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Application of high-order Higdon non-reflecting boundary conditions to linear shallow water models

Beny Neta[1,*,†], Vince van Joolen[2], John R. Dea[1] and Dan Givoli[3]

[1]*Department of Applied Mathematics, Naval Postgraduate School, Monterey, CA 93943, U.S.A.*
[2]*Department of Mathematics, United States Naval Academy, 121 Blake Road, Anapolis, MD 21402, U.S.A.*
[3]*Department of Aerospace Engineering and Asher Center for Space Research, Technion, Israel Institute of Technology, Haifa 32000, Israel*

SUMMARY

A shallow water model with linear time-dependent dispersive waves in an unbounded domain is considered. The domain is truncated with artificial boundaries $\mathscr{B}$ where a sequence of high-order non-reflecting boundary conditions (NRBCs) proposed by Higdon are applied. Methods devised by Givoli and Neta that afford easy implementation of Higdon NRBCs are refined in order to reduce computational expenses. The new refinement makes the computational effort associated with the boundary treatment quadratic rather than exponential (as in the original scheme) with the order. This allows for implementation of NRBCs of higher orders than previously. A numerical example for a semi-infinite channel truncated on one side is presented. Finite difference schemes are used throughout. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Various applications require computational solutions of dispersive wave problems in domains that are much smaller than the actual domains in which the governing equations hold. The use of non-reflecting boundary conditions (NRBCs) is one method of solving these problems. Here, the original

*Correspondence to: Beny Neta, Department of Applied Mathematics, Naval Postgraduate School, Monterey, CA 93943, U.S.A.
†E-mail: bneta@nps.edu, byneta@gmail.com

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domain is truncated by introducing an artificial boundary $\mathscr{B}$, which encloses the computational domain Ω. An NRBC is applied to $\mathscr{B}$ in order to minimize reflections that result when waves that propagate inside of Ω impinge on $\mathscr{B}$. Boundary conditions that generate no spurious reflection are called 'perfectly non-reflecting,' 'perfectly absorbing,' or simply 'exact' and are reviewed in [1]. Most NRBCs are approximate and generate some reflection. However, if the reflection is small (relative to the order of magnitude of the discretization error) the NRBC is considered adequate.

In the last three decades many NRBCs have been proposed [2]. The complexity of designing accurate NRBCs varies between three types of linear wave problems: time harmonic, non-dispersive time dependent, and dispersive. The prototype governing equations for these are, respectively, the Helmholtz equation, the scalar wave equation, and the Klein–Gordon equation. Effective, exact, and high-order NRBCs are available for time harmonic wave problems; see [3, 4]. Time-dependent wave problems are more involved. Hagstrom and Hariharan [5] proposed a high-order asymptotic NRBC for two-dimensional domains. Additionally, they, along with Grote and Keller [6], constructed exact NRBCs for three-dimensional waves where $\mathscr{B}$ is a sphere.

Dispersive wave problems, in which waves of different frequencies propagate at different speeds, are the most difficult to handle. Their solutions consist of an infinite superposition of single waves, each characterized by its wave number component (or, equivalently, by its phase speed component). Here we apply an NRBC proposed by Higdon [7] on the artificial boundary. A scheme to discretize high-order Higdon NRBCs that was developed by the authors [8–10] is refined and used in this paper. To be more precise, Givoli and Neta have developed two different formulations. In [8] the high-derivative (HD) formulation is introduced and in [9] the auxiliary variable (AV) formulation is given. The HD formulation involves quantities in 'inner layers' of the domain, and also in previous time steps. The advantages of HD over AV (see also Givoli [11]) are also significant:

- HD is simpler (less subtle), and seems to be more stable.
- HD is simpler to code. Only one unknown variable, u.
- HD requires no corner conditions for the problem to be well posed. On the other hand, AV does need corner conditions, see for example [12, 13].
- HD is quite general and can be applied to nonlinear problems, with variable coefficients, etc. On the other hand, AV cannot (since the corner condition relies on the fact that the equation is linear with constant coefficients).

