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A New Scheme for Trajectory Propagation

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Abstract

In this paper we develop a new numerical method to integrate the equations of motion of a celestial body. The idea is to replace the differential equation for the fast moving component by an equation for the energy per unit mass. We use a simple first-order explicit method for the approximation of the new system. It is shown that the radial error is much smaller than that of some numerical schemes. It will be of interest to have a more extensive comparison with state-of-the-art methods currently in use for long-term trajectory propagation. The evaluation of energy is also more accurate than in other known schemes. This method also conserves the energy per unit mass in the case of perturbation-free flight. The idea can be extended to higher-order methods and implicit schemes.

Introduction

In this paper we develop a method for the solution of the equations of motion of an object acted upon by several gravitational masses. In general the motion (in Cartesian coordinates) can be described by a special class (for which y' is missing) of second order initial value problems (IVPs)

$$y''(t) = f(t, y(t)), \quad y(0) = y_0, \quad y'(0) = y'_0, \qquad t > 0 \tag{1}$$

There is a vast literature for the numerical solution of this problem as well as for the general second order IVPs (useful when using polar coordinates)

$$y''(t) = f(t, y(t), y'(t)), \quad y(0) = y_0, \quad y'(0) = y'_0, \qquad t > 0 \tag{2}$$

See for example the excellent books by Lambert [12, 13], Burrage [6] and Butcher [7]. The numerical integration methods for equation (1) can be divided into two distinct classes: (a) problems for which the solution period is known (even approximately) in advance and that knowledge is being exploited; (b) problems for which the period is not known ([1]). For the first class, special methods are available based on *a priori* knowledge of the frequency or a range of frequencies, see Bettis [5], Steifel and Bettis [21], Gautschi [11], Neta and Ford [16], Lyche [15] and others. See Neta [17] for more information and references.

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Here we only consider some methods of the second class for which the period is unknown. Numerical methods of Runge-Kutta type as well as linear multistep methods can be found in the literature. See also Der [9] and the references there.

Before we continue, we need several definitions. Let y_i and y_i' be an approximation to $y(t_i)$ and $y'(t_i)$, respectively, and let f_i be an approximate value for $f(t_i, y_i, y_i')$. A multistep (or k-step) method to solve the second order IVP is given by

$$\sum_{i=0}^{k} a_i y_{n+i} = h^2 \sum_{i=0}^{k} b_i f_{n+i} \tag{3}$$

The method is called explicit if $b_k = 0$, and otherwise it is called implicit. We define the characteristic polynomials ρ and σ as (see also [12])

$$\rho(\zeta) = \sum_{i=0}^{k} a_i \zeta^i \tag{4}$$

$$\sigma(\zeta) = \sum_{i=0}^{k} b_i \zeta^i \tag{5}$$

The order of the method is defined to be p if for an adequately smooth arbitrary test function $\zeta(t)$

$$\sum_{i=0}^{k} a_i \zeta(t + ih) - h^2 \sum_{i=0}^{k} b_i \zeta''(t + ih) = C_{p+2} h^{p+2} \zeta^{(p+2)}(x) + O(h^{p+3})$$

where C_{p+2} is the error constant. Clearly the coefficients are arbitrary up to a multiplicative constant. We remove this arbitrariness by choosing $a_k = 1$. We also require that a_0 and b_0 not vanish simultaneously, i.e. $|a_0| + |b_0| \neq 0$ otherwise, we just shift the summation index and get a method of step $k - 1$. Similarly we have to assume that ρ and σ have no common factor (irreducibility), or else we have a method of smaller step (after dividing by that common factor.) In order for the method to be of order at least one, we have to satisfy $\rho(1) = \rho'(1) = 0$, $\rho''(1) = 2\sigma(1)$ (consistency). The method is assumed to satisfy also the following:

1. $\sum_{i=0}^{k} |b_i| \neq 0$,
2. The method is zero-stable, i.e. no root of the first characteristic polynomial has modulus greater than one, and every root of modulus one has multiplicity not greater than two.

