for all } v^h \in S_h. \tag{6}$$

Let $\hat{u}^h$ be any function in S_h, then

$$\begin{aligned}\langle v^h, (u^h-\hat{u}^h)_t\rangle &\\ &+\int_0^t a(t-\tau)\langle\sigma(u_x^h)-\sigma(\hat{u}_x^h), v_x^h\rangle\, d\tau\\ &=\langle v^h, (u-\hat{u}^h)_t\rangle\\ &+\int_0^t a(t-\tau)\langle\sigma(u_x)-\sigma(\hat{u}_x^h), v_x^h\rangle\, d\tau\\ &\text{for all } v^h\in S_h.\end{aligned} \tag{7}$$

Let

$$e(x,t)=u^h(x,t)-\hat{u}^h(x,t) \tag{8}$$

$$E(x,t)=u(x,t)-\hat{u}^h(x,t). \tag{9}$$

Since $e\in S_h$ we can let $v^h=e$. Using the mean value theorem for the function σ,

$$\sigma(w)-\sigma(z)=\sigma'(\xi)(w-z) \tag{10}$$

one obtains from (7)–(9),

$$\begin{aligned}\langle e(x,t), e_t(x,t)\rangle &\\ &+\int_0^t a(t-\tau)\langle\sigma'(\xi(x,\tau))\, e_x(x,\tau), e_x(x,t)\rangle\, d\tau\\ &=\langle e(x,t), E_t(x,t)\rangle\\ &+\int_0^t a(t-\tau)\langle\sigma'(\xi(x,\tau))\, E_x(x,\tau), e_x(x,t) > d\tau.\end{aligned} \tag{11}$$

Note that,

$$\langle e(x,t), e_t(x,t)\rangle=\frac{1}{2}\frac{d}{dt}\|e(\cdot,t)\|_0^2. \tag{12}$$

Combining (10), (1.15), and (11) one has,

$$\frac{d}{dt}\|e(\cdot,t)\|_0^2 + 2\int_0^t a(t-\tau)\langle\sigma'(\xi(x,\tau))\, e_x(x,\tau), e_x(x,t) > d\tau \leqslant 2\int_0^1 e(x,t)\, E_t(x,t)\, dx + 2\gamma\int_0^t a(t-\tau)\int_0^1 e_x(x,t)\, E_x(x,\tau)\, dx d\tau. \tag{13}$$

Using Schwarz's inequality on the last integral on the right and integrating with respect to t one obtains:

$$\|e(\cdot,T)\|_0^2 - \|e(\cdot,0)\|_0^2 + 2\int_0^T dt\int_0^t d\tau\, a(t-\tau)\int_0^1 dx\, \sigma'(\xi(x,\tau))\, e_x(x,\tau)\, e_x(x,t) \leqslant \int_0^T dt\int_0^1 e(x,t)\, E_t(x,t)\, dx + 2\gamma\int_0^1 dx\int_0^T dt\, e_x(x,t)\left\{\int_0^t a^2(t-\tau)\, d\tau\right\}^{1/2}\left\{\int_0^t E_x^2(x,\tau)\, d\tau\right\}^{1/2}. \tag{14}$$

Using Schwarz's inequality on the integrals on the right and (4) it follows that

$$\|e(\cdot,T)\|_0^2 + \frac{\beta A^2}{a(0)}\int_0^1 dx\left\{\int_0^T |e_x(x,\tau)|\, d\tau\right\}^2 \leqslant \|e(\cdot,0)\|_0^2 + 2\int_0^1 dx\left\{\int_0^T |e(x,t)|\, dt\right\}\cdot\{\sup_t |E_t(x,t)|\} + 2\gamma\int_0^1 dx\left\{\int_0^T |e_x(x,t)|\, dt\right\}\sup_t\left\{\int_0^t a^2(t-s)\, ds\right\}^{1/2}\cdot\left\{\int_0^T E_x^2(x,\tau)\, d\tau\right\}^{1/2}. \tag{15}$$

