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A family of optimal quartic-order multiple-zero finders with a weight function of the principal kth root of a derivative-to-derivative ratio and their basins of attraction

Young Hee Geum[a,1], Young Ik Kim[a,*], Beny Neta[b]

[a] Department of Applied Mathematics, Dankook University, Cheonan, 330-714, Republic of Korea
[b] Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA 93943, United States

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Abstract

Multiple-zero finders with optimal quartic convergence for nonlinear equations are proposed in this paper with a weight function of the principal kth root of a derivative-to-derivative ratio. The optimality of the proposed multiple-zero finders is checked for their consistency based on Kung–Traub's conjecture established in 1974. Through various test equations, relevant numerical experiments strongly support the claimed theory in this paper. Also investigated are extraneous fixed points of the iterative maps associated with the proposed methods. Their dynamics is explored along with illustrated basins of attraction for various polynomials.

Keywords: Multiple-zero finder; Quartic-order method; Asymptotic error constant; Principal root

1. Introduction

Efficient zero-finding techniques for nonlinear equations $f(x) = 0$ have been investigated to solve a variety of scientific and engineering problems including the prediction of weather forecast, the ground trace of an artificial satellite as well as the location of an object via global positioning system coordinates. The most widely accepted classical Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

usually solves the nonlinear equation $f(x) = 0$ without difficulty, provided that a good initial guess x_0 is chosen near the zero α. For α with given multiplicity of $m \geq 1$, modified Newton's method [38,40] in the following form

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

* Corresponding author.
E-mail addresses: conpana@empas.com (Y.H. Geum), yikbell@yahoo.co.kr (Y.I. Kim), bneta@nps.edu (B. Neta).
[1] Research Professor.

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is frequently used by many researchers. It is known that numerical scheme (1.2) is a second-order one-point optimal [23] method. There are relatively very few methods for multiple roots when the multiplicity is known. See, e.g., a method of order 1.5 [47], a method of order 2 [40], third order methods [14,15,18,45,36,28,39,9,19,16,22,30], in addition to the fourth order methods [33,29,24,25,21,49].

This paper proposes a family of new efficient and optimal quartic-order multiple-zero finders with multiplicity of $m \geq 1$. By use of the kth root of a derivative-to-derivative ratio, we extend modified Newton's method to design two-point optimal quartic-order multiple-zero finders with evaluations of two derivatives and one function per iteration. The optimality will be checked on the basis of Kung–Traub's conjecture [23] that any multipoint method [37] without memory can reach its convergence order of at most 2^{r-1} for r functional evaluations.

This paper is comprised of six sections. Following this introductory section, Section 2 briefly states a review of existing studies on multiple-zero finders. Described in Section 3 is methodology and convergence analysis for newly proposed multiple-zero finders. A main theorem is established to derive asymptotic error constants and error equations by use of weight functions dependent upon the principal kth root of a derivative-to-derivative ratio. In Section 4, special forms of weight functions are considered based on polynomials and rational functions with labeled case numbers. Section 5 discusses the extraneous fixed points and related dynamics of the method. Tabulated in Section 6 are computational results for a variety of numerical examples. Table 9 compares the magnitudes of $e_n = x_n - \alpha$ of the proposed methods with those of typical existing methods. Interesting basins of attraction associated with the proposed methods are displayed with detailed analyses and comments. Stated at the end are overall conclusion and future work.

2. Preliminary review of existing multiple-root finders

Typical quartic-order multiple-zero finders can be found in papers [17,24,25,42]. Interesting works of Liu et al. [25], Soleymani et al. [42] and one of the methods in Li et al. [24] are introduced here, respectively in (2.1)–(2.3):

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - mG_f(w) \cdot \dfrac{f(x_n)}{f'(x_n)}, \quad w = \sqrt[m-1]{\dfrac{f'(y_n)}{f'(x_n)}},\ m > 1, \\ \text{where } G_f \text{ is sufficiently differentiable at 0 with } G_f(0) = 0,\ G'_f(0) = 1,\ G''_f(0) = \dfrac{4m}{m-1}. \end{cases} \tag{2.1}$$

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n + \dfrac{4md}{d(m^2+2m-4) - m^2 v}\left[1 - \dfrac{m^3(m-2)}{16d^2}\left(v - \dfrac{m+2}{m}d\right)^2\right] \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ \quad v = \dfrac{f'(y_n)}{f'(x_n)},\ d = \left(\dfrac{m}{m+2}\right)^m. \end{cases} \tag{2.2}$$

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \left(a_3 + \dfrac{1}{b_1 + b_2 v}\right) \cdot \dfrac{f(x_n)}{f'(x_n)}, \quad v = \dfrac{f'(y_n)}{f'(x_n)}, \\ \text{where } a_3 = -\dfrac{m(m-2)}{2},\ b_1 = -\dfrac{1}{m},\ b_2 = \dfrac{1}{md},\ d = \left(\dfrac{m}{m+2}\right)^m. \end{cases} \tag{2.3}$$

Besides that, Zhou et al. [49] proposed the following fourth-order iterative scheme:

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - Z_f(v) \cdot \dfrac{f(x_n)}{f'(x_n)}, \quad \text{with } v = \dfrac{f'(y_n)}{f'(x_n)},\ Z_f(\cdot) \in C^2(\mathbb{R}). \end{cases}$$

In particular, $Z_f(v)$ in the form of a first-order rational function is considered as follows:

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \left(\dfrac{b+cv}{1+av}\right)\cdot\dfrac{f(x_n)}{f'(x_n)}, \quad a = -\dfrac{1}{d},\ b = -\dfrac{m^2}{2},\ c = \dfrac{m(m-2)}{2d},\ d = \left(\dfrac{m}{m+2}\right)^m. \end{cases} \tag{2.4}$$

Remark 2.1. We find that Method (2.3) is identical with Method (2.4) in view of the relation

$$a_3 + \frac{1}{b_1 + b_2 v} = -\frac{m^2}{2} - \frac{mv}{d-v} = \frac{b+cv}{1+av}$$

via direct substitution of $b_1,\ b_2,\ a_3$ and due to the fact that

$$a = \frac{b_2}{b_1} = -\frac{1}{d}, \qquad b = a_3 + \frac{1}{b_1} = -\frac{m^2}{2}, \qquad c = \frac{a_3 b_2}{b_1} = \frac{m(m-2)}{2d}, \qquad d = \left(\frac{m}{m+2}\right)^m.$$

Convergence behavior of existing methods (2.1)–(2.3) for various test equations will be compared later in Section 6 with proposed methods to be shown in the next section.

3. Methodology and convergence analysis

Let a function $f : \mathbb{C} \to \mathbb{C}$ have a multiple zero α with integer multiplicity $m \geq 1$ and be analytic [1] in a small neighborhood of α. Then new iterative methods are proposed below to find an approximate zero α of multiplicity m, given an initial guess x_0 sufficiently close to α:

$$\begin{cases} y_n = x_n - \gamma\dfrac{f(x_n)}{f'(x_n)}, \quad \gamma \in \mathbb{R}, \\ x_{n+1} = x_n - Q_f(s)\cdot\dfrac{f(x_n)}{f'(x_n)}, \quad s = \left[\dfrac{f'(y_n)}{f'(x_n)}\right]^{\frac{1}{k}},\ k \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\gamma \in \mathbb{R}$ is a parameter; $Q_f : \mathbb{C} \to \mathbb{C}$ is analytic in a neighborhood of λ with $\lambda \in \mathbb{R}$ to be determined later for optimal quartic-order convergence. Since s is a one-to-k multiple-valued function, we consider its principal analytic branch [1]. Hence, it is convenient to treat s as a principal root given by $s = \exp[\frac{1}{k}\mathrm{Log}(\frac{f'(y_n)}{f'(x_n)})]$, with $\mathrm{Log}(\frac{f'(y_n)}{f'(x_n)}) = \mathrm{Log}\left|\frac{f'(y_n)}{f'(x_n)}\right| + i\ \mathrm{Arg}(\frac{f'(y_n)}{f'(x_n)})$ for $-\pi < \mathrm{Arg}(\frac{f'(y_n)}{f'(x_n)}) \leq \pi$; this convention of Arg(z) for $z \in \mathbb{C}$ agrees with that of Log[z] command of Mathematica [48] to be adopted in numerical experiments of Section 6. By means of further inspection of s, we find that $\lambda \in \mathbb{R}$ is characterized in such a way that $s = \left|\frac{f'(y_n)}{f'(x_n)}\right|^{\frac{1}{k}} \cdot \exp[\frac{i}{k}\ \mathrm{Arg}(\frac{f'(y_n)}{f'(x_n)})] = \lambda + O(e_n)$.

Definition 3.1 (*Error Equation, Asymptotic Error Constant, Order of Convergence*)**.** Let $x_0, x_1, \ldots, x_n, \ldots$ be a sequence converging to α and $e_n = x_n - \alpha$ be the n^{th} iterate error. If there exist real numbers $p \in \mathbb{R}$ and $b \in \mathbb{R} - \{0\}$ such that the following error equation holds

$$e_{n+1} = b\ {e_n}^p + O(e_n^{p+1}), \tag{3.2}$$

then b or $|b|$ is called the asymptotic error constant and p is called the order of convergence [44].

In this paper, we investigate the optimal convergence of proposed methods (3.1). We here establish a main theorem describing the convergence analysis regarding proposed methods (3.1) and find out how to select the parameter γ and the weight function Q_f for optimal fourth-order convergence. Three functional evaluations in (3.1) are evidently qualified for the possibility of optimal order of four in the sense of Kung–Traub. Hence, it suffices to determine the constant parameter γ and relevant properties of the weight function Q_f for fourth-order convergence.

Applying the Taylor's series expansion of f about α, we get the following relations:

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!} {e_n}^m [1 + A_1 e_n + A_2 e_n^2 + A_3 e_n^3 + A_4 e_n^4 + O(e_n^5)], \tag{3.3}$$

$$f'(x_n) = \frac{f^{(m)}(\alpha)}{(m-1)!} {e_n}^{m-1} [1 + B_1 e_n + B_2 e_n^2 + B_3 e_n^3 + B_4 e_n^4 + O(e_n^5)], \tag{3.4}$$

where $A_k = \frac{m!}{(m+k)!}\theta_k$, $B_k = \frac{(m-1)!}{(m+k-1)!}\theta_k$ and $\theta_k = \frac{f^{(m+k)}(\alpha)}{f^{(m)}(\alpha)}$ for $k = 1, 2, 3, 4$.
Dividing (3.3) by (3.4), we have

$$\frac{f(x_n)}{f'(x_n)} = \frac{1}{m}[e_n - K_1 e_n^2 - K_2 e_n^3 + K_3 e_n^4 + O(e_n^5)], \tag{3.5}$$

where $K_1 = -A_1 + B_1$, $K_2 = -A_2 + A_1 B_1 - B_1^2 + B_2$ and $K_3 = -A_3 + A_2 B_1 - A_1 B_1^2 + B_1^3 + A_1 B_2 - 2B_1 B_2 + B_3$.