2. STATEMENT OF THE PROBLEM

A linearized form of the shallow water equations (SWEs) is modeled in a semi-infinite channel (Figure 1). For simplicity, the channel has a flat bottom and there is no advection. Rotational (Coriolis) effects are considered. A Cartesian coordinate system (x, y) is introduced and the channel is oriented in the x-direction. The width and depth of the channel are b and H, respectively. Neumann boundary conditions $\partial\eta/\partial y = 0$ are specified on the north and south boundaries, Γ_N and Γ_S. A Dirichlet boundary condition $\eta(0, y, t) = \eta_W(y, t)$ is prescribed on the west boundary Γ_W. As $x \to \infty$, the solution is bounded and does not include incoming waves. The initial conditions are $\eta(x, y, 0) = \eta_0$ and $(\partial\eta/\partial t)(x, y, 0) = w_0$ where η_0 and w_0 have local support. An artificial east boundary Γ_E at $x = x_E$ truncates the semi-infinite domain dividing it into two subdomains: an exterior domain $\mathscr{D}$, and a finite computational domain Ω bounded by Γ_N, Γ_S, Γ_E, and Γ_W.

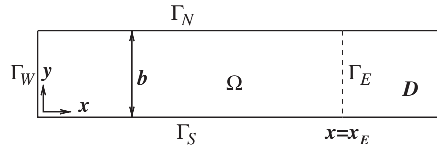

Figure 1. Semi-infinite channel.

3. THE INTERIOR SCHEME

We use the homogeneous Klein–Gordon equation

$$\frac{\partial^2 \eta}{\partial t^2} - C_0^2 \nabla^2 \eta + f^2 \eta = 0 \tag{1}$$

to model the linearized SWE in the truncated domain [14]. Here, η is the perturbation to the water elevation, C_0 is the reference wave speed, and f is a dispersion parameter due to Coriolis forces. Higdon proved that discrete NRBCs are stable if the interior scheme is the standard *second-order centered* difference scheme [7]. Thus, we discretize (1) and solve explicitly for $\eta_{p,q}^{n+1}$ as follows:

$$\eta_{p,q}^{n+1} = \left(\frac{C_0 \Delta t}{\Delta x}\right)^2 (\eta_{p-1,q}^n - 2\eta_{p,q}^n + \eta_{p+1,q}^n) + \left(\frac{C_0 \Delta t}{\Delta y}\right)^2 (\eta_{p,q-1}^n - 2\eta_{p,q}^n + \eta_{p,q+1}^n) + [2 - (f\Delta t)^2]\eta_{p,q}^n - \eta_{p,q}^{n-1} \tag{2}$$

where $\eta_{p,q}^n$ is the finite difference (FD) approximation of $\eta(x, y, t)$ at grid point (x_p, y_q) at time t_n and $C_0 = \sqrt{gH}$.

4. HIGDON'S NRBCs

For a straight boundary normal to the x-direction, the Higdon NRBC of order J is

$$H_J : \left[\prod_{j=1}^{J} \left(\frac{\partial}{\partial t} + C_j \frac{\partial}{\partial x}\right)\right] \eta(x, y, t) = 0 \tag{3}$$

where t is time, C_j is a set of parameters signifying phase speeds in the x-direction. Equation (3) involves up to Jth-order normal and temporal derivatives and is exact for all combinations of waves that propagate with x-direction phase speeds $C_1, \ldots, C_J$. These NRBCs were presented and analyzed in a sequence of papers [15–18] for non-dispersive acoustic, elastic, and dispersive waves.

Many examples in meteorological literature are based on H_1 with a specially chosen C_1 that remains constant, e.g. Pearson [19]. If we set $C_1 = C_0$ we get the classical Sommerfeld-like NRBC. In schemes devised by Orlanski [20], C_1 is changed dynamically and locally for each time-step based on the solution at the previous time-step; see also [21–23]. For other parameter choices,

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Higdon NRBCs are equivalent to NRBCs derived from rational approximations of the dispersion relation [7].

Higdon NRBCs are advantageous because they are robust. Higdon showed that the *reflection coefficient* is a product of J factors, *each of which is smaller than* 1 [7]. This implies that the reflection coefficient becomes smaller as the order J is increased. A good choice for the C_j minimizes reflection, but a non-optimal C_j will still reduce reflection if J is increased; see, for example, [8].