The direct application of linear k-step methods (3) to problem (1), rather than the application of a conventional linear multistep method to an equivalent first-order system is usually recommended (Ash [2]). Clearly if one is working with polar coordinates, then the system of equations **does** have dependence on the first derivative, see equation (8) below. In that case the scheme becomes

$$\sum_{i=0}^{k} a_i y_{n+i} = h \sum_{i=0}^{k} b_i f_{n+i} \tag{6}$$

and the order is p if for an adequately smooth arbitrary test function $\zeta(t)$

$$\sum_{i=0}^{k} a_i \zeta(t + ih) - h \sum_{i=0}^{k} b_i \zeta'(t + ih) = C_{p+1} h^{p+1} \zeta^{(p+1)}(x) + O(h^{p+2})$$

where C_{p+1} is the error constant. The method is consistent if it is of order at least one, i.e. $\rho(1) = 0$, $\rho'(1) = \sigma(1)$. The method is zero-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root of modulus one is simple.

Our idea here is to develop a new method that conserves the energy per unit mass in the case of perturbation-free flight and use the energy in other cases to approximate the angular variation.

In the next section we introduce the method for perturbation-free flight and discuss its properties. In the following section, the generalization to cases where the energy is not conserved is given. We close with numerical experiments for both cases and compare the solution to Runge-Kutta-Fehlberg (RKF45) as implemented in Matlab.

Development of the Method

In this section we describe our new scheme to solve the system of equations

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -\frac{k}{(x^2+y^2)^{3/2}}x \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -\frac{k}{(x^2+y^2)^{3/2}}y \end{aligned} \tag{7}$$

where $k = 3.986004415 \cdot 10^{14}\ \mathrm{m^3\ sec^{-2}}$ is Earth's gravitational parameter, see e.g. Vallado [22]. This is a system of IVPs similar to (1). This is a special case where the orbit is in the xy-plane. In general we have a third equation for z similar to the above, and the denominator of all three equations will include z^2.

It is well known that Cowell's method gives a numerical solution that spirals inward. Encke's method improves this result but requires more work [3]. See also van Dooren [23] for a way to stabilize Cowell's method.

It can be easily shown (see e.g. Bate et al. [3]) that the energy and angular momentum are conserved for a perturbation-free flight. The conservation of one of these quantities will be used to approximate the fast changing variable. It is thus important to rewrite the system in polar coordinates.

$$\begin{aligned} \frac{\mathrm{d}^2r}{\mathrm{d}t^2} &= r\left(\frac{d\theta}{dt}\right)^2 - \frac{k}{r^2} \\ \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} &= -\frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} \end{aligned} \tag{8}$$

This system is similar to equation (2) since the first order derivatives are present. If the radius r doesn't change much in time (this, for example, excludes cometary flights) then we can use any method for the first equation. The value of θ corresponding to r will be obtained by integrating the relation (conservation of energy per unit mass, E)

$$E = \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{k}{r} = E(0) \tag{9}$$

It can be shown that E is constant, and thus it can be computed from the initial condition.

Remark: In the three-dimensional case, we will have to use spherical coordinates

$$\frac{d^2r}{dt^2} = r\left[\left(\frac{d\theta}{dt}\right)^2 \cos^2\phi + \left(\frac{d\phi}{dt}\right)^2\right] - \frac{k}{r^2}$$

$$\frac{d^2\theta}{dt^2}\cos\phi = -\frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt}\cos\phi + 2\frac{d\theta}{dt}\frac{d\phi}{dt}\sin\phi$$

$$\frac{d^2\phi}{dt^2} = -\frac{2}{r}\frac{dr}{dt}\frac{d\phi}{dt} - \left(\frac{d\theta}{dt}\right)^2 \cos\phi\sin\phi \tag{10}$$

Again we assume that only one out of the three variables changes fast. This one will be approximated using the energy conservation condition, which is

$$E = \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\phi}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\cos\phi\right)^2\right] - \frac{k}{r} = E(0) \tag{11}$$

One can use **any** scheme to solve the system numerically. Here we chose a first-order explicit method. For example, we use here a second order Taylor series method. Let r_i be an approximation to $r(t_i)$ and similarly for $\theta(t)$ and the derivatives of each. Then the method is

$$r_{i+1} = r_i + \Delta t\,\dot{r}_i + \frac{1}{2}(\Delta t)^2\ddot{r}_i \tag{12}$$

where

$$\dot{r}_{i+1} = \dot{r}_i + \Delta t\,\ddot{r}_i \tag{13}$$

and $\ddot{r}_i$ is obtained from the differential equation. In a similar fashion we find

$$\theta_{i+1} = \theta_i + \Delta t\dot{\theta}_i + \frac{1}{2}(\Delta t)^2\ddot{\theta}_i \tag{14}$$

where $\dot{\theta}_{i+1}$ is computed from equation (9)

$$\frac{d\theta}{dt} = \frac{1}{r}\sqrt{2\left(\frac{k}{r} + E(0)\right) - \left(\frac{dr}{dt}\right)^2} \tag{15}$$