Instead of using the parameter γ directly, we introduce a new parameter given by $t = 1 - \gamma/m$ for algebraic convenience of shortened expressions. Then, from the above relation (3.5), we obtain

$$y_n = \alpha + te + K_1(1-t)e_n^2 + K_2(1-t)e_n^3 + K_3(1-t)e_n^4 + O(e_n^5). \tag{3.6}$$

Our aim is to determine a fixed value of t for the desired quartic convergence of the proposed scheme. Expanding $f'(y_n)$ about the multiple zero α leads us to the following relation:

$$\begin{aligned} f'(y_n) = \frac{f^{(m)}(\alpha)e_n^{m-1}}{(m-1)!} \Big\{ & t^{m-2}[t + (B_1 t^2 + K_1(m-1)(1-t))]e_n \\ & + \frac{1}{2}t^{m-3}[K_1^2(m-2)(m-1)(1-t)^2 - 2B_1 K_1 m(1-t)t^2 + 2t(B_2 t^3 + K_2(m-1)(1-t))]e_n^2 \\ & + \frac{1}{6}t^{m-4}[K_1^3(m-3)(m-2)(m-1)(1-t)^3 + 3B_1 K_1^2(m-1)m(1-t)^2 t^2 \\ & + 6K_1(1-t)t(K_2(m-2)(m-1)(1-t) + B_2(m+1)t^3) + 6t^2(K_3(m-1)(1-t) \\ & + t(B_1 K_1 m(1-t) + B_3 t^3))]e_n^3 \Big\} + O(e_n^4). \end{aligned} \tag{3.7}$$

For later use, we now conveniently denote

$$s = \left[\frac{f'(y_n)}{f'(x_n)}\right]^{\frac{1}{k}}. \tag{3.8}$$

Then, applying Taylor's expansion or multinomial expansion, we get the above expression s as follows:

$$\begin{aligned} s = {} & t^{\frac{m-1}{k}} + \frac{(t-1)t^{\frac{m-1-k}{k}}(K_1 - K_1 m + B_1 t)}{k} e_n - \frac{(t-1)t^{\frac{m-1-2k}{k}}}{2k^2}[K_1^2(1+k-m)(m-1)(t-1) \\ & + 2B_1 K_1 t[1 - m + t(m-1+k)] + t[-B_1^2(t-1)t + k\{2K_2(m-1) + (B_1^2 - 2B_2)t(t+1)\}]]e_n^2 \\ & + \frac{(t-1)t^{\frac{m-1-3k}{k}}}{6k^3}[-K_1^3(1+k-m)(1+2k-m)(m-1)(t-1)^2 \\ & + 3B_1 K_1^2(m-1)(t-1)t[1+k-m+(-1+k+m)t] \\ & + 3K_1 t\{2kK_2(1+k-m)(m-1) - (m-1)t[B_1^2(1+k) + 2k(-B_2 + K_2(1+k-m))] \\ & + 2B_1^2(-1+k+m)t^2 + [B_1^2(k-1) - 2B_2 k](-1+2k+m)t^3\} + t^2\{B_1^3 t[(t-1)^2 - 3k(t^2-1) \\ & + 2k^2(1+t+t^2)] + 6k^2[K_3(1-m) + B_3 t(1+t+t^2)] - 6B_1 k[K_2(1-m+(-1+k+m)t) \\ & + B_2 t(1 + k(1+t) + (k-1)t^2)]\}]e_n^3 + O(e_n^4). \end{aligned} \tag{3.9}$$

We now let $\lambda = t^{\frac{m-1}{k}}$ for notational simplicity. It is worth to carefully treat the branch point $t = 0$ of λ. If $t = 0$, s in (3.8) can be written more compactly as:

$$\begin{aligned} s = e_n^{\frac{m-1}{k}} \Big[& K_1^{\frac{m-1}{k}} + \frac{K_1^{\frac{m-1}{k}-1}}{k}(-B_1K_1 + K_2(m-1))e_n + \frac{K_1^{\frac{m-1}{k}-2}}{2k^2}((B_1^2(1+k) - 2kB_2)K_1^2 \\ & + 2kB_1K_1^3 + (m-1)(2K_1(-B_1K_2 + kK_3) + K_2^2(-1-k+m)))e_n^2 \\ & + \frac{K_1^{\frac{m-1}{k}-3}}{6k^3}(-6kB_1^2K_1^4 + K_1^2(m-1)(3(-2B_2k + B_1^2(1+k))K_2 + 6k(-B_1K_3 + kK_4)) \\ & + K_1(m-1)(3B_1K_2^2(k-m+1) - 6kK_2K_3(k-m+1)) + K_2^3(k-m+1)(1+2k-m)(m-1) \\ & + K_1^3(-6B_3k^2 + 6B_1B_2k(1+k) - B_1^3(1+k)(1+2k) + 6kB_1K_2(k+m-1)))e_n^3 + O(e_n^4)\Big]. \end{aligned} \tag{3.10}$$

As a result, we will use s in (3.9) and (3.10) respectively for $t \neq 0$ and $t = 0$. If $t \neq 0$, then we find that $\lambda = t^{\frac{m-1}{k}}$ is well defined for all values of $m \in \mathbb{N}$. Expanding Taylor series of $Q_f(s)$ about λ up to third-order terms we find:

$$Q_f(s) = Q_0 + Q_1(s-\lambda) + Q_2(s-\lambda)^2 + Q_3(s-\lambda)^3 + O(e_n^4). \tag{3.11}$$

Hence by substituting (3.3)–(3.11) into the proposed method (3.1), we obtain the error equation as

$$e_{n+1} = e_n - Q_f(s) \cdot \frac{f(x_n)}{f'(x_n)} = \psi_1 e_n + \psi_2 e_n^2 + \psi_3 e_n^3 + \psi_4 e_n^4 + O(e_n^5), \tag{3.12}$$

where $\psi_1 = 1 - \frac{Q_0}{m}$ and the coefficients $\psi_i (2 \le i \le 4)$ generally depend on the parameters t, $Q_j (j = 0, 1 \cdots)$, $\theta_i (i = 1, 2, \ldots)$ and the function $f(x)$. Solving $\psi_1 = 0$, $\psi_2 = 0$ independently of θ_i for Q_0 and Q_1, we get

$$Q_0 = m,\ Q_1 = \frac{mkt^{\frac{1-m+k}{k}}}{(t-1)[t+1+m(t-1)]}. \tag{3.13}$$

Substituting (3.13) into $\psi_3 = 0$ and simplifying, we have:

$$\begin{aligned} \Big[& -\frac{Q_2(t-1)^2 t^{-2+2(m-1)/k}(1+m(t-1)+t)^2}{k^2m^3(m+1)^2} - \frac{(m-1)(1+t) - (-3+m)(1+m)t^2 + (1+m)^2t^3}{2m^2(1+m)^2 t(1+m(t-1)+t)} \\ & - \frac{(t-1)(1+m(t-1)+t)}{2km^2(m+1)^2 t}\Big]\theta_1^2 - \frac{t(m-(m+2)t)}{m(m+1)(m+2)(1+m(t-1)+t)}\theta_2 = 0. \end{aligned} \tag{3.14}$$

We first let $\psi_3 = \psi_{31}\theta_1^2 + \psi_{32}\theta_2$. To make $\psi_3 = 0$ independently of θ_1 and θ_2, we solve $\psi_{31} = 0$ and $\psi_{32} = 0$ simultaneously of t and Q_2. As a result, with the use of notation

$$\rho = \left(\frac{m}{m+2}\right)^{\frac{m-1}{k}}, \tag{3.15}$$

we obtain:

$$t = \frac{m}{m+2}, \quad \lambda = \rho, \quad Q_0 = m, \quad Q_1 = -\frac{km^2(m+2)}{4\rho}, \quad Q_2 = \frac{km^2(m+2)(1+k+km)}{8\rho^2}. \tag{3.16}$$

If $t = 0$, i.e., if $\gamma = m$, then we readily choose $k = m - 1 \neq 0$ or $m = 1$ via special treatment from (3.6) in order that s might start with a lowest integer-order term in e_n. This eventually leads us to summarized results together with the corresponding error equation shown in (3.17) or (3.18):

$$\begin{cases} m \neq 1, \quad \gamma = m, \quad Q_0 = m, \quad Q_1 = m, \quad Q_2 = \dfrac{2m^2}{m-1}, \quad k = m-1, \\ e_{n+1} = \Big[\Big(\dfrac{m^3+8m^2+m+2}{2(m-1)^2m^3(m+1)^3} - \dfrac{Q_3}{m^4(m+1)^3}\Big)\theta_1^3 - \dfrac{1}{(m-1)m(m+1)^2(m+2)}\theta_1\theta_2\Big]e_n^4 + O(e_n^5). \end{cases} \tag{3.17}$$

or

$$\begin{cases} m=1, \quad \gamma=1, \quad Q_0=1, \quad Q_1=-\dfrac{k}{2}, \quad k\in\mathbb{N}, \\ e_{n+1}=\left[\left(\dfrac{k+1}{4k}-\dfrac{Q_2}{k^2}\right)\theta_1^2+\dfrac{1}{12}\theta_2\right]e_n^3+O(e_n^4). \end{cases} \tag{3.18}$$

Remark 3.1. It is seen that method (2.1) can be obtained if we take $Q_f = m(1+G)$ when $\gamma = m$ via (3.17) under the restriction of f to $\mathbb{R}$. One can see that (3.18) describes only a family of cubic-order simple-zero finders. In addition, existing Methods (2.2), (2.3) are special cases when $k=1$ is chosen from proposed methods (3.1). Note also that they satisfy (3.16) when each of their respective weight functions is represented as $Q_f(s)$ with $k=1$.

In this investigation, we will limit ourselves only to the case for nonzero $t=\frac{m}{m+2}$ (i.e., $\gamma=\frac{2m}{m+2}$) as shown in (3.16). Substituting (3.13) and (3.16) into ψ_4, we find

$$\psi_4=\phi_1\theta_1^3+\phi_2\theta_1\theta_2+\phi_3\theta_3, \tag{3.19}$$

where

$$\phi_1=\frac{16+24(1+m)+k^2(8+16m+12m^2+14m^3+14m^4+6m^5+m^6)}{3k^2m^5(m+1)^3(m+2)^2}+\frac{64\rho^3Q_3}{k^3m^2(m+1)^3(m+2)^3},$$

$$\rho=\left(\frac{m}{m+2}\right)^{\frac{m-1}{k}},$$

$$\phi_2=-\frac{1}{m(m+1)^2(m+2)}, \quad \text{and} \quad \phi_3=\frac{m}{(m+1)(m+2)^3(m+3)}.$$

The consequence of the analysis carried out thus far immediately leads us to the following theorem.

Theorem 3.2. *Let $f:\mathbb{C}\to\mathbb{C}$ have a zero α of multiplicity $m\in\mathbb{N}$ and be analytic in a small neighborhood of α. Let $k\in\mathbb{N}$ be given. Let $\gamma=\frac{2m}{m+2}$, $\rho=(\frac{m}{m+2})^{\frac{m-1}{k}}$ and $\theta_j=\frac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}$ for $j\in\mathbb{N}$. Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α. Let $Q_f:\mathbb{C}\to\mathbb{C}$ be analytic in a neighborhood of ρ. Let $Q_j=\frac{1}{j!}\frac{d^j}{ds^j}Q_f(s)|_{s=\rho}$ for $1\le j\le 3$. Suppose that $Q_0=m$, $Q_1=-\frac{km^2(m+2)}{4\rho}$ and $Q_2=\frac{km^2(m+2)(1+k+km)}{8\rho^2}$ hold. Then iterative methods (3.1) are of optimal fourth-order and possess the following error equation:*

$$e_{n+1}=(\phi_1\theta_1^3+\phi_2\theta_1\theta_2+\phi_3\theta_3)e_n^4+O(e_n^5), \tag{3.20}$$

where $\phi_i(1\le i\le 3)$ are given in (3.19).

4. Special cases of weight functions with $\gamma=\frac{2m}{m+2}$

Using relations (3.9), (3.11) and (3.16), the Taylor-polynomial form of $Q_f(s)$ is easily given by

$$Q_f(s)=m-\frac{km^2(m+2)}{4\rho}(s-\rho)+\frac{km^2(m+2)(1+k+km)}{8\rho^2}+Q_3(s-\rho)^3, \tag{4.1}$$

where s and ρ are introduced respectively in (3.8) and (3.15). Special cases of $Q_f(S)$ are considered here. In each case, relevant coefficients are determined based on relation (3.16).