Another advantage to Higdon NRBCs is that they apply to a variety of wave problems, in one, two, and three dimensions and in various configurations. They can be used, without any difficulty, for *dispersive* wave problems and for problems with stratification [14, 24, 25]. Most other available NRBCs are either designed for non-dispersive media (as in acoustics and electromagnetics) or are of low order (as in meteorology and oceanography).

5. DISCRETIZATION OF HIGDON'S NRBCs

The Higdon condition H_J is a product of J operations of the form $\partial/\partial t + C_j\partial/\partial x$. Consider the following FD approximations:

$$\frac{\partial}{\partial t} \simeq \frac{I - S_t^-}{\Delta t}, \quad \frac{\partial}{\partial x} \simeq \frac{I - S_x^-}{\Delta x} \tag{4}$$

where Δt and Δx are the time-step size and grid spacing in the x-direction, respectively, I is the identity operator, and S_t^- and S_x^- are backward shift operators defined as

$$S_t^- \eta_{pq}^n = \eta_{pq}^{n-1}, \quad S_x^- \eta_{pq}^n = \eta_{p-1,q}^n \tag{5}$$

We use (3) and (4) to obtain

$$\left[\prod_{j=1}^{J}\left(\frac{I - S_t^-}{\Delta t} + C_j \frac{I - S_x^-}{\Delta x}\right)\right]\eta_{Eq}^n = 0 \tag{6}$$

where the index E corresponds to a grid point on the boundary Γ_{E}. Collecting the terms I, S_t^-, and S_x^- (after multiplying (6) by Δt) yields

$$\prod_{j=1}^{J}\left[\left(1 + C_j\frac{\Delta t}{\Delta x}\right)I - S_t^- - C_j\frac{\Delta t}{\Delta x}S_x^-\right]\eta_{Eq}^n = 0 \tag{7}$$

Performing the following substitutions

$$a_j = 1 - c_j, \quad b_j = -1 \quad \text{and} \quad c_j = -C_j\frac{\Delta t}{\Delta x} \tag{8}$$

allows us to rewrite (7) as

$$\prod_{j=1}^{J}(a_j I + b_j S_t^- + c_j S_x^-)\eta_{Eq}^n = 0 \tag{9}$$

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In an expanded form, H_J is represented as the summation of 3^J terms:

$$\sum_{m=0}^{3^J-1} A_m P_m \eta^n_{Eq} = 0 \tag{10}$$

where A_m is a product of a_j, b_j, and/or c_j. Similarly P_m is made up of a combination of operators, I, S_t^-, and/or S_x^-. Equation (10) should be used as follows: Let $m=0$, then $A_0 = \prod_{j=1}^{J} a_j$, $P_0 = I$, and

$$A_0 \eta^n_{Eq} = -\sum_{m=1}^{3^J-1} A_m P_m \eta^n_{Eq}$$

We solve the above for η^n_{Eq} for all points on the boundary Γ_{E}.

From (10) it is evident that a high-order discretization must be allocated a great deal of computational time, since the number of operations associated with (10) is $O(3^J)$, see Givoli and Neta [8]. For example, a Higdon NRBC of order $J=10$ requires the calculation and summation of over 59 000 terms. Note that the operators P_m contain terms of the form $(S_t^-)^\beta (S_x^-)^\gamma$ for $0 \leqslant \beta + \gamma \leqslant J$. Since the number of integer combinations of β and γ that satisfies the inequality is $(J+1)(J+2)/2$; thus, the number of different possibilities is only $O(J^2)$. By setting the C_j's equal to a single value C, we slightly modify (8):

$$a = 1 - c, \quad b = -1 \quad \text{and} \quad c = -C\frac{\Delta t}{\Delta x} \tag{11}$$

Now (9) becomes

$$(aI + bS_t^- + cS_x^-)^J \eta^n_{Eq} = 0 \tag{12}$$

Using a trinomial expansion, we have the following equation:

$$\sum_{\beta=0}^{J} \sum_{\gamma=0}^{J-\beta} \frac{J!}{\alpha!\beta!\gamma!} a^\alpha b^\beta c^\gamma S_t^{-\beta} S_x^{-\gamma} \eta^n_{Eq} = 0 \tag{13}$$

where $\alpha = J - \beta - \gamma$. This results in reducing the complexity from $O(3^J)$ to $O(J^2)$.