Using Schwarz's inequality again and noting that $\|e(\cdot,T)\|_0^2 \geqslant 0$, one has

$$\frac{\beta A^2}{a(0)}\, |||e_x|||_0^2 \leqslant \|e(\cdot,0)\|_0^2$$

$$+ 2 \left\{ \int_0^1 dx \left\{ \int_0^T |e(x,t)|\, dt \right\}^2 \right\}^{1/2}$$

$$\times \left\{ \int_0^1 dx \{\sup_t |E_t(x,t)|\}^2\, dx \right\}^{1/2}$$

$$+ 2\gamma M \left\{ \int_0^1 dx \left\{ \int_0^T dt\, |e_x(x,t)| \right\}^2 \right\}^{1/2}$$

$$\cdot \left\{ \int_0^1 dx \int_0^T E_x^2(x,s)\, ds \right\}^{1/2}, \qquad (16)$$

where

$$M = \sup_t \left\{ \int_0^t a^2(t-\tau)\, d\tau \right\}^{1/2}. \qquad (17)$$

Since $2ab \leqslant \varepsilon a^2 + b^2/\varepsilon$ for $\varepsilon > 0$, one has after combining like terms

$$((\beta A^2/a(0)) - \delta) |\!|\!| e_x |\!|\!|_0^2 - \varepsilon\, |\!|\!| e |\!|\!|_0^2 \leqslant \| e(\cdot, 0)\|_0^2 + (1/\varepsilon)\, |\!|\!| E_t |\!|\!|_0^2 + (\gamma^2 M^2/\delta)\, |\!|\!| E_x |\!|\!|_0^2. \qquad (18)$$

Using Poincare's inequality

$$\| e_x \|_0 \geqslant C_p \| e \|_0 \qquad (19)$$

one can show that

$$|\!|\!| e_x |\!|\!|_0 \geqslant C_p\, |\!|\!| e |\!|\!|_0 \qquad (20)$$

for possibly different constant. Let $\delta > 0$, $\varepsilon > 0$ be chosen so that

$$\beta A^2/a(0)) - \delta - \varepsilon C_p^{-2} = 1 + (1/C_p^2). \qquad (21)$$

Note that

$$|\!|\!| e |\!|\!|_1^2 = |\!|\!| e_x |\!|\!|^2 + |\!|\!| e |\!|\!|^2 \leqslant (1 + (1/C_p^2)) |\!|\!| e_x |\!|\!|^2. \qquad (22)$$

Thus

$$|\!|\!| e |\!|\!|_1^2 \leqslant \| e(\cdot, 0)\|_0^2 + (1/\varepsilon)\, |\!|\!| E_t |\!|\!|_0^2 + (\gamma^2 M^2/\delta)\, |\!|\!| E_x |\!|\!|_0^2.$$

Using (2.8) and noting that (see [16])

$$\| e(\cdot, 0)\|_0 \leqslant C_2 h^r \| u_0 \|_r,$$

one obtains,

$$\|\!|e|\!\|_1^2 \leqslant C_2^2 h^{2r} \|u_0\|_r^2 + (C_0^2 h^{2r}/\varepsilon) \square u_t \square_r^2 \\ + (\gamma^2 M/\delta)\, C_1^2 h^{2(r-1)} \blacksquare u \blacksquare_r^2.$$

Taking square roots of both sides yields,

$$\|\!|e|\!\|_1 \leqslant h^{r-1} \{C_2^2 h^2 \|u_0\|_r^2 + (C_0^2 h^2/\varepsilon) \square u_t \square_r^2 \\ + (\gamma^2 M C_1^2/\delta) \blacksquare u \blacksquare_r^2\}^{1/2}. \tag{23}$$

Combining (23) with (2.8) and the triangle inequality we obtain (5).

Note that the result means that the error for the finite-element approximation is optimal in the sense that it is of the same order as that for the best approximation to the actual solution in the space S_h.

4. Numerical Solution

This section is devoted to the numerical solution of (2.9)–(2.10). Let $\phi_1(x), \ldots, \phi_N(x)$ be a basis for S_h (where N is the dimension of S_h). Therefore, any $u^h \in S_h$ can be represented by

$$u^h(x, t) = \sum_{i=1}^{N} u_i(t)\, \phi_i(x). \tag{1}$$

Since (2.9)–(2.10) are valid for all $v^h \in S_h$, one can let $v^h = \phi_j$. This yields the following system for the vector of weights $\mathbf{u}(t)$:

$$M\dot{\mathbf{u}}(t) + \int_0^t a(t-\tau)\, \mathbf{K}(\mathbf{u}(\tau))\, d\tau = \mathbf{F}(t), \tag{2}$$

$$M\mathbf{u}(0) = \mathbf{W}, \tag{3}$$

where the entries of the so called mass matrix M are

$$M_{ij} = \langle \phi_j(x), \phi_i(x) \rangle, \qquad 1 \leqslant i, j \leqslant N. \tag{4}$$