Although a variety forms of weight functions $Q_f(s)$ are available in view of Taylor-polynomial forms shown by (4.1), we will limit ourselves to considering several cases of weight functions comprising low-order polynomials or simple rational functions.

Case 1: Second-order polynomial weight functions: $Q_3=0$

$$Q_f(s)=m-\frac{km^2(m+2)}{4\rho}(s-\rho)+\frac{km^2(m+2)(1+k+km)}{8\rho^2}(s-\rho)^2. \tag{4.2}$$

Table 1
Typical $Q_f(s)$ of Case 1 with k, ρ.

CN	k	ρ	$Q_f(s)$
1A	m	$(\frac{m}{m+2})^{\frac{m-1}{m}}$	$m - \frac{m^3(m+2)}{4\rho}(s-\rho) + \frac{m^3(m+2)(1+m+m^2)}{8\rho^2}(s-\rho)^2$
1B	$m-1$	$\frac{m}{m+2}$	$m - \frac{(m-1)m^2(m+2)}{4\rho}(s-\rho) + \frac{(m-1)m^4(m+2)}{8\rho^2}(s-\rho)^2$
1C	$m+1$	$(\frac{m}{m+2})^{\frac{m-1}{m+1}}$	$m - \frac{m^2(m+1)(m+2)}{4\rho}(s-\rho) + \frac{m^2(m+1)(m+2)(m^2+2m+2)}{8\rho^2}(s-\rho)^2$
1D	$m+2$	$(\frac{m}{m+2})^{\frac{m-1}{m+2}}$	$m - \frac{m^2(m+2)^2}{4\rho}(s-\rho) + \frac{m^2(m+2^2)(m^2+3m+3)}{8\rho^2}(s-\rho)^2$
1E	$m+3$	$(\frac{m}{m+2})^{\frac{m-1}{m+3}}$	$m - \frac{m^2(m+2)(m+3)}{4\rho}(s-\rho) + \frac{m^2(m+2^3)(m+3)}{8\rho^2}(s-\rho)^2$

Table 2
Typical $Q_f(s)$ of Case 2 with a_i, b_j, Q_3.

CN	a_1	a_2	a_3	b_2	b_3	$Q_f(s)$	Q_3
2A	m	$\frac{\mu}{4\rho}$	0	$\frac{\delta}{2\rho}$	0	$\frac{m+a_2(s-\rho)}{1+b_2(s-\rho)}$	$-\frac{m\tau\delta^2}{16\rho^3}$
2B	m	0	0	$\frac{\tau}{4\rho}$	$-\frac{\tau\mu}{16m\rho^2}$	$\frac{m}{1+b_2(s-\rho)+b_3(s-\rho)^2}$	$\frac{\tau^2(-2m\delta+\mu)}{64\rho^3}$
2C	m	0	$\frac{\tau\mu}{16\rho^2}$	$\frac{\tau}{4\rho}$	0	$\frac{m+a_3(s-\rho)^2}{1+b_2(s-\rho)}$	$-\frac{m\tau^2\delta}{32\rho^3}$

Here, $\mu = m(2+2k-km^2)$, $\delta = 1+k+km$ and $\tau = km(m+2)$.

We list typical second-order $Q_f(s)$ with interesting combinations of k, ρ in Table 1, where CN stands for the corresponding case identification number.

Case 2: Second-order rational weight functions

$$Q_f(s) = \frac{a_1 + a_2(s-\rho) + a_3(s-\rho)^2}{1 + b_2(s-\rho) + b_3(s-\rho)^2}, \tag{4.3}$$

where $a_1 = m, a_2 = \frac{\mu}{4\rho} + \frac{4\rho}{\tau}(-a_3 + b_3 m)$ with $\mu = m(2+2k-km^2)$, $\tau = km(m+2)$ and $b_2 = \frac{\delta}{2\rho} + \frac{4\rho}{m\tau}(-a_3 + b_3 m)$ with $\delta = 1+k+km$ are determined using (3.16) with a_3 and b_3 as parameters to be selected. Although we explicitly obtain relations $a_2 = a_2(a_3, b_3)$ and $b_2 = b_2(a_3, b_3)$ in terms of two parameters a_3 and b_3, one should note that they allow us to conveniently solve for any two parameters in terms of other two remaining parameters out of four parameters a_2, b_2, a_3, b_3.

In fact, in Table 2, we list typical $Q_f(s)$ with interesting combinations of a_i, b_j, Q_3. Sub-cases of {**2A**, **2B**, **2C**} are conveniently designated as {**2A1**, **2B1**, **2C1**}, {**2A2**, **2B2**, **2C2**} and {**2A3**, **2B3**, **2C3**}, respectively for $k = m$, $k = m-1$ and $k = m+3$.

Remark 4.1. (a) When $m = 2, k = m-1 = 1$, sub-case numbers **2A2**, **2B2**, **2C2** all denote the same $Q_f(s) = \frac{2}{4s-1}$ due to the values of $\rho = \frac{1}{2}, \mu = 0, \delta = 4, \tau = 8, a_2 = a_3 = b_3 = 0, b_2 = 4$ in Table 2. Besides that, (2.2) shows that $Q_f(s) = -\frac{4m(\frac{m}{m+2})^m}{(\frac{m}{m+2})^m(m^2+2m-4)-m^2 s}\left[1 - \frac{m^3(m-2)}{16(\frac{m}{m+2})^{2m}}\{s - (\frac{m}{m+2})^{m-1}\}^2\right] = \frac{2}{4s-1}$ when $m = 2$. Likewise, (2.3) shows that $Q_s(s) = \frac{2}{4s-1}$ when $m = 2$.

(b) When $k = m = 2$, sub-case number **2B1** also denotes $Q_f(s) = \frac{2}{4s-1}$.

(c) As a result, sub-case numbers **2B1**, **2A2**, **2B2**, **2C2** all denote the same $Q_f(s) = \frac{2}{4s-1}$, when $m = 2$.

In the next section, we will discuss the extraneous fixed points [20,46] of Q_f and relevant dynamics of the proposed method. The dynamics behind basins of attraction was initiated by Stewart [43] and followed by works of Chun et al. [31,10,11], Chicharro et al. [7], Cordero et al. [13] and Neta et al. [32], in addition to the works of Amat et al. [3,4,2,5], Scott et al. [41] and Chun et al. [8]. The only papers comparing basins of attraction for methods to obtain multiple roots are due to Neta et al. [34], Neta and Chun [30,31], and Chun and Neta [10]. More recent results on basins of attraction can be found in [6,26,27].

5. Extraneous fixed points

In general, multipoint iterative methods finding a zero α of a nonlinear equation $f(x) = 0$ can be written as

$$x_{n+1} = R_f(x_n), \quad n = 0, 1, \ldots, \tag{5.1}$$

where a fixed point ξ of R_f is α. The iteration function R_f, however, might possess other fixed points $\xi \neq \alpha$. Such fixed points are called the *extraneous fixed points* of the iteration function R_f. Extraneous fixed points may form attractive, indifferent or repulsive cycles as well as other periodic orbits to display chaotic dynamics of the method under investigation. This motivates our technical selection of appropriate parameters among free parameters a_2, b_2, a_3, b_3 of Q_f in Case 2 of the preceding section via intensive analysis of the extraneous fixed points under some constraints.

The proposed method (3.1) can be put in the form:

$$x_{n+1} = R_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n), \tag{5.2}$$

where $H_f(x_n) = Q_f(s)$ can be regarded as a weight function of the classical Newton's method. It is obvious that α is a fixed point of R_f. The points $\xi \neq \alpha$ for which $H_f(\xi) = 0$ are extraneous fixed points of R_f. It is interesting to vary k with some choices of free parameters in Cases 1 and 2 and observe the resulting dynamics of the method related to the extraneous fixed points.

If we look at H_f in Case 1, it varies with one integer parameter $k \geq 1$. Thus all of its listed five sub-cases are dynamically different from each other. Similarly, if we look at H_f in Case 2, it varies with one integer parameter $k \geq 1$ as well as with two of four parameters a_2, b_2, a_3, b_3. Thus all of its sub-cases are different dynamically. According to a study of Neta et al. [31], the dynamics of members associated with Case 1 of a polynomial-type weight function for $k \in \{1, m-1\}$ did not give good results. Thus for this reason, it is reasonable in this study to exclude Case 1 for further investigation of its dynamics.

We limit ourselves to paying a special attention to Case 2 of a rational-type weight function in order to explore further properties of extraneous fixed points and relevant dynamics.

By closely following the works of Chun et al. [31,10,12] and Neta et al. [31,32,35], we construct $H_f(x_n) = Q_f(s)$ in (5.2) as follows:

$$H_f(x_n) = \frac{m + a_2(s-\rho) + a_3(s-\rho)^2}{1 + b_2(s-\rho) + b_3(s-\rho)^2}, \tag{5.3}$$

where s is given by (3.8) and parameters a_2, b_2, a_3, b_3 are described in (4.3). We now apply a polynomial $f(z) = (z^2-1)^m$ to $H_f(x_n)$ and construct a weight function $H(z)$, with a change of a variable $\zeta = z^2$, in the form of

$$H(z) = \frac{F(\zeta)}{D(\zeta)}, \tag{5.4}$$

where $F(\zeta) = m + a_2(\omega - \rho) + a_3(\omega-\rho)^2$, $D(\zeta) = 1 + b_2(\omega-\rho) + b_3(\omega-\rho)^2$, $\omega = \left[\Delta(m+1+\zeta^{-1})\{(m+1)^2 - \zeta^{-1}\}^{m-1}\right]^{\frac{1}{k}}$ and $\Delta = (m+2)^{1-2m}$.

It is interesting to investigate the complex dynamics of the iterative map R_p, as constructed with the above $H(z)$, of the form

$$z_{n+1} = R_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)} H(z_n), \tag{5.5}$$

in connection with the basins of attraction for a variety of polynomials $p(z_n)$ and a weight function $H(z_n)$. Indeed, $R_p(z)$ represents the classical Newton's method with a weight function $H(z)$ and may possess its fixed points as zeros of $p(z)$ or extraneous fixed points associated with $H(z)$. As a result, basins of attraction for the fixed points or the extraneous fixed points as well as their periodic orbits may make an impact on the complicated and chaotic complex dynamics whose visual description for various polynomials will be shown in the latter part of Section 6.

We now inspect the zeros ζ of F in (5.4) more closely. The extraneous fixed points ξ of iterative map R_p (5.5) can be found from the zeros ζ of F via relation $\xi = \zeta^{1/2}$, provided that $D(\zeta) \neq 0$. Note that F is a finite sum of fractional powers in ζ. It must be emphasized that any general algebraic ways of zero-finding of $F(\zeta)$ seem to be infeasible. In order to find the zeros ζ of F, we need to solve for ζ simultaneously by eliminating ω from the following two relations:

$$\begin{cases} F(\zeta) = m + a_2(\omega - \rho) + a_3(\omega - \rho)^2 = 0, \\ T(\zeta) = \omega^k \zeta^m - \Delta(1 + (m+1)\zeta)\{(m+1)^2\zeta - 1\}^{m-1} = 0, \end{cases} \tag{5.6}$$

where T is introduced for convenience of notation. From the first equation $F(\zeta) = 0$, we can solve for $\omega = \omega(a_2, a_3, \rho)$ in terms of parameters a_2, a_3 and the constant $\rho = \rho(m, k)$. Substitution of ω into the second equation $T(\zeta) = 0$ yields the desired zeros $\zeta = \zeta(m, k)$ in terms of parameters m and k. Based on a close inspection of (5.6), it is worthwhile for us to describe some properties about zeros ζ of F in the following remark.