We can take the positive sign. The $\ddot{\theta}_i$ can be computed from the differential equation. The algorithm is then as follows:

- Initialize
  —Set $t_0 = 0$
  —Choose Δt
  —Use the initial conditions for r_0, $\dot{r}_0$, θ_0 and $\dot{\theta}_0$
  —Evaluate the energy per unit mass using equation (9)
- Every time step
  Here we loop i from 0
  —Evaluate $\ddot{r}_i$ and $\ddot{\theta}_i$ using equation (8),
  —Evaluate r_{i+1} using equation (12),
  —Evaluate θ_{i+1} using equation (14),
  —Evaluate $\dot{r}_{i+1}$ using equation (13),
  —Evaluate $\dot{\theta}_{i+1}$ using equation (15),
- Plot numerical solution

It is important to point out that the method is explicit and therefore doesn't require any iteration.

Properties of the Scheme

The method introduced here is based on Taylor series. "Taylor algorithm may be regarded as a one-step explicit method which involves higher order total derivatives." ([12], p. 46) these methods are special one-step cases of a more general class of multistep methods containing higher total derivatives, called Obrechkoff methods (see [20], [18], and [19].) Since the order of the differential equation is second and we kept second-order terms in the Taylor expansion, our method is first-order accurate. To see this, we rewrite the method in matrix form. Let $\mathbf{z}$ and $\mathbf{f}(\mathbf{z})$ be the vectors

$$\mathbf{z} = \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} \tag{16}$$

$$\mathbf{f}(\mathbf{z}) = \begin{bmatrix} \dot{r} \\ \dot{\theta} \\ r\dot{\theta}^2 - \dfrac{k}{r^2} \\ -\dfrac{2}{r}\dot{r}\dot{\theta} \end{bmatrix} \tag{17}$$

then equation (8) can be written as

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}). \tag{18}$$

Now let $\mathbf{z}_i$ and $\mathbf{f}_i$ be the vectors $\mathbf{z}$ and $\mathbf{f}(\mathbf{z})$ evaluated at t_i, then the numerical scheme can be written in matrix form

$$\mathbf{z}_{i+1} = \mathbf{z}_i + \Delta t \Gamma \mathbf{f}_i \tag{19}$$

where Γ is a block upper triangle matrix with 2×2 blocks given by

$$\Gamma = \begin{pmatrix} I & \frac{1}{2}\Delta t I \\ 0 & I \end{pmatrix} \tag{20}$$

and I is a 2×2 identity matrix. It is clear from equation (19) that the method is first-order and explicit. By definition, the method is zero stable, since the first characteristic polynomial is

$$\rho(\zeta) = \zeta - 1$$

which has only one root of unit magnitude. To check absolute stability, we follow Lambert [13] and consider the test system

$$\mathbf{y}' = A\mathbf{y} \tag{21}$$

where the eigenvalues λ_i, $i = 1, \ldots, 4$ of the constant 4×4 matrix A (assumed distinct) have negative real parts. Therefore the general solution of the test equation satisfies

$$\|\mathbf{y}(t)\| \to 0 \qquad t \to \infty$$

We now ask under what conditions the numerical solution for this test equation satisfy

$$\|\mathbf{z}_n\| \to 0 \qquad t \to \infty$$

Our method when applied to the test equation becomes

$$\mathbf{z}_{n+1} - (I + \Delta t \Gamma A)\mathbf{z}_n = 0 \tag{22}$$

Since the eigenvalues of A have negative real parts, the question is how the matrix Γ affects these eigenvalues. We will now obtain a condition on the absolute stability of the method.

Lemma: For a system of four first order ODEs resulting from two second-order systems, one can choose the matrix A as the 2×2 block matrix

$$A = \begin{pmatrix} 0 & I \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \\ e & f & g & p \end{pmatrix} \tag{23}$$

The assumption that all eigenvalues have a negative real part implies

$$\begin{aligned} \text{tr}(A) = \text{tr}(\beta) &< 0 \\ \det(\beta) - \text{tr}(\alpha) &> 0 \\ \det(\alpha) = \det(A) &> 0 \\ a_5 &> 0 \end{aligned} \tag{24}$$

where $a_5 = fc - bg + ap - ed$, $\det(A)$ is the determinant of A and $\text{tr}(A)$ is the trace of the matrix, i.e. the sum of the diagonal elements.