The entries of **K**, **F**, **W** are

$$K_j(\mathbf{u}(\tau)) = \langle \sigma(u_x^h(x, \tau)), \phi_j' \rangle, \qquad 1 \leqslant j \leqslant N, \tag{5}$$

$$F_j(t) = \langle f(x, t), \phi_j \rangle, \qquad 1 \leqslant j \leqslant N, \tag{6}$$

$$W_j = \langle u_0(x), \phi_j \rangle, \qquad 1 \leqslant j \leqslant N. \tag{7}$$

To solve the system (2)–(3) we use Taylor's expansion. Let

$$\mathbf{u}(t+\Delta t)=\mathbf{u}(t)+(\Delta t)\,\dot{\mathbf{u}}(t)+\tfrac{1}{2}(\Delta t)^2\,\ddot{\mathbf{u}}(t)+O((\Delta t)^3). \tag{8}$$

Differentiating (2) with respect to t, one has

$$M\ddot{\mathbf{u}}(t)+\int_0^t a_t(t-\tau)\,\mathbf{K}(\mathbf{u}(\tau))\,d\tau+a(0)\,\mathbf{K}(\mathbf{u}(t))=\dot{\mathbf{F}}(t), \tag{9}$$

where

$$\dot{F}_j(t)=\langle f_t(x,t),\phi_j\rangle, \qquad 1\leqslant j\leqslant N. \tag{10}$$

Multiplying (8) by the matrix M and using (2) and (9) one obtains after neglecting terms of order $(\Delta t)^3$,

$$M[\mathbf{u}(t+\Delta t)-\mathbf{u}(t)]=(\Delta t)[\mathbf{F}(t)+\tfrac{1}{2}(\Delta t)\,\dot{\mathbf{F}}(t)]-\tfrac{1}{2}(\Delta t)^2\,a(0)\,\mathbf{K}(\boldsymbol{u}(t))$$
$$-(\Delta t)\int_0^t [a(t-\tau)+\tfrac{1}{2}(\Delta t)\,a_t(t-\tau)]\,\mathbf{K}(\mathbf{u}(\tau))\,d\tau. \tag{11}$$

Using the trapezoidal rule $(0=t_0<t_1<t_2<\cdots<t_j=t)$,

$$\int_0^t \mathbf{Q}(\tau)\,d\tau=\sum_{i=1}^{j}\frac{\mathbf{Q}(t_i)+\mathbf{Q}(t_{i-1})}{2}\,(\Delta t)+O((\Delta t)^2) \tag{12}$$

for the integral on the right one has after neglecting terms of order $(\Delta t)^3$,

$$M(\mathbf{u}^{j+1}-\mathbf{u}^j)=(\Delta t)[\mathbf{F}^j+\tfrac{1}{2}(\Delta t)\,\dot{\mathbf{F}}^j]-\tfrac{1}{2}(\Delta t)^2\,a(0)\,\mathbf{K}(\mathbf{u}^j)$$
$$-\tfrac{1}{2}(\Delta t)^2\sum_{i=1}^{j}[\mathbf{K}(\mathbf{u}^{i-1})\,a(t_j-t_{i-1})+\mathbf{K}(\mathbf{u}^i)\,a(t_j-t_i)],$$
$$j=0,1,2,\ldots, \tag{13}$$

where $\mathbf{V}^i=\mathbf{V}(t_i)$ and $t_i=i(\Delta t)$.

One can rewrite (13) in the form

$$M(\mathbf{u}^{j+1}-\mathbf{u}^j)=(\Delta t)[\mathbf{F}^j+\tfrac{1}{2}(\Delta t)\,\dot{\mathbf{F}}^j]-(\Delta t)^2\sum_{i=0}^{j}\zeta_i a((j-i)(\Delta t))\,\mathbf{K}(\mathbf{u}^i),$$
$$j=0,1,2,\ldots, \tag{14}$$

where

$$\zeta_i=\tfrac{1}{2}, \qquad \text{for} \quad i=0,$$
$$=1, \qquad \text{otherwise}.$$

Remark 1. The only starting value needed is $\mathbf{u}^0=\mathbf{u}(0)=\mathbf{M}^{-1}\mathbf{W}$.