Remark 5.1. (1) The first equation of (5.6) immediately ensures that no zeros ζ of F, i.e., no extraneous fixed points ξ exist when $a_2 = a_3 = 0$, which occurs in **Case 2B** as shown in Table 2. In addition, $a_2 = a_3 = 0$ occurs when $\mu = m(2 + 2k - km^2) = 0$ for a pair of values $(m, k) = (2, 1)$. Hence, no extraneous fixed points ξ exist for Methods **Case 2A** and **Case2C** when a particular pair of values $(m, k) = (2, 1)$ is employed.

(2) In view of the relations $a_2 = \frac{\mu}{4\rho} + \frac{4\rho}{\tau}(-a_3 + b_3 m)$ and $b_2 = \frac{\delta}{2\rho} + \frac{4\rho}{m\tau}(-a_3 + b_3 m)$, we find $b_2 = \frac{a_2}{m} + \frac{\tau}{4\rho}$ and $b_3 = \frac{1}{m}(a_3 + \frac{(4a_2\rho - \mu)}{16\rho^2}\tau)$.

(3) The second equation T defines a polynomial of degree m in ζ. Hence we may apply known polynomial root-finding methods to the root-finding of F.

(4) The leading term of T is given by $\left[\omega^k - (\frac{m+1}{m+2})^{2m-1}\right]\zeta^m$.

(5) T contains at most m roots whose values are dependent on both m and k.

(6) The maximal number of extraneous fixed points ξ of iterative map (5.5) associated with $H(z)$ is $2m$, since $F(\zeta)$ contains at most m roots.

(7) In general, not all the roots ζ of T satisfy $H(\zeta) = 0$. Hence back substitution of ζ is required to check whether they satisfy $H(\zeta) = 0$. The desired roots ζ are then used to find the extraneous roots ξ by taking $\xi = \zeta^{\frac{1}{2}}$ counting all their analytic branches.

In view of Remark 5.1-(1), no extraneous fixed points ξ of the iterative map (5.5) exist for **Method 2B**. As a result, we are only interested in solving (5.6) for zeros ζ of F associated with **Method 2A** and **Method 2C**. From the first equation of (5.6), we solve for ω:

$$\omega = \begin{cases} \rho\left(1 - \dfrac{4m}{\mu}\right), & \text{for } \textbf{Method 2A}, \\ \rho\left(1 \pm 4i\sqrt{\dfrac{m}{\mu\tau}}\right), & \text{for } \textbf{Method 2C}. \end{cases} \tag{5.7}$$

Substituting this ω into $T(\zeta) = 0$, the second equation of (5.6), we can express $T(\zeta)$ explicitly by means of m, k, ζ. For given values of m and k, the zeros ζ can be found, among which suitable ones satisfying $H(\zeta) = 0$ give desired extraneous fixed points $\xi = \zeta^{1/2}$. We especially wish to explore the dynamical behavior of (5.5) with selected pairs of values of (m, k) for $k \in \{m - 1, m, m + 3\}$ and $m \in \{2, 3, 4, 5\}$. Tables 3–5 list $F(\zeta)$, extraneous fixed points ξ for such selected values of m and k. For **Method 2C** with $\omega = \rho\left(1 - 4i\sqrt{\frac{m}{\mu\tau}}\right)$, we find that fourteen purely imaginary extraneous fixed points $\pm 0.790826i$, $\pm 0.831866i$, $\pm 0.838400i$, $\pm 0.846759i$, $\pm 0.87271i$, $\pm 0.875406i$, $\pm 0.879442i$ are indeed attractive when the values of (m, k) are given by (3, 6), (4, 3), (4, 4), (4, 7), (5, 4), (5, 5), (5, 8), respectively. All other extraneous fixed points ξ associated with H in this exploration are found to be repulsive.

In the latter part of Section 6, we will illustrate the complex dynamics behind the basins of attraction for iterative maps (5.5) when applied to various polynomials. For convenience and later use, we denote 12 iterative map names for **Method 2A** with given pairs of values $(m, k) = (2, 1), (2, 2), \ldots, (5, 8)$ in Table 3 respectively by **GKN2Am2k1**, **GKN2Am2k2**, ..., **GKN2Am5k8**. Similar map notations are also employed for **Method 2B** and **Method 2C**.

We wish to compare the dynamical behavior of (5.5) with that of another complex iterative map associated with an existing method. The dynamics of Method (2.1) was studied by Neta et al. [31], which did not give good results. Thus for the sake of comparison, we are interested in the dynamics of Method (2.3) (to be abbreviated by **LCN6** later

Table 3
$F(\zeta)$ with **Method 2A** and extraneous fixed points ξ for $2 \le m \le 5$.

m	k	$F(\zeta)$	$\xi = \zeta^{1/2}$	No. of ξ
	1	m	$-^{\star}$	0
2	2	$\frac{3}{\sqrt{2}} - \frac{1}{8}\sqrt{(9-1/\zeta)(3+1/\zeta)}$	$\pm 0.191563 \pm 0.158752i$	4
	5	$3 \cdot 2^{-6/5} - 2^{-6/5}((9-1/\zeta)(3+1/\zeta))^{1/5}$	$\pm 0.202398 \pm 0.164549i$	4
	2	$\frac{4}{5} - 5^{-3/2}\sqrt{(-16+1/\zeta)^2(4+1/\zeta)}$	$\pm 0.318721 \pm 0.237713i, \pm 0.202474$	
3	3	$\frac{23}{19}(\frac{3}{5})^{2/3} - 5^{-5/3}((-16+1/\zeta)^2(4+1/\zeta))^{1/3}$	$\pm 0.319208 \pm 0.237756i, \pm 0.202507$	6
	6	$\frac{11}{10}(\frac{3}{5})^{1/3} - 5^{-5/6}((-16+1/\zeta)^2(4+1/\zeta))^{1/6}$	$\pm 0.319497 \pm 0.237782i, \pm 0.202527$	
	3	$\frac{11}{15} - 6^{-7/3}((25-1/\zeta)^3(5+1/\zeta))^{1/3}$	$\pm 0.166518 \pm 0.0278804i, \pm 0.364110 \pm 0.250396i$	
4	4	$\frac{29}{27}(\frac{2}{3})^{3/4} - 6^{-7/4}((25-1/\zeta)^3(5+1/\zeta))^{1/4}$	$\pm 0.166519 \pm 0.0278800i, \pm 0.364152 \pm 0.250395i$	8
	7	$\frac{25}{24}(\frac{2}{3})^{3/7} - \frac{1}{6}((25-1/\zeta)^3(5+1/\zeta))^{1/7}$	$\pm 0.166520 \pm 0.0278797i, \pm 0.364188 \pm 0.250394i$	
	4	$\frac{47}{63} - 7^{-9/4}((-36+1/\zeta)^4(6+1/\zeta))^{1/4}$	$\pm 0.144767 \pm 0.0387239i$, $\pm 0.389575 \pm 0.253811i$, ± 0.13234	
5	5	$\frac{117}{113}(\frac{5}{7})^{4/5} - 7^{-9/5}((-36+1/\zeta)^4(6+1/\zeta))^{1/5}$	$\pm 0.144768 \pm 0.0387239i$, $\pm 0.389583 \pm 0.253811i$, ± 0.13234	10
	8	$\frac{93}{91}\sqrt{\frac{5}{7}} - 7^{-9/8}((-36+1/\zeta)^4(6+1/\zeta))^{1/8}$	$\pm 0.144768 \pm 0.0387238i$, $\pm 0.389590 \pm 0.253810i$, ± 0.132341	

$-^{\star}$: no suitable value.

Table 4
$F(\zeta)$ with **Method 2C** when $\omega = \rho\big(1 - 4i\sqrt{\frac{m}{\mu\tau}}\big)$ and extraneous fixed points ξ for $2 \le m \le 5$.

m	k	$F(\zeta)$	$\xi = \zeta^{1/2}$	No. of ξ
	1	m	$-$	0
2	2	$-\frac{1}{2} + \frac{1}{\sqrt{2}} - \frac{1}{8}\sqrt{(9-1/\zeta)(3+1/\zeta)}$	$\pm 0.337740, \pm 0.601200i$	4
	5	$2^{-1/5}(1 - \frac{\sqrt{5}}{10} - \frac{1}{2}((9-1/\zeta)(3+1/\zeta))^{1/5})$	$\pm 0.349353, \pm 0.675194i$	4
	2	$\frac{1}{25}(15-\sqrt{10}) - \frac{1}{25\sqrt{5}}\sqrt{(-16+\frac{1}{\zeta})^2(4+\frac{1}{\zeta})}$	$\pm 0.748517i, \pm 0.216849, \pm 0.342619$	
3	3	$(\frac{3}{5})^{\frac{2}{3}} - \frac{4 \cdot 5^{-7/6}}{3^{1/3}\sqrt{19}} - 5^{-\frac{5}{3}}((-16+\frac{1}{\zeta})^2(4+\frac{1}{\zeta}))^{\frac{1}{3}}$	$\pm 0.769926i, \pm 0.216456, \pm 0.346197$	6
	6	$\frac{14}{3^{2/3}5^{4/3}} - 5^{-5/6}(-16+\frac{1}{\zeta})^2(4+\frac{1}{\zeta})^{1/6}$	$\pm 0.790826i, \pm 0.216119, \pm 0.349447$	
	3	$\frac{2}{3} - \frac{1}{9\sqrt{5}} - 6^{-7/3}((25-1/\zeta)^3(5+1/\zeta))^{1/3}$	$\pm 0.831866i, \pm 0.171227 \pm 0.0257895i, \pm 0.360331$	
4	4	$\frac{17}{2^{1/4}3^{11/4}} - 6^{-7/4}((25-1/\zeta)^3(5+1/\zeta))^{1/4}$	$\pm 0.838400i, \pm 0.171195 \pm 0.0258051i, \pm 0.361263$	8
	7	$\frac{84-\sqrt{7}}{42 \cdot 2^{4/7}3^{3/7}} - \frac{1}{6}((25-1/\zeta)^3(5+1/\zeta))^{\frac{1}{7}}$	$\pm 0.846759i, \pm 0.171155 \pm 0.0258243i, \pm 0.362428$	
	4	$\frac{5}{7} - \frac{\sqrt{2/7}}{21} - 7^{-\frac{9}{4}}((36-1/\zeta)^4(6+1/\zeta))^{1/4}$	$\pm 0.87271i$, $\pm 0.147045 \pm 0.0371392i$, ± 0.134114, ± 0.37286	
5	5	$(\frac{5}{7})^{\frac{4}{5}} - \frac{4 \cdot 7^{-\frac{13}{10}}}{5^{\frac{1}{5}}\sqrt{113}} - 7^{-\frac{9}{5}}((36-1/\zeta)^4(6+1/\zeta))^{\frac{1}{5}}$	$\pm 0.875406i$, $\pm 0.147039 \pm 0.0371435i$, ± 0.134109, ± 0.373212	10
	8	$\sqrt{\frac{5}{7}} - \frac{1}{7\sqrt{91}} - 7^{-\frac{9}{8}}((36-1/\zeta)^4(6+1/\zeta))^{\frac{1}{8}}$	$\pm 0.879442i$, $\pm 0.147030 \pm 0.0371499i$, ± 0.134102, ± 0.373735	

in Section 6) that contains a first-order rational weight function, rather than that of Method (2.2) with a second-order rational weight function. By repeating a similar analysis that we have done so far, iterative method (2.3) can be put in the form:

$$x_{n+1} = \mathscr{R}_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}\mathscr{H}_f(x_n), \tag{5.8}$$

where $\mathscr{H}_f(x_n) = \big(a_3 + \frac{1}{b_1+b_2 s}\big)$, $s = \frac{f'(y_n)}{f'(x_n)}$, $a_3 = -\frac{m(m-2)}{2}$, $b_1 = -\frac{1}{m}$, $b_2 = \frac{1}{md}$, $d = (\frac{m}{m+2})^m$.