To obtain an explicit formula for $\eta^n_{E,q}$, one must solve (10). Due to its algebraic complexity, Higdon did this for orders up to $J=3$ only. The Givoli/Neta formulation theoretically allowed us to find an explicit formula for any order, but was constrained to $J=9$ due to computational considerations [24]. The formulation in (13) lifts the computational expenses and allows the exploration of Higdon NRBC's of higher orders. Since the interior scheme (2) and the boundary scheme (10) are explicit, the whole scheme is explicit.

6. NUMERICAL EXAMPLE—WAVE GUIDE

A channel with width $b=5$ and depth $H=0.1$ is considered. The medium has a uniform density $\rho=1$. A gravitational parameter $g=10$ and a dispersion parameter $f=1$ are used. The initial

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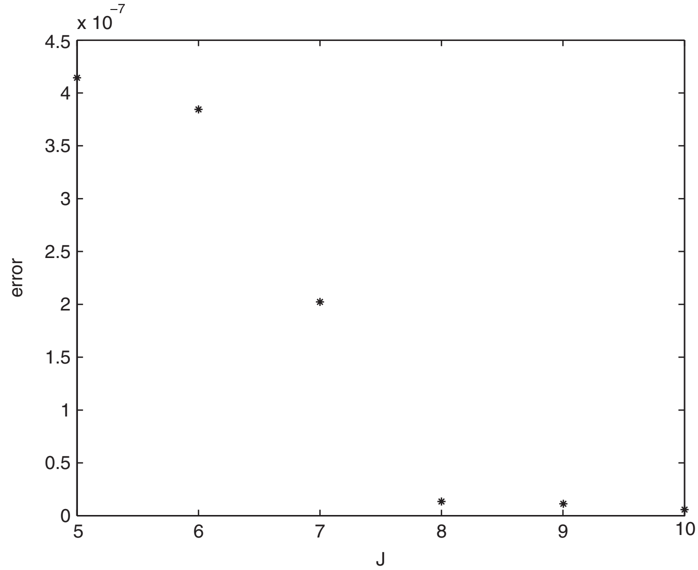

Figure 2. Error at time $t = 1000$ as a function of J.

values are zero everywhere, and a boundary function:

$$\eta_W(y,t) = \begin{cases} 0.005\cos\left[\dfrac{\pi}{2r}(y-2.5)\right] & \text{if } 1\leqslant y\leqslant 4 \text{ and } 0\leqslant t\leqslant 5 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

is used to generate the wave pulse depicted in Plate 1.

The artificial boundary is introduced at $x = 5$ and Ω is defined on a 5×5 square with a 20×20 mesh (i.e. $\Delta x = \Delta y = 0.25$). An extended domain $\mathscr{D}$ with a 10×5 rectangle on a 40×20 mesh is used to calculate a reference solution η_0. A time increment of $\Delta t = 0.125$ is used. An artificial boundary is also imposed on $\mathscr{D}$ at $x = 10$, but run times are sufficiently short so that spurious reflections will not pollute Ω.

In the first case a Higdon NRBC with $J = 10$ is constructed with parameters $C = \sqrt{gH} = 1$. The numerical solution η_{10} is compared with η_0 to obtain an error measurement:

$$\|e(t)\| = \sqrt{\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{[\eta_0(x_i,y_j,t)-\eta_{10}(x_i,y_j,t)]^2}{N_x N_y}} \tag{15}$$

where N_x and N_y are the number of grid points in the x- and y-directions, respectively.

Plate 2 shows solutions for η_0 and η_{10} at the end of the run, $t = 14$. The top-left and top-right plots depict η_0 on the truncated domain Ω and extended domain $\mathscr{D}$, respectively. The bottom-left plot corresponds to η_{10} on the truncated domain Ω. The bottom-right plot presents the pointwise error differences in the truncated domain. The error as measured by (15) is 1.1384×10^{-5}.