Remark 2. Once $\mathbf{u}^0$ and $\mathbf{K}(\mathbf{u}^0)$ are computed we can solve the *linear* system (14) for $\mathbf{u}^1 - \mathbf{u}^0$ and then compute $\mathbf{u}^1$. Note that for $j = 0$ the summation in (14) is null.

Remark 3. The linear system in (14) is banded. Since M is positive definite and symmetric no pivoting is needed. In fact, one has to factor the matrix M only once (to obtain $\mathbf{u}^0$) For all other time steps only back substitution is required.

Remark 4. Storage requirements. Let $B =$ band width of M ($B = 3$ if linear elements are used),
$NT =$ number of time steps $= T/(\Delta t)$. One can store M in $(B+1)/2$ N-vectors (after factorization).
$(NT+1)$ N-vectors are needed for $\mathbf{K}(\mathbf{u}^i)$ $i = 0, 1, \ldots, NT$, one N-vector (only current is saved) is required for $\mathbf{F}^i + \frac{1}{2}(\Delta t)\,\dot{\mathbf{F}}^i$ and
two N-vectors are required for $\mathbf{u}^j$ and $\mathbf{u}^{j+1}$.

5. Numerical Experiments

We have developed a computer program to obtain the numerical solution to the nonlinear integro-differential equation based upon the algorithm described in Section 4. The computer program also gives the error in the norm (1.19). These numerical experiments confirmed our theoretical result concerning the rate of convergence of the approximant to the exact solution as the mesh spacing h tends to zero.

Case 1.

$$a(t) = e^{-t}, \qquad \sigma(u_x) = u_x^2, \qquad u_0(x) = e^{-x},$$
$$f(x, t) = e^{-(x+t)} + 2e^{-2x}(e^{-t} - e^{-2t}).$$

Exact solution: $u(x, t) = e^{-(x+t)}$.

Case 2.

$$a(t) = e^{-t}, \qquad \sigma(u_x) = 1 + u_x^2, \qquad u_0(x) = e^{-x},$$
$$f(x, t) = e^{-(x+t)} + 2e^{-2x}(e^{-t} - e^{-2t}).$$

Exact solution: $u(x, t) = e^{-(x+t)}$.

Case 3.

$$a(t) = e^{-2t}, \qquad \sigma(u_x) = u_x^2, \qquad u_0(x) = \sin x,$$
$$f(x, t) = \cos(x + t) + \tfrac{1}{4}[\sin 2(x + t) - \cos 2(x + t)$$
$$- e^{-2t}(\sin 2x - \cos 2x)].$$

TABLE I

h	Energy norm (1.19)	Rate of convergence
1/2	0.18488	1.8
1/3	0.08908	1.6
1/4	0.05632	2.0
1/5	0.03607	2.1
1/10	0.00840	

$$\left\{ \int_0^T \int_0^1 (E_t^2 + E_x^2)\, dx\, dt \right\}^{1/2}$$

TABLE II

h	Energy norm (1.19)	Rate of convergence
1/2	2.2213	1.6
1/3	1.16074	2.1
1/4	0.63792	2.5
1/5	0.36795	2.4
1/10	0.06971	

TABLE III

h	Energy norm (1.19)	Rate of convergence
1/2	0.05182	2.1
1/3	0.02211	2.0
1/4	0.012437	2.2
1/5	0.007612	2.2
1/10	0.001657	

TABLE IV

h	Energy norm (1.19)	Rate of convergence
1/2	0.11758	2.0
1/3	0.05224	1.9
1/4	0.03024	2.3
1/5	0.01810	2.5
1/10	0.0032	

Exact solution: $u(x, t) = \sin(x + t)$.

Case 4.

$$a(t) = e^{-2t}, \qquad \sigma(u_x) = 1 + u_x^2, \qquad u_0(x) = \sin x,$$
$$f(x, t) = \cos(x + t) + \tfrac{1}{4}[\sin 2(x + t) - \cos 2(x + t) - e^{2t}(\sin 2x - \cos 2x)].$$

Exact solution: $u(x, t) = \sin(x, t)$.

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