Table 5
$F(\zeta)$ with **Method 2C** when $\omega = \rho\big(1 + 4i\sqrt{\frac{m}{\mu\tau}}\,\big)$ and extraneous fixed points ξ for $2 \le m \le 5$.

m	k	$F(\zeta)$	$\xi = \zeta^{1/2}$	No. of ξ
	1	m	–	0
2	2	$\frac{1}{2} + \frac{1}{\sqrt{2}} - \frac{1}{8}\sqrt{(9-1/\zeta)(3+1/\zeta)}$	$\pm 0.289943 \pm 0.196945i$	4
	5	$2^{-1/5}(1 + \frac{\sqrt{5}}{10} - \frac{1}{2}((9-1/\zeta)(3+1/\zeta))^{1/5})$	$\pm 0.298027 \pm 0.198635i$	4
	2	$\frac{1}{25}(15+\sqrt{10}) - \frac{1}{25\sqrt{5}}\sqrt{(-16+\frac{1}{\zeta})^2(4+\frac{1}{\zeta})}$	$\pm 0.371481 \pm 0.238198i, \pm 0.205350$	
3	3	$(\frac{3}{5})^{\frac{2}{3}} + \frac{4\cdot 5^{-7/6}}{3^{1/3}\sqrt{19}} - 5^{-\frac{5}{3}}((-16+\frac{1}{\zeta})^2(4+\frac{1}{\zeta}))^{\frac{1}{3}}$	$\pm 0.370779 \pm 0.238249i, \pm 0.205320$	6
	6	$\frac{16}{3^{2/3}5^{4/3}} - 5^{-5/6}(-16+\frac{1}{\zeta})^2(4+\frac{1}{\zeta})^{1/6}$	$\pm 0.369900 \pm 0.238310i, \pm 0.205282$	
	3	$\frac{2}{3} + \frac{1}{9\sqrt{5}} - 6^{-7/3}((25-1/\zeta)^3(5+1/\zeta))^{1/3}$	$\pm 0.167184 \pm 0.0276061i, \pm 0.400090 \pm 0.247313i$	
4	4	$\frac{19}{2^{1/4}3^{11/4}} - 6^{-7/4}((25-1/\zeta)^3(5+1/\zeta))^{1/4}$	$\pm 0.167178 \pm 0.0276085i, \pm 0.399730 \pm 0.247364i$	8
	7	$\frac{84+\sqrt{7}}{42\cdot 2^{4/7}3^{3/7}} - \frac{1}{6}((25-1/\zeta)^3(5+1/\zeta))^{\frac{1}{7}}$	$\pm 0.167170 \pm 0.0276117i, \pm 0.399248 \pm 0.247431i$	
	4	$\frac{5}{7} + \frac{\sqrt{2/7}}{21} - 7^{-\frac{9}{4}}((36-1/\zeta)^4(6+1/\zeta))^{1/4}$	$\pm 0.145017 \pm 0.0385588i, \quad \pm 0.417034 \pm 0.250374i, \pm 0.132531$	
5	5	$(\frac{5}{7})^{\frac{4}{5}} + \frac{4\cdot 7^{-\frac{13}{10}}}{5^{\frac{1}{5}}\sqrt{113}} - 7^{-\frac{9}{5}}((36-1/\zeta)^4(6+1/\zeta))^{\frac{1}{5}}$	$\pm 0.145015 \pm 0.0385597i, \quad \pm 0.416870 \pm 0.250402i, \pm 0.132530$	10
	8	$\sqrt{\frac{5}{7}} + \frac{1}{7\sqrt{91}} - 7^{-\frac{9}{8}}((36-1/\zeta)^4(6+1/\zeta))^{\frac{1}{8}}$	$\pm 0.145013 \pm 0.038561i, \quad \pm 0.416620 \pm 0.250443i, \pm 0.132529$	

Table 6
Extraneous fixed points ξ of $\mathscr{H}$ for $3 \le m \le 5$.

m	ξ	No. of ξ
3	$\pm 0.202285, \ \pm 0.316011 \pm 0.237459i$	6
4	$\pm 0.166502 \pm 0.0278872i, \ \pm 0.363341 \pm 0.25042i$	8
5	$\pm 0.132338, \ \pm 0.144764 \pm 0.0387259i, \ \pm 0.389278 \pm 0.253836i$	10

Like iterative map R_p (5.5), the complex iterative map $\mathscr{R}_p$ associated with $\mathscr{R}_f$ can be written as

$$z_{n+1} = \mathscr{R}_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)}\mathscr{H}(z_n). \tag{5.9}$$

In addition, the weight function $\mathscr{H}(z)$ associated with $\mathscr{H}_f(x_n)$ for $f(z) = (z^2-1)^m$ is found to be:

$$\mathscr{H}(z) = -\frac{m}{2} \cdot \frac{\mathscr{F}(z)}{-d(m+2)^{2m}z^{2m} + (m+2)(1+(m+1)z^2)(-1+(m+1)^2z^2)^{m-1}}, \tag{5.10}$$

provided that $z \notin \{0, \pm 1, \pm\frac{1}{m+1}\}$, where $\mathscr{F}(z) = -dm(m+2)^{2m}z^{2m} + (m^2-4)(1+(m+1)z^2)(-1+(1+m)^2z^2)^{m-1}$ and $d = (\frac{m}{m+2})^m$.
Observe that $\mathscr{F}(z)$ is a polynomial of degree $2m$. For $m = 2$, $\mathscr{H}(z)$ does not possess any extraneous fixed point. Otherwise, it possesses $2m$ extraneous fixed points. Typical extraneous fixed points of $\mathscr{H}(z)$ are shown for $3 \le m \le 5$ in Table 6. They are all found to be repulsive.

6. Numerical experiments and basins of attraction

In this section we first deal with computational characteristics of proposed method (3.1) for a variety of test functions in comparison with other existing methods. Later on in the latter part of this section, we will explore and display the complex dynamics behind the basins of attraction of iterative maps (5.5) along with concluding remarks.

A variety of numerical experiments have been carried out with Mathematica programming to confirm the developed theory. In these experiments, we have moderately adopted the minimum number of precision digits as 112, via

Mathematica command $\$MinPrecision = 112$, to achieve the specified accuracy. In case that α is not exact, it is replaced by a more accurate value which has larger number of significant digits than the assigned $\$MinPrecision = 112$.

Definition 6.1 (*Computational Convergence Order*)**.** Assume that theoretical asymptotic error constant $\eta = \lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^p}$ and convergence order $p \geq 1$ are known. Define $p_n = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$ as the computational convergence order. Note that $\lim_{n\to\infty} p_n = p$.

Remark 6.1. Note that p_n requires knowledge at two points x_n, x_{n-1}, while the usual COC (computational order of convergence) $\frac{\log(|x_n - x_{n-1}|/|x_{n-1}-x_{n-2}|)}{\log(|x_{n-1}-x_{n-2}|/|x_{n-2}-x_{n-3}|)}$ does require knowledge at four points $x_n, x_{n-1}, x_{n-2}, x_{n-3}$. Hence p_n can be handled with a less number of working precision digits than the usual COC whose number of working precision digits is at least p times as large as that of p_n.

Computed values of x_n are accurate up to $\$MinPrecision$ significant digits. If α has the same accuracy of $\$MinPrecision$ as that of x_n, then $e_n = x_n - \alpha$ would be nearly zero and hence computing $|e_{n+1}|/e_n^p|$ would unfavorably break down. To clearly observe the convergence behavior, we desire α to have more significant digits that are Φ digits higher than $MinPrecision. To supply such α, a set of following Mathematica commands are used:

$$
\begin{aligned}
&sol = FindRoot[f(x), \{x, x_0\}, PrecisionGoal \to \Phi + \$MinPrecision,\\
&\qquad WorkingPrecision \to 2 * \$MinPrecision];\\
&\qquad\qquad \alpha = sol[[1, 2]].
\end{aligned}
$$

In this experiment, we assign $\Phi = 16$. As a result, the numbers of significant digits of x_n and α are found to be 112 and 128, respectively. Nonetheless, the limited paper space allows us to list both of them only up to 15 significant digits. We set the error bound ϵ to $\frac{1}{2} \times 10^{-80}$ satisfying $|x_n - \alpha| < \epsilon$.

Iterative methods (3.1) with cases **1A**, **1B**, **1C**, **1D**, **1E** and **2A**, **2B**, **2C** were respectively identified by **W1A**, **W1B**, **W1C**, **W1D**, **W1E and W2A**, **W2B**, **W2C**, being W-prefixed. Methods **W1C**, **W1D**, **W2C3** have been successfully applied to the test functions $F_1 - F_3$ below:

$$
\begin{cases}
\mathbf{W1C}: F_1(x) = \left[x - \sqrt{3}x^3\cos\left(\frac{\pi x}{6}\right) + \frac{10}{x^2+1} - 4\right]^2 (x-3)^3, \quad m = 5,\ \alpha = 3\\
\mathbf{W1D}: F_2(x) = \left[6x - 2i - 3\pi + 3\cos\left(x - \frac{i}{3}\right)\log(x^4+1)\right]\cos\left(x - \frac{i}{3}\right),\\
\quad m = 2,\ \alpha = \frac{i}{3} + \frac{\pi}{2},\ i = \sqrt{-1}\\
\mathbf{W2C3}: F_3(x) = [x\sin^{-1}(x-1) + e^{x^2} - 4]^3, \quad m = 3,\ \alpha \approx 1.15736504704271,\\
\text{where } \log z (z \in \mathbb{C}) \text{ represents a principal analytic branch such that } -\pi < Im(\log z) \leq \pi.
\end{cases}
$$

As seen in Table 8, they clearly confirmed quartic-order convergence. The values of computational asymptotic error constant agree up to 10 significant digits with η. As expected, the computational convergence order well approaches 4.

Additional test functions in Table 7 are used to display the convergence behavior of proposed scheme (3.1).

In Table 9, we compare numerical errors $|x_n - \alpha|$ of proposed methods **W1A**, **W2A1**, **W2B2**, **W2C3** with those of existing optimal fourth-order multiple-zero finders. Abbreviations **Liu**, **Sol**, **LCN6** denote existing optimal fourth-order multiple-zero finders obtained by Liu et al. (2.1) with $G_f(w) = w + \frac{2m}{m-1}w^2$, Soleymani et al. (2.2), Li et al. (2.3), respectively.

The least errors within the prescribed error bound are highlighted in bold face. Most of existing methods other than Method **Liu** exhibit similar performance. After two iterations, in view of strict comparison, Method **W2C3** shows slightly better convergence for f_2, f_3, f_6, f_7 and **W2B2** for f_1, f_5, while method **Sol** for f_4. It is worth to observe that methods **Sol**, **W2B2** and **LCN6** display the same results for f_6 due to Remark 4.1(a). It seems that Method **Liu** is less accurate for the current set of test functions employed in this experiment. It might be worthwhile to note that $\gamma = m$ is used in Method **Liu**, while $\gamma = \frac{2m}{m+2}$ in other remaining methods. By inspecting the asymptotic error constant $\eta(\theta_i, m, Q_f) = \frac{|x_{n+1}-\alpha|}{|x_n - \alpha|^p}$ when p is known, we find that the local convergence is dependent on the function $f(x)$, an initial value x_0, the multiplicity m, the zero itself and the weight function Q_f. Accordingly, for a given set of test functions, one method is hardly expected to always show better performance than the others.