At the western corners we have used the values from the western boundary condition. Then we computed the solution on the north and south edges using the Neumann boundary conditions.

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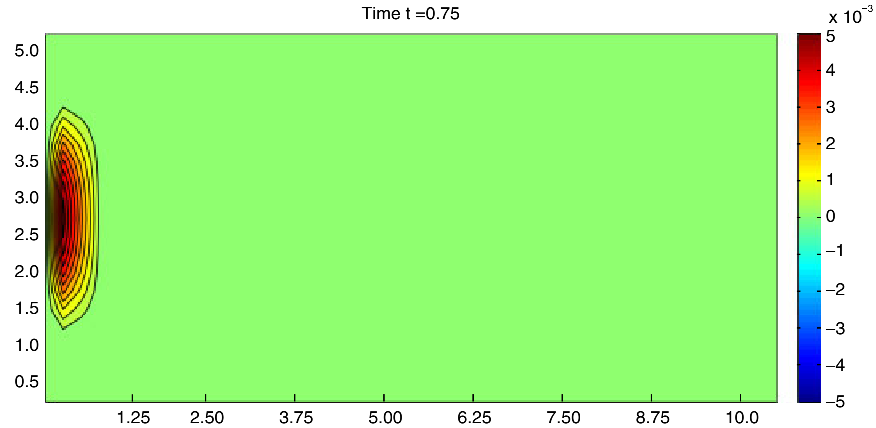

Plate 1. Initial wave pulse depicted on extended plot for $t = 0.75$.

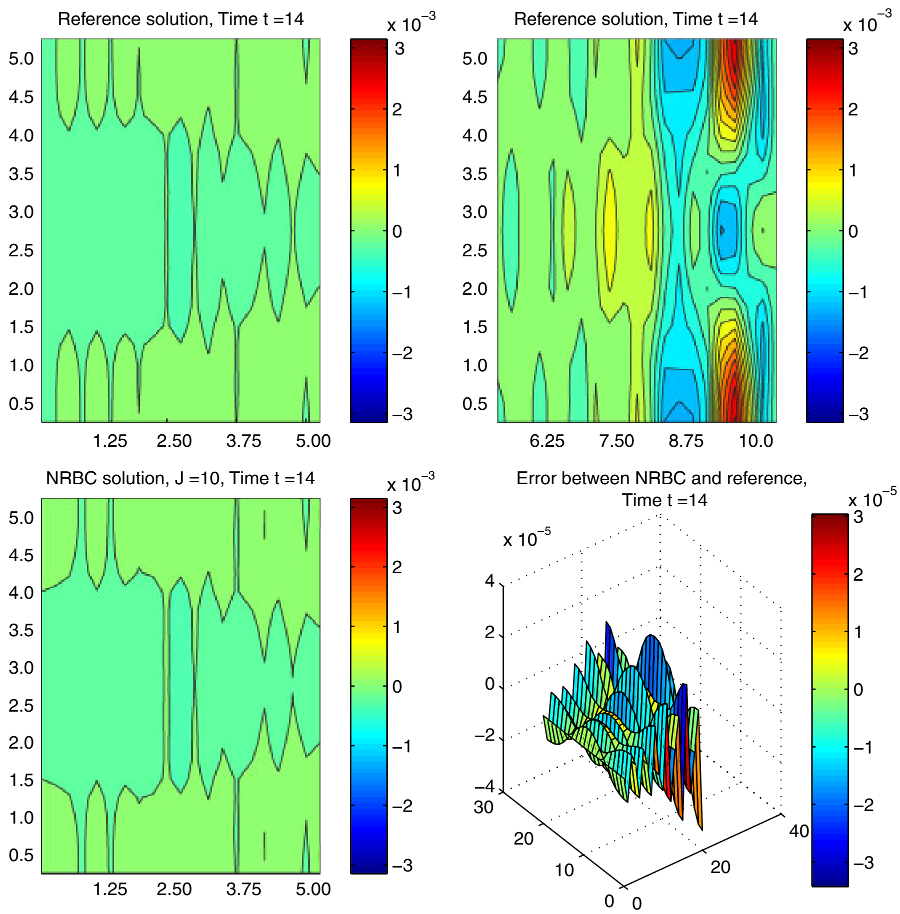

Plate 2. Comparing solutions for a 14 s run.