Proof: The eigenvalues of A could be all real and negative, or could be complex conjugate pairs with negative real parts. In either case, the fourth-order characteristic polynomial is

$$\lambda^4 - (c + p)\lambda^3 + (cp - dg - a - f)\lambda^2 + a_5\lambda + \det(\alpha) = 0 \tag{25}$$

This polynomial can be factored into

$$(\lambda^2 + a_1\lambda + a_2)(\lambda^2 + a_3\lambda + a_4) \tag{26}$$

The first factor gives the eigenvalues λ_1, λ_2 possibly a complex pair, i.e., $\lambda_2 = \bar{\lambda_1}$ and the second factor gives λ_3, λ_4. In any case $\Re(\lambda_i) < 0$, $i = 1, 2, 3, 4$. Note that

$$\begin{aligned} \lambda_1 + \lambda_2 &= -a_1 \\ \lambda_1\lambda_2 &= a_2 \\ \lambda_3 + \lambda_4 &= -a_3 \\ \lambda_3\lambda_4 &= a_4 \end{aligned} \tag{27}$$

and therefore $a_i > 0$, $i = 1, 2, 3, 4$. If we expand equation (26), we have

$$\lambda^4 + (a_1 + a_3)\lambda^3 + (a_1a_3 + a_2 + a_4)\lambda^2 + (a_2a_3 + a_1a_4)\lambda + a_2a_4 = 0 \tag{28}$$

All the coefficients of this polynomial are positive, since all a_i are positive. Comparing equations (28) with (25) and using the definition of trace and determinant of a matrix, we have equation (24).

Theorem: The method is absolutely stable for all h satisfying

$$h \leq \frac{2(a_1 + a_3)}{\mathrm{tr}(\alpha)}, \quad \text{if } \mathrm{tr}(\alpha) > 0 \tag{29}$$

If the trace is nonpositive, then there is no restriction on the step size.

Proof: The matrix ΓA will have eigenvalues with negative real part if all the coefficients of the fourth degree characteristic polynomial stay positive. It is easy to show that the characteristic polynomial ΓA is

$$\mu^4 - \left(\mathrm{tr}(\beta) + \frac{\Delta t}{2}\mathrm{tr}(\alpha)\right)\mu^3 + \left(\det(\beta) - \mathrm{tr}(\alpha) + \frac{\Delta t}{2}a_5 + \frac{(\Delta t)^2}{4}\det(\alpha)\right)\mu^2 + (a_5 + \Delta t \det(\alpha))\mu + \det(\alpha) = 0 \tag{30}$$

Since a_5 and $\det(\alpha)$ are positive, and since $\mathrm{tr}(\beta) < 0$, the only coefficient that may change sign is the coefficient of the cubic term when $\mathrm{tr}(\alpha) > 0$, i.e.

$$-\left(\mathrm{tr}(\beta) + \frac{\Delta t}{2}\mathrm{tr}(\alpha)\right) > 0 \tag{31}$$

This implies

$$\Delta t \leq \frac{2|\mathrm{tr}(\beta)|}{\mathrm{tr}(\alpha)} \tag{32}$$

which is what we have to prove.

Generalization

The equations here are expressed more generally as

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{k}{r^2} + a_r$$
$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t} + a_\theta \tag{33}$$

In this case the energy is not conserved. It can be shown by combining the equations (33) and using the definition of energy that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{r}a_r + r^2\dot{\theta}a_\theta \tag{34}$$

We will use Crank Nicolson [8] to approximate this to second-order and get the approximation for $\dot{\theta}$. The Crank Nicolson [8] scheme for equation (34) is

$$E_{i+1} = E_i + \frac{\Delta t}{2}\{\dot{r}_i a_r|_i + r_i^2\dot{\theta}_i a_\theta|_i + \dot{r}_{i+1}a_r|_{i+1} + r_{i+1}^2\dot{\theta}_{i+1}a_\theta|_{i+1}\} \tag{35}$$

Now substitute for E_{i+1} from equation (9) and solve this for $\dot{\theta}_{i+1}$ to get the quadratic equation

$$\dot{\theta}_{i+1}^2 - \Delta t a_\theta|_{i+1}\dot{\theta}_{i+1} = \frac{1}{r_{i+1}^2}\left\{2\left(\frac{k}{r_{i+1}} + E_i\right) - \dot{r}_{i+1}^2 + \Delta t[\dot{E}_i + \dot{r}_{i+1}a_r|_{i+1}]\right\} \tag{36}$$

Note that if a_r or a_θ depend on $\dot{\theta}$ then equation (36) may become a more complicated nonlinear equation (see later). Note also that if $a_r = a_\theta = 0$, we get the same equation as equation (15). Now the algorithm requires the update of a_r and a_θ.