Table 7
Additional test functions $f_i(x)$, zeros α and initial guesses x_0.

i	$f_i(x)$	α	m	x_0
1	$x^3[x^4 + \log(x+1)]$	0	4	0.11
2	$(4 + \cos x - \cos 3x - 4x)^2(1 - x + \cos x - \cos^3 x)$	$\approx$1.27016002579975	3	1.35
3	$[\sin(x^2 - 4) - x\log(x^2 + \pi - 3)]^3(x^2 - 4 + \pi)$	$\sqrt{4-\pi}$	4	1.0
4	$(x^2 + e^{-x^2} + x\sin x - 2)^6$	$\approx$0.916952932621001	6	1.0
5	$[\cos(\frac{2\pi}{x}) + x^3 + 9]^5$	-2	5	-1.9
6	$[(x-3)^2 + \frac{2}{25} + \frac{1}{x^3}\log\{(x-3)^2 + +\frac{27}{25}\}][(x-3)^2 + \frac{2}{25}]$	$3 - \frac{i\sqrt{2}}{5}$	2	$2.97 - 0.33i$
7	$(\frac{x^2}{1-\log x} - \frac{1}{\sqrt{x}})^7$	1	7	0.9

Here $\log z$ ($z \in \mathbb{C}$) represents a principal analytic branch with $-\pi \le \mathrm{Im}(\log z) < \pi$.

Table 8
Convergence for test functions $F_1(x) - F_3(x)$ with methods **W1C**, **W1D**, **W2C3**.

MT	F	n	x_n	$\|F(x_n)\|$	$\|x_n - \alpha\|$	$\|e_n/e_{n-1}^4\|$	η	p_n
		0	2.7	0.795179	0.3			
		1	3.00412039385887	9.073×10^{-10}	4.120×10^{-3}	0.5086905999	0.2152569195	3.28569
W1C	F_1	2	3.00000000006141	5.408×10^{-49}	6.140×10^{-11}	0.2130459405		4.00188
		3	3.00000000000000	1.664×10^{-205}	3.061×10^{-42}	0.2152569195		4.00000
		4	3.00000000000000	0.0×10^{-557}	0.0×10^{-111}			
		0	$\binom{1.5}{0.3}*$	0.0127611	0.0782511			
		1	$\binom{1.57044483302481}{0.333832082157843}$	8.226×10^{-7}	6.101×10^{-4}	16.27362171	12.18343574	3.88639
W1D	F_2	2	$\binom{1.57079632679357}{0.333333333332306}$	6.2172×10^{-24}	1.678×10^{-12}	12.11228598		4.00079
		3	$\binom{1.57079632679490}{0.333333333333333}$	2.066×10^{-92}	9.678×10^{-47}	12.18343574		4.00000
		4	$\binom{1.57079632679490}{0.333333333333333}$	0.0×10^{-222}	0.0×10^{-111}			
		0	1.2	0.0988196	0.0426350			
		1	1.15737235256885	4.096×10^{-13}	7.305×10^{-6}	2.210997983	2.563142231	4.04684
W2C3	F_3	2	1.15736504704271	4.087×10^{-58}	7.300×10^{-21}	2.563075141		4.00000
		3	1.15736504704271	4.056×10^{-238}	7.281×10^{-81}	2.563142231		4.00000
		4	1.15736504704271	0.0×10^{-334}	0.0×10^{-111}			

MT = method, $*\binom{1.5}{0.3} = 1.5 + 0.3i$.

We introduce the efficiency index [25] defined by $EI = p^{1/d}$ where p is the order of convergence and d is the number of distinct functional or derivative evaluations per iteration. The proposed methods (3.1) as well as all other listed methods have same EIs of $4^{1/3} \approx 1.587$ being optimal in the sense of Kung–Traub's conjecture. This paper constructs optimal quartic-order multiple-zero finders with a weight function dependent upon the principal kth root of $\frac{f'(y_n)}{f'(x_n)}$ and derives their relevant error equations. It is worth to note that existing methods (2.1)–(2.3) are special cases of our proposed methods as stated in Remark 3.1.

It is very important for us to discuss initial values influencing the convergence behavior of iterative methods. We find that iterative map (5.5) as Newton's method with a weight function $H(z)$ requires good initial values close to zero α. It is, however, a difficult task to determine how close the initial values are to zero α, since initial values are generally dependent upon computational precision, error bound and the given function $f(x)$ under consideration. Among various ways of selecting stable initial values, visual illustration of basins of attraction is a greatly effective one. Since the area of convergence can be seen on the basins of attraction, it would be reasonable to say that larger area of convergence indicates a better method. Clearly a quantitative analysis is necessary for measuring the size of area of convergence. To this end, we provide Table 10 featuring a statistical data describing the average number of iterations

Table 9
Comparison of $|x_n - \alpha|$ for various multiple-zero finders.

$f, x_0; m$	$\|x_n - \alpha\|$	**Liu**	**Sol**	**LCN6**	**W1A**	**W2A1**	**W2B2**	**W2C3**
$f_1, 0.11; 4$	$\|x_1 - \alpha\|$	2.26e−3	2.18e−5	2.18e−5	2.17e−5	2.18e−5	2.17e−5	2.18e−5
	$\|x_2 - \alpha\|$	2.03e−13	2.49e−20	2.46e−20	2.43e−20	2.46e−20	2.42e−20	2.49e−20
	$\|x_3 - \alpha\|$	8.95e−27	4.21e−80	3.98e−80	3.82e−80	3.98e−80	**3.72e−80**	4.20e−80
$f_2, 1.35; 3$	$\|x_1 - \alpha\|$	4.64e−5	2.13e−5	2.20e−5	2.31e−5	2.20e−5	2.26e−5	2.12e−5
	$\|x_2 - \alpha\|$	1.00e−17	1.58e−19	1.89e−19	2.43e−19	1.88e−19	2.14e−19	1.53e−19
	$\|x_3 - \alpha\|$	2.19e−68	4.74e−76	1.02e−75	2.99e−75	1.01e−75	1.74e−75	**4.23e−76**
$f_3, 1.0; 4$	$\|x_1 - \alpha\|$	3.74e−5	1.98e−4	2.00e−5	2.07e−5	2.00e−5	2.12e−5	1.91e−5
	$\|x_2 - \alpha\|$	4.22e−18	3.39e−15	1.61e−19	1.93e−19	1.61e−19	2.16e−19	1.25e−19
	$\|x_3 - \alpha\|$	6.85e−70	2.90e−58	6.78e−76	1.43e−75	6.72e−76	2.33e−75	**2.36e−76**
$f_4, 1.0; 6$	$\|x_1 - \alpha\|$	6.09e−5	3.08e−5	3.29e−5	3.37e−5	3.29e−5	3.76e−5	3.09e−5
	$\|x_2 - \alpha\|$	2.70e−17	7.99e−19	1.11e−18	1.25e−18	1.11e−18	2.16e−18	8.15e−19
	$\|x_3 - \alpha\|$	1.05e−66	**3.62e−73**	1.45e−72	2.43e−72	1.45e−72	2.34e−71	3.92e−73
$f_5, -1.9; 5$	$\|x_1 - \alpha\|$	8.57e−3	1.15e−5	1.02e−5	9.54e−6	1.02e−5	8.03e−6	1.14e−5
	$\|x_2 - \alpha\|$	2.15e−9	1.06e−21	5.01e−22	3.13e−22	5.02e−22	8.22e−23	1.03e−21
	$\|x_3 - \alpha\|$	8.67e−36	7.93e−86	2.86e−87	3.65e−88	2.90e−87	**9.01e−91**	6.81e−86
$f_6, 2.97,$	$\|x_1 - \alpha\|$	5.16e−4	6.95e−5	6.95e−5	9.44e−5	6.90e−5	6.95e−5	6.30e−5
$-0.33i; 2$	$\|x_2 - \alpha\|$	8.24e−12	2.58e−16	2.58e−16	1.20e−15	2.48e−16	2.58e−16	1.58e−16
	$\|x_3 - \alpha\|$	5.34e−43	4.96e−62	4.96e−62	3.23e−59	4.19e−62	4.96e−62	**6.39e−63**
$f_7, 0.9; 7$	$\|x_1 - \alpha\|$	2.24e−2	3.55e−6	8.11e−6	1.22e−5	8.09e−6	3.92e−5	2.78e−6
	$\|x_2 - \alpha\|$	9.79e−7	5.17e−23	2.03e−21	1.14e−20	2.00e−21	2.02e−18	1.98e−23
	$\|x_3 - \alpha\|$	3.74e−24	2.31e−90	7.94e−84	8.97e−81	7.62e−84	1.45e−71	**5.15e−92**

Here, 2.26e−3 denotes 2.26×10^{-3}.

Table 10
Average number of iterations per point for each example (1–6) and each of the 7 methods.

Example	2A $k = m - 1$	2A $k = m$	2A $k = m + 3$	2B $k = m - 1$	2B $k = m$	2B $k = m + 3$	2C $k = m - 1$	2C $k = m$	2C $k = m + 3$	LCN6
1 $m = 2$	4.42	3.41	3.40	4.42	4.42	4.36	4.42	3.36	3.35	4.42
2 $m = 3$	4.36	4.36	4.36	8.08	8.34	8.57	4.28	4.24	4.20	4.41
3 $m = 3$	5.96	5.97	5.99	13.05	–	–	6.63	6.42	6.12	5.93
4 $m = 4$	4.63	4.63	4.63	–	–	–	4.58	4.57	4.56	4.62
5 $m = 5$	3.74	3.74	3.74	–	–	–	3.69	3.69	3.69	3.80
6 $m = 5$	6.18	6.19	6.19	–	–	–	6.10	6.05	5.95	6.15
Average	4.88	4.72	4.72	–	–	–	4.95	4.72	4.64	4.89

per point. In the following 6 examples, we take a 6 by 6 square centered at the origin and containing all the zeros of the given functions. We assume that all zeros are of the same multiplicity m. We then take 360,000 equally spaced points in the square as initial points for the iterative methods. We color the point based on the root it converged to. This way we can find out if the method converged within the maximum number of iteration allowed and if it converged to the root closer to the initial point.

We now are ready to discuss the complex dynamics of iterative maps (5.5) and (5.9) applied to various polynomials. To continue our discussion, let us first identify three members of iterative maps associated with Case 2 for iterative map (5.5) by **GKN2A**, **GKN2B** and **GKN2C** respectively with a choice of parameter values in Table 2. In addition, we identify an iterative map (5.9) associated with **LCN6**.