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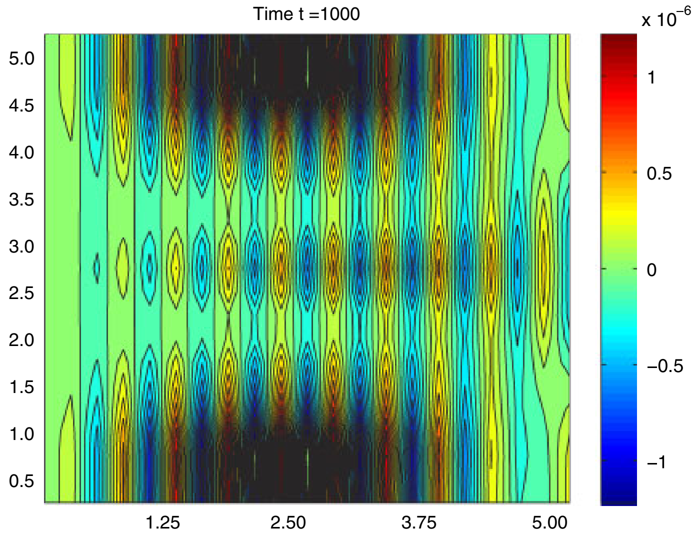

Plate 3. Extended time ($t = 1000$) solution for $J = 5$.

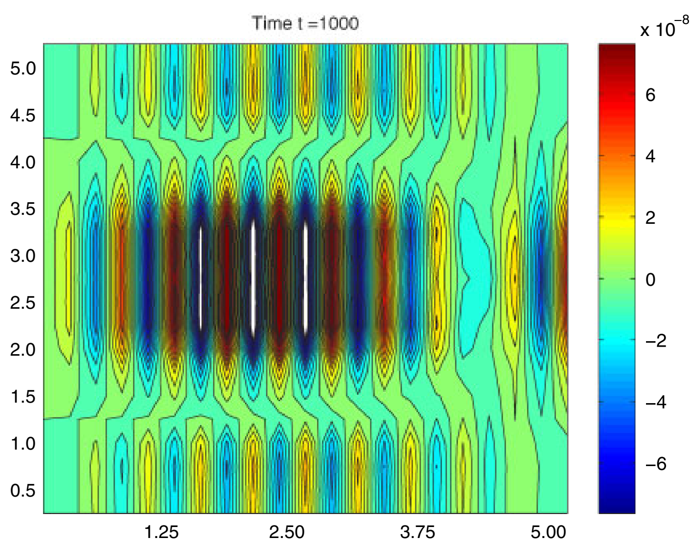

Plate 4. Extended time ($t = 1000$) solution for $J = 10$.

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At the north/south eastern corners we have used the nonreflecting boundary conditions using the points on the north/south edges.

At $t=14$, the pulse has passed through the artificial boundary and is well into the extended domain. An inspection of the left-hand plots serves as a quick qualitative check of the results. Visually, the solutions for η_{10} appear to be nearly identical to the reference solution η_0. Careful observation reveals slight differences. A numerical measure of these differences is presented in the bottom-right plot. Prior implementations of the Higdon NRBC allowed discretization up through $J=9$. The new method presented in this paper lifts this restriction.

In our next experiment, we have run the same test with $J=5$ and 10 for a long time integration ($t_{\text{final}}=1000$). The following Plates 3 and 4 show the solution at $t=1000$ for $J=5$ and 10, respectively. As can be seen, by this time the waves have exited the truncated domain and the ripples seen in the figures are of order 10^{-6} (for $J=5$) and 10^{-8} (for $J=10$). These figures show that the solution at $t=1000$ is essentially zero. In the next figure (Figure 2) we have plotted the error at time $t=1000$ as a function of J ($5\leqslant J\leqslant 10$) using (15) with η_0 is replaced by η_J and $\eta_{10}=0$. This shows that the scheme is stable for very long time integrations.

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