- Initialize
  —Set $t_0 = 0$
  —Choose Δt
  —Use the initial conditions for r_0, $\dot{r}_0$, θ_0 and $\dot{\theta}_0$
  —Evaluate the a_r and a_θ at t_0
  —Evaluate the energy per unit mass using equation (9)
  —Evaluate the time derivative of the energy per unit mass using equation (34)
- Every time step
  Here we loop on i from zero
  —Evaluate $\ddot{r}_i$ and $\ddot{\theta}_i$ using equation (33),
  —Evaluate r_{i+1} using equation (12),
  —Evaluate θ_{i+1} using equation (14),
  —Evaluate $\dot{r}_{i+1}$ using equation (13),
  —Evaluate $\dot{\theta}_{i+1}$ by solving equation (36), and a_r and a_θ
  —Evaluate the energy per unit mass using equation (9)
  —Evaluate the time derivative of the energy per unit mass using equation (34)
- Plot numerical solution

Note that if a_r or a_θ depend on $\dot{\theta}$ then equation (36) may become a more complicated nonlinear equation. Thus it may require a bit more work to obtain $\dot{\theta}_{i+1}$. For example, if $a_\theta = -\delta\dot{\theta}$ then equation (36) is still quadratic in the form

$$(1 + \delta\Delta t)\dot{\theta}_{i+1}^2 = \frac{1}{r_{i+1}^2}\left\{2\left(\frac{k}{r_{i+1}} + E_i\right) - \dot{r}_{i+1}^2 + \Delta t[\dot{E}_i + \dot{r}_{i+1}a_r|_{i+1}]\right\} \tag{37}$$

If a_θ is independent of $\dot{\theta}$, but $a_r = -\epsilon\dot{\theta}$ then

$$\dot{\theta}_{i+1}^2 - \Delta t\left(a_\theta|_{i+1} - \epsilon\frac{\dot{r}_{i+1}}{r_{i+1}^2}\right)\dot{\theta}_{i+1} = \frac{1}{r_{i+1}^2}\left\{2\left(\frac{k}{r_{i+1}} + E_i\right) - \dot{r}_{i+1}^2 + \Delta t\dot{E}_i\right\} \tag{38}$$

Numerical Experiments

In this section we compare our method (based on Taylor series second-order approximation) to a second-order approximation of the original equations and the fourth-order Runge-Kutta-Fehlberg (RKF45) as implemented in Matlab. We experiment with perturbation-free as well as non-conserving cases. We should emphasize here that one can use any integrator to approximate the radial and energy equations. In the first example we solve the system (8) under the initial conditions

$$\begin{aligned} r(0) &= R + 10^5 \\ \dot{r}(0) &= 0 \\ \theta(0) &= 0 \\ \dot{\theta}(0) &= \frac{8000}{r(0)} \end{aligned} \tag{39}$$

where $R = 6.3781363 \times 10^6$ m is the radius of Earth.

We compared our new first-order method to a second-order Taylor series and the Runga-Kutta-Fehlberg (RKF45) as implemented in Matlab. The integration step is

$\Delta t = 1$ sec (except RKF45 which has a variable step size) and the final time of integration is 6000 minutes. The value of r decreases (solution spiralling inward) but at a different rate for two of the methods and increases for the RKF45. The solution using Taylor series method loses 3%, RKF45 method **gains** 1.3% and our first order method loses only 0.5% in 100 hours. It is of no surprise that the energy is conserved with our method, what is interesting is that the angular momentum is almost constant (drops by 0.02%), even though we haven't tried to conserve it. In Fig. 1 we show the graph of energy as computed by Taylor series method and our scheme. Note that the straight horizontal line is the energy for our new scheme. This is followed by a plot of energy (Fig. 2) as computed by RKF45 and a plot of the angular momentum (Fig. 3). Again the straight line is the graph of angular momentum as computed by our new scheme. Note that the energy using RKF45 increases but the maximum is lower than the second-order Taylor series method.

What happens if the perturbation is proportional to r? In our second example, we took the case $a_r = -\beta r$, with $\beta = 10^{-8}$. Now the energy is negatively proportional to r^2, since

$$\frac{dE}{dt} = -\beta r\dot{r} = -\frac{1}{2}\beta\frac{d}{dt}(r^2)$$

In the next two figures, we have plotted the radial distance and the energy at the end of each period (Fig. 4). Notice that the maxima of r are the minima of E as expected.