Example 1. As a first example, we have taken a quadratic polynomial raised to the power of 2 with all real roots:

$$p_1(z) = (z^2 - 1)^2. \tag{6.1}$$

Clearly the roots are ± 1 with multiplicity 2. Basins of attraction for iterative maps **GKN2A**, **GKN2B** and **GKN2C** for $k = m - 1,\ m,\ m + 3$ and **LCN6** are illustrated in Fig. 1. Each basin is painted in a different color. At a root its

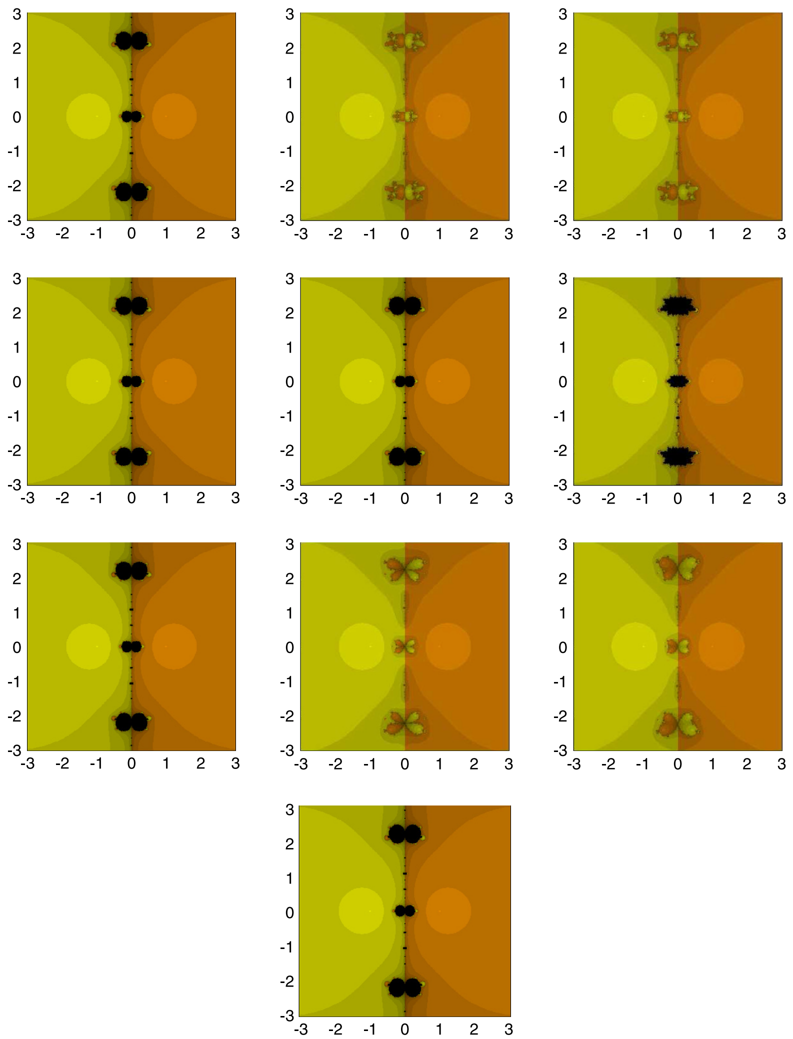

Fig. 1. The top left for 2A $k = m - 1$, top center for 2A $k = m$, and top right for 2A $k = m + 3$, second row left for 2B $k = m - 1$, second row center for 2B $k = m$, second row right for 2B $k = m + 3$, third row left for 2C $k = m - 1$, third row center for 2C $k = m$, third row right for 2C $k = m + 3$ and the bottom for LCN6 for the roots of the polynomial $(z^2 - 1)^2$.

color is white, while getting darker for more iterations required for convergence within the iteration limit. At black points, we recognize that the corresponding iterative maps did not converge within the iteration limit of 40 currently prescribed in this experiment. Based on the displayed results in Fig. 1, we find that **GKN2A** and **GKN2C** with $k = m$ and $k = m + 3$ are best, since there are no black points. If we look at the first row of Table 10, we find that these are the cases with the lowest number of iterations per point. On the other hand the case with $k = m + 3$ requires much more CPU time than the one with $k = m$. **LCN6** took less CPU time than **GKN2A** with $k = m + 3$, **GKN2B** and **GKN2C** for those values of k we tried (see Table 11).

Example 2. In our second example, we have taken a cubic polynomial raised to the power of 3:

$$p_2(z) = (z^3 + 4z^2 - 10)^3. \tag{6.2}$$

Basins of attraction for **GKN2A**, **GKN2B**, **GKN2C** and **LCN6** are illustrated in Fig. 2. We can see now that the only cases with black points are **GKN2B** with the 3 k values we tried. In terms of the average number of iterations

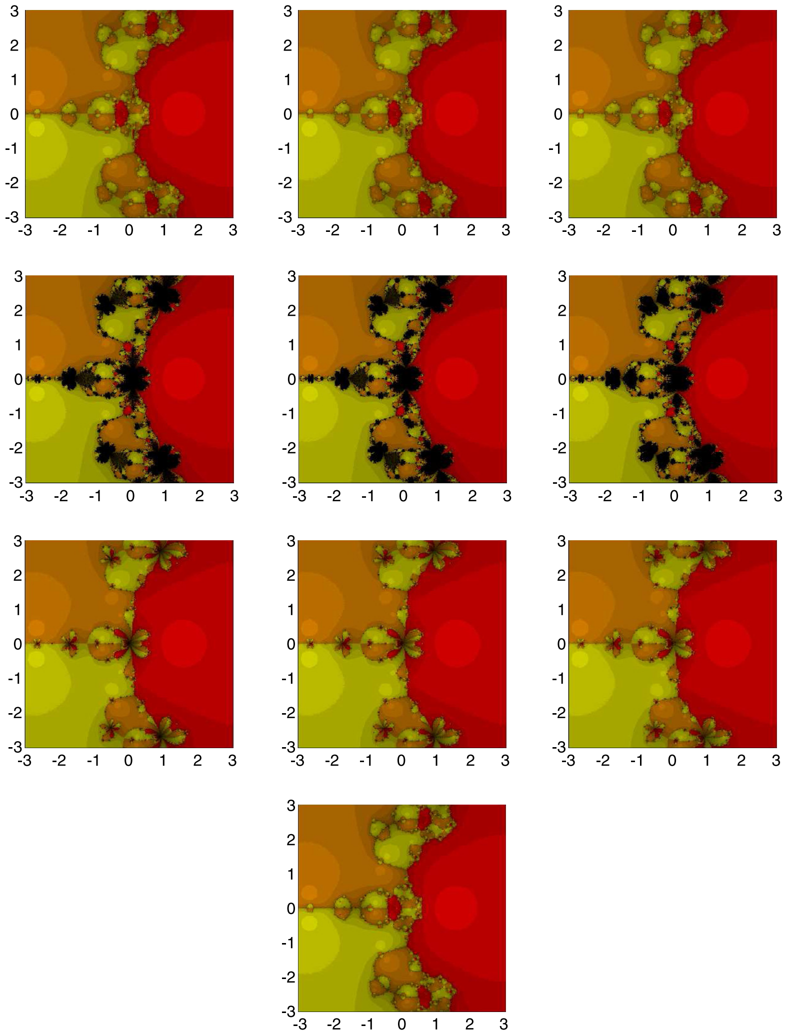

Fig. 2. The top left for 2A $k = m - 1$, top center for 2A $k = m$, and top right for 2A $k = m + 3$, second row left for 2B $k = m - 1$, second row center for 2B $k = m$, second row right for 2B $k = m + 3$, third row left for 2C $k = m - 1$, third row center for 2C $k = m$, third row right for 2C $k = m + 3$ and the bottom for LCN6 for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$.

Table 11
CPU time (in seconds) required for each example (1–6) and each of the 7 methods using a Dell Multiplex-990.

Example	2A $k = m - 1$	2A $k = m$	2A $k = m + 3$	2B $k = m - 1$	2B $k = m$	2B $k = m + 3$	2C $k = m - 1$	2C $k = m$	2C $k = m + 3$	LCN6
1 $m = 2$	186.19	170.19	779.11	291.45	469.54	1009.95	278.22	363.56	777.74	253.54
2 $m = 3$	642.06	1276.63	1270.01	1204.48	2521.55	2440.01	626.28	1233.04	1218.63	499.71
3 $m = 3$	884.43	1774.49	1774.23	1845.62	–	–	959.88	1837.56	1721.48	663.30
4 $m = 4$	1172.47	1289.78	1310.64	–	–	–	1145.54	1272.71	1265.27	458.11
5 $m = 5$	875.45	972.38	937.06	–	–	–	841.20	968.51	886.16	298.49
6 $m = 5$	1578.43	1824.22	1701.97	–	–	–	1516.67	1701.92	1612.95	658.02
Average	889.85	1217.95	1295.50	–	–	–	894.63	1229.88	1247.04	471.86

(see Table 10) **GKN2C** for any k requires slightly less than **GKN2A** and **LCN6** and much less than **GKN2B**. In terms of CPU time, **LCN6** is much faster than any of the other methods followed by **GKN2C** with $k = m - 1$ and **GKN2A** with $k = m - 1$. In the following examples we will not show **GKN2B** with $k = m,\ m + 3$.

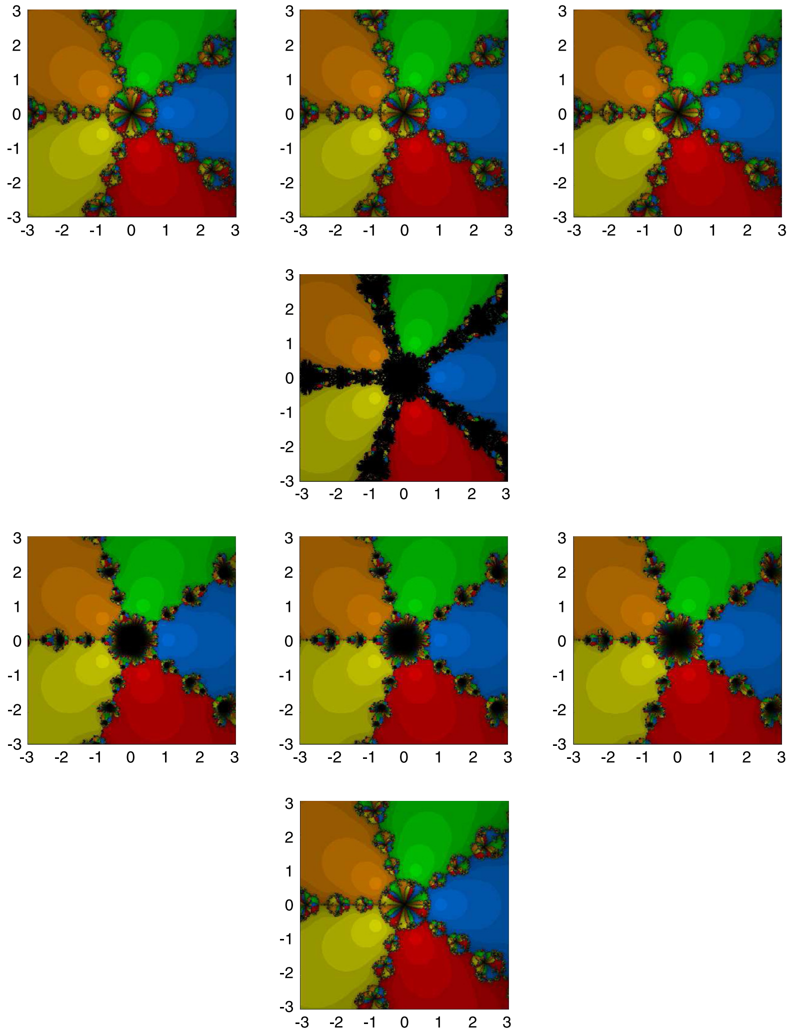

Fig. 3. The top left for 2A $k = m - 1$, top center for 2A $k = m$, and top right for 2A $k = m + 3$, second row for 2B $k = m - 1$, third row left for 2C $k = m - 1$, third row center for 2C $k = m$, third row right for 2C $k = m + 3$ and the bottom row for LCN6 for the roots of the polynomial $(z^5 - 1)^3$.

Example 3. As a third example, we have taken a quintic polynomial raised to the power of 3:

$$p_3(z) = (z^5 - 1)^3. \tag{6.3}$$

Basins of attraction for **GKN2A**, **GKN2B** (with $k = m - 1$), **GKN2C** and **LCN6** are illustrated in Fig. 3. **GKN2B** and **GKN2C** are the only ones having black points. All the other plots seem to be the same. If we examine the average number of iterations per point (see Table 10), we find that the range is 5.93–5.99 for **GKN2A** and **LCN6** whereas **GKN2C** requires slightly more 6.12–6.63. Again, no appreciable difference between **GKN2A** and **LCN6**. On the other hand, the CPU time for **LCN6** was the smallest (663 s) followed by **GKN2A** with $k = m - 1$ (884 s) and **GKN2C** with $k = m - 1$ (960 s). We will not show **GKN2B** for the rest of the examples since they did not perform as well as the others.