We have also plotted the energy (Fig. 5) using our scheme and Taylor series approximation. The extremal values with our scheme (right) stay constant, but with the Taylor series (left), the values are growing. A similar trend can be seen with RKF45.

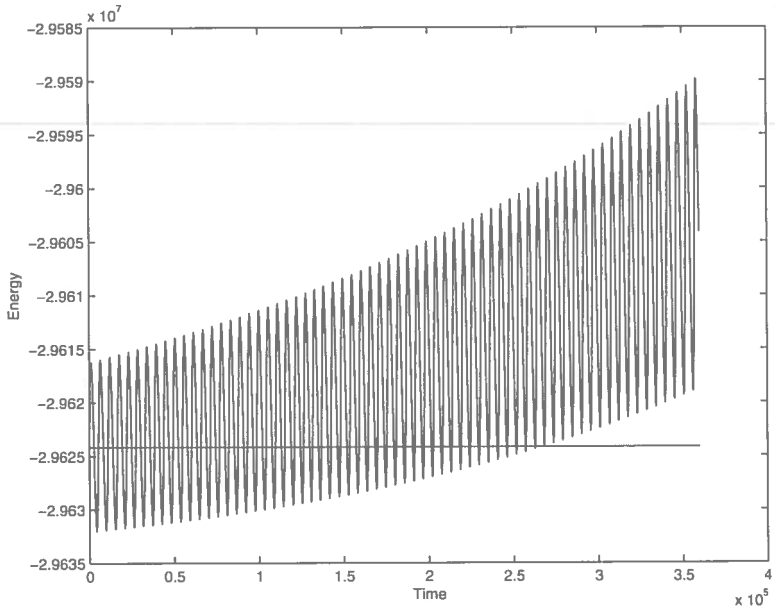

FIG. 1. Energy as a Function of Tiime for the Conservative Case. The Line Shows the Energy for our Scheme and the Curve for the Second-Order Taylor Series.

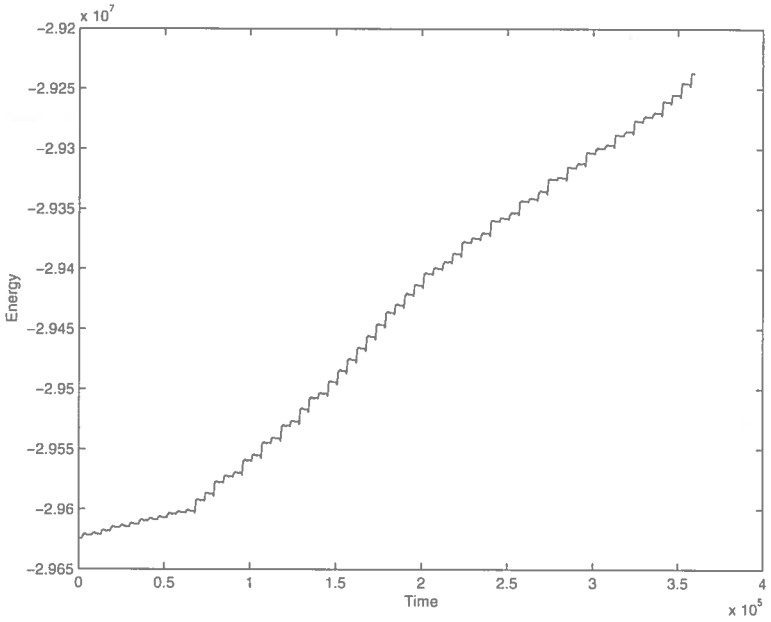

FIG. 2. Energy as a Function of Time for the Conservative Case Using RKF45 Method.

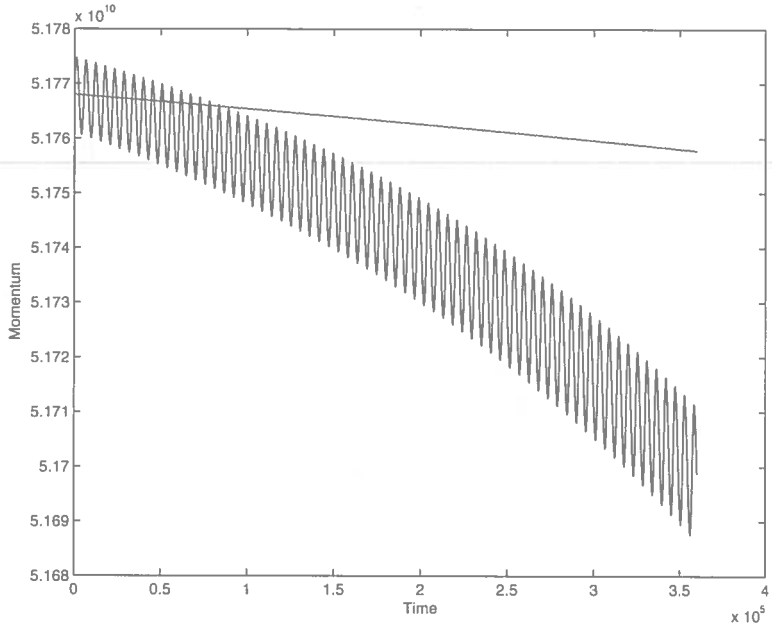

FIG. 3. Angular Momentum as a Function of Time for the Conservative Case. The Line Shows the Angular Momentum for our Scheme and the Curve for the Second-Order Taylor Series.

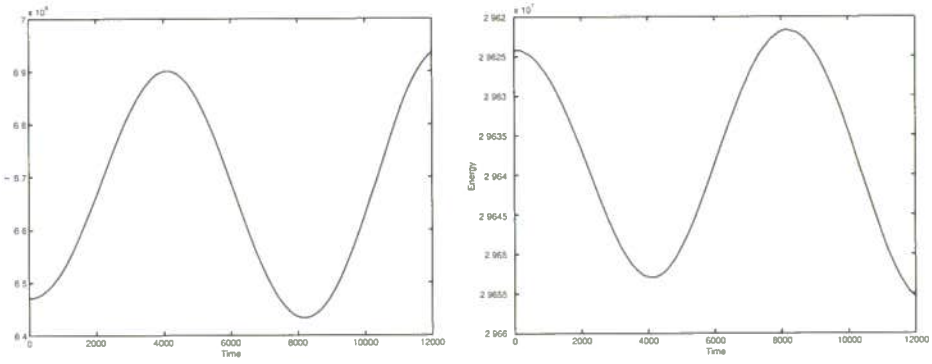

FIG. 4. r (Left) and E (Right) as a Function of Time for the Case $a_r = -\beta r$, $a_\theta = 0$ Using our New Scheme.

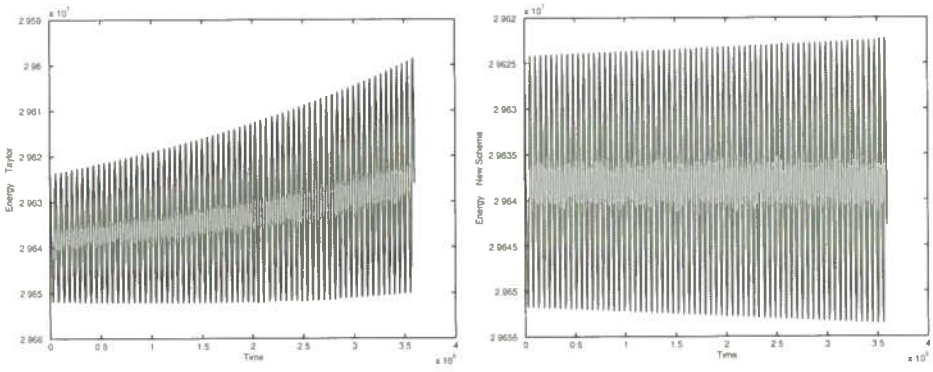

FIG. 5. Energy as a Function of Time for the Case $a_r = -\beta r$, $a_\theta = 0$ Using Taylor Series (Left) and our New Scheme (Right).

In our third example, we included a small perturbation $a_r = -\alpha \dot{r}$ with $\alpha = 10^{-2}$ and a_θ still zero. Clearly the energy is not conserved. The rate of change of energy in this case is negatively proportional to the square of the rate of change of r, i.e.

$$\dot{E} = -\alpha(\dot{r})^2$$

Therefore the energy is monotonically decreasing as can be seen on the left side of Fig. 6. Notice that the energy becomes constant at some time. This is because r becomes a constant and stays there. In fact one can get the analytic solution for $\dot{\theta} = cr^{-2}$ and then the r equation becomes

$$\ddot{r} + \alpha \dot{r} = cr^{-3} - kr^{-2}$$

This equation has an asymptotic solution $r = c/k$, which can be seen in the graph of r, on the right side of Fig. 6.