Example 4. As a fourth example, we have taken a different cubic polynomial raised to the power of 4:

$$p_4(z) = (z^3 - z)^4. \tag{6.4}$$

Now all the roots are real. Basins of attraction for **GKN2A**, **GKN2C** and **LCN6** are illustrated in Fig. 4. It is clear that the basins of attraction for all 7 methods are similar. The average number of iterations per point is also the

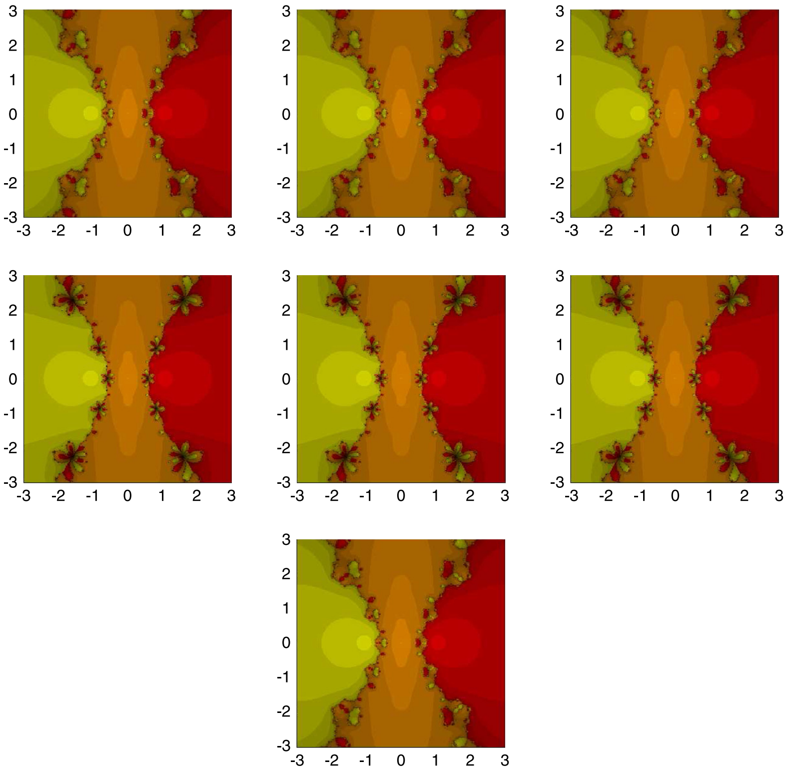

Fig. 4. The top left for 2A $k = m - 1$, top center for 2A $k = m$, and top right for 2A $k = m + 3$, second row left for 2C $k = m - 1$, second row center for 2C $k = m$, second row right for 2C $k = m + 3$ and the bottom for LCN6 for the roots of the polynomial $(z^3 - z)^4$.

same (4.56–4.63 iterations on average) but **LCN6** is much faster (458.11 s) followed in a distance by **GKN2C** with $k = m - 1$ (1145 s) and **GKN2A** with $k = m - 1$ (1172 s). We believe that the cost is associated with computing the kth root.

Example 5. As a fifth example, we have taken a quadratic polynomial raised to the power of 5:

$$p_5(z) = (z^2 - 1)^5. \tag{6.5}$$

Basins of attraction for **GKN2A**, **GKN2C** and **LCN6** are illustrated in Fig. 5. Again there is no visible difference between the plots. Consulting Table 10 we find that **GKN2A** requires 3.74 iteration per point (on average) vs. 3.80 for **LCN6**. The lowest is **GKN2C** with 3.69 iterations per point (on average). As before the difference is in the CPU time required. The algorithm **LCN6** requires about 298.49 s followed by **GKN2C** with $k = m - 1$ with 841.20 s and **GKN2A** with $k = m - 1$ with 875.45 s. The other 4 require more than 900 s. Notice that even though the roots for this polynomial are the same as those in example 1 (except for the multiplicity) we do **not** find black points in the basins for this example.

Example 6. As a last example, we have taken a quartic polynomial raised to the power of 5:

$$p_6(z) = (z^4 - 1)^5. \tag{6.6}$$

Basins of attraction for **GKN2A**, **GKN2C** and **LCN6** are illustrated in Fig. 6. There is no visible difference between the plots for **GKN2A** and **LCN6**. There are black points in **GKN2C** for the three k values we chose. The average number of iterations per point (see Table 10) is about the same for all cases. The CPU time for all new methods is about 2.5 times the CPU time used by **LCN6**.

To summarize, we have averaged the number of iterations per point and the CPU time for all examples. We can conclude that on average (on these 6 examples) **LCN6** is the fastest followed by **GKN2A** and **GKN2C** both with

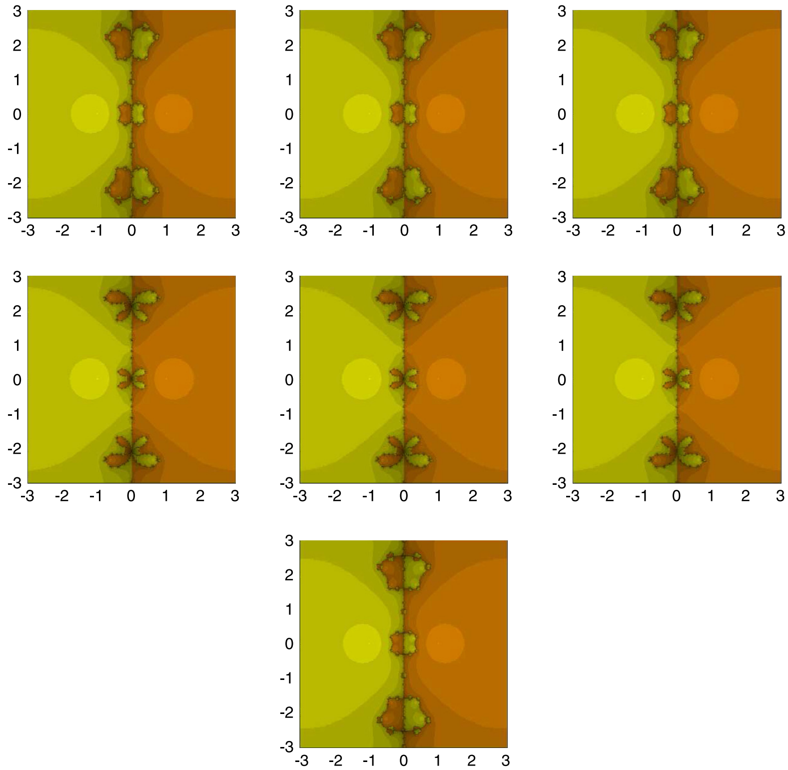

Fig. 5. The top left for 2A $k = m - 1$, top center for 2A $k = m$, and top right for 2A $k = m + 3$, second row left for 2C $k = m - 1$, second row center for 2C $k = m$, second row right for 2C $k = m + 3$ and the bottom for LCN6 for the roots of the polynomial $(z^2 - 1)^5$.

Table 12
Number of points requiring 40 iterations for each example (1–6) and each of the 7 methods.

Example	2A $k = m - 1$	2A $k = m$	2A $k = m + 3$	2B $k = m - 1$	2B $k = m$	2B $k = m + 3$	2C $k = m - 1$	2C $k = m$	2C $k = m + 3$	LCN6
1 $m = 2$	10 289	1	1	10 289	10 289	9 565	10 289	1	1	10 289
2 $m = 3$	1	1	1	33 380	37 324	42 807	7	1	1	91
3 $m = 3$	293	282	315	83 434	–	–	10 327	7453	3667	227
4 $m = 4$	0	0	0	–	–	–	0	0	0	0
5 $m = 5$	3	1	1	–	–	–	1	1	1	601
6 $m = 5$	281	281	225	–	–	–	3525	2705	1585	1217
Average	1811.17	94.33	90.5	–	–	–	4024.83	1693.5	875.83	2070.8

$k = m - 1$. In terms of the average number of iterations per point, all seven methods are about the same (4.6–4.9). Another piece of information we collected here in Table 12 is the number of points requiring 40 iterations for each example and each method. It turns out that **GKN2A** with $k = m + 3$ has the lowest number of points on average (90) followed by **GKN2A** with $k = m$ with 94 points. The highest was **GKN2C** with $k = m - 1$ and the second highest is **LCN6**. We have thus found a more robust method than **LCN6** which on average has the lowest number of points requiring 40 iterations, namely **GKN2A** with $k = m + 3$ and $k = m$. Unfortunately these are also the ones that use most CPU time.

By controlling parameters present in the structure of the weight function Q_f of our proposed methods (3.1), we have improved not only the computational errors but also the dynamical behavior affecting the structure of basins of attraction with the presence of various extraneous fixed points of iterative map (5.5). A technique of varying the extraneous fixed points is to adjust various parameters of the weight function of (5.5) when applied to a well-known polynomial $p(z) = (z^2 - 1)^m$ as employed by [31,35,46]. In our future work developing a new family of multiple-zero

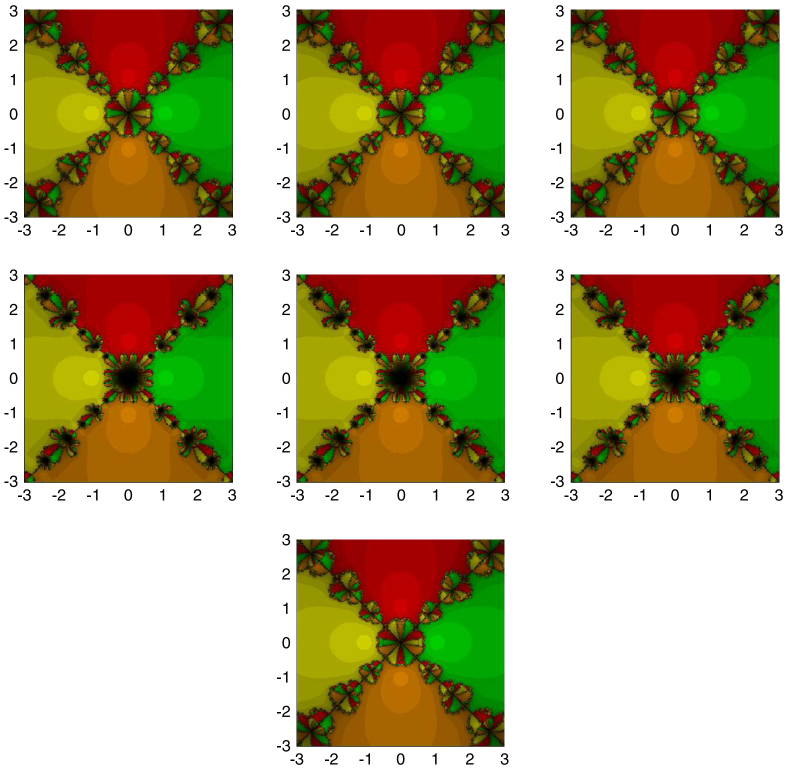

Fig. 6. The top left for 2A $k = m - 1$, top center for 2A $k = m$, and top right for 2A $k = m + 3$, second row left for 2C $k = m - 1$, second row center for 2C $k = m$, second row right for 2C $k = m + 3$ and the bottom for LCN6 for the roots of the polynomial $(z^4 - 1)^5$.

finders, our current approach based on the principal analytic branch of the kth root of a derivative-to-derivative ratio would play a crucial role in designing a higher-order family of multiple-zero finders as well as in enhancing relevant dynamics associated with basins of attraction when applied to a wide variety of complex polynomials by controlling free parameters present in a structure of the desired weight function.

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