A fourth example is a case where the perturbation is $a_r = 0$ and $a_\theta = -\alpha\dot{\theta}$ with $\alpha = 10^{-8}$. In this case the rate of change of energy is $\dfrac{dE}{dt} = -\alpha r^2(\dot{\theta})^2$ which is always nonpositive, thus the energy should decrease monotonically. The graph of energy (Fig. 7) shows that for our scheme, the energy is decreasing monotonically, but for the Taylor series method we only get the correct trend.

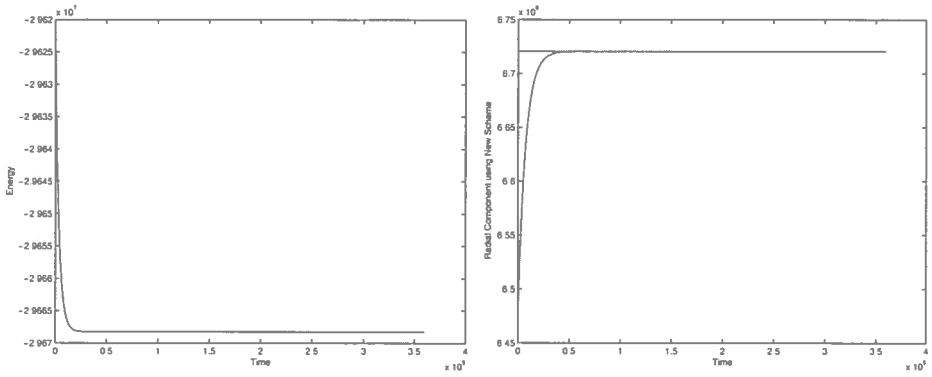

FIG. 6. Energy and r as a Function of Time for the Case $a_r = -\alpha\dot{r}$, $a_\theta = 0$ Using our Scheme.

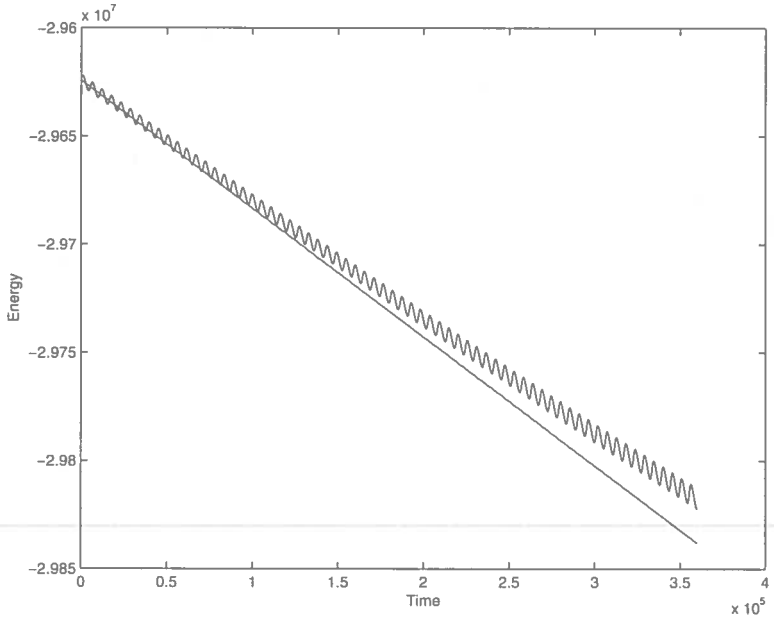

FIG. 7. Energy as a Function of Time for the Case $a_r = 0$, $a_\theta = -\alpha\dot{\theta}$. Taylor Series (Curve) Versus our New Scheme (Line).

Conclusion

Here we developed a new scheme to integrate the equations of motion of a celestial body. The scheme conserves energy in the case of perturbation free-flight as was demonstrated in the first example. The Runge-Kutta-Fehlberg as implemented in Matlab is unable to conserve the energy and angular momentum. In the nonconservative cases, we have shown the benefit of our scheme as compared to Taylor series second-order scheme and Runge-Kutta-Fehlberg fourth-order method. One can use a different numerical scheme to approximate the differential equations. The idea here is to use the energy conservation (or the rate of change of energy in non-

conservative cases) to replace one of the differential equations in the system. In future research we show how to apply our idea with implicit approximation for real world problems.

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