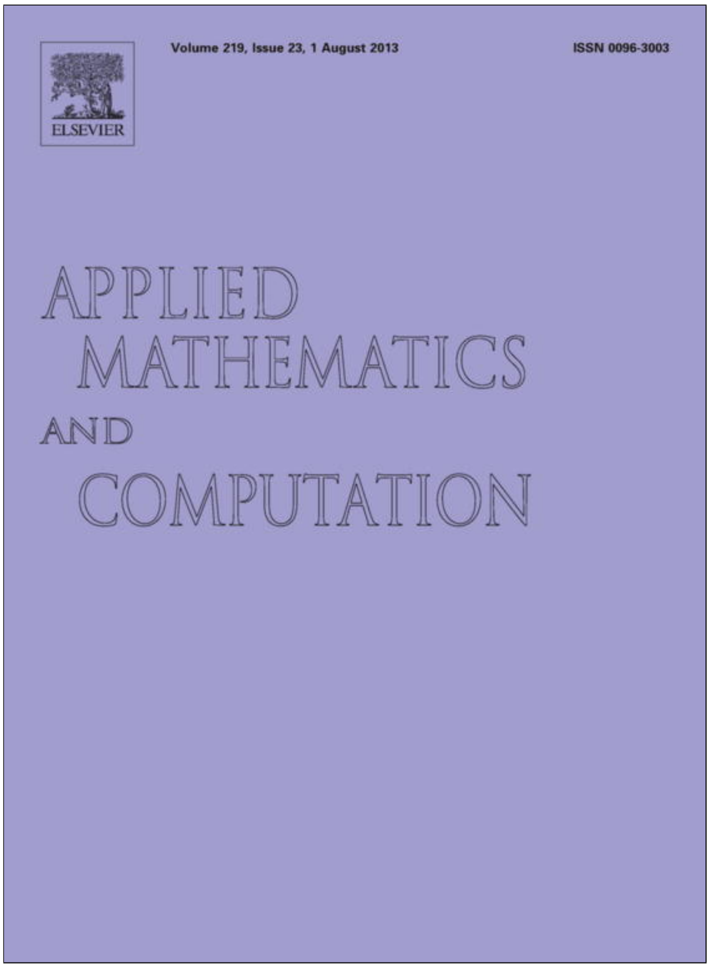
Volume 219, Issue 23, 1 August 2013
ISSN 0096-3003
ELSEVIER
APPLIED MATHEMATICS AND COMPUTATION

Applied Mathematics and Computation 219 (2013) 10987–11004

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On a family of Laguerre methods to find multiple roots of nonlinear equations

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ARTICLE INFO

Keywords:
Iterative methods
Order of convergence
Rational maps
Basin of attraction
Julia sets
Conjugacy classes

ABSTRACT

There are several methods for solving a nonlinear algebraic equation having roots of a given multiplicity m. Here we compare a family of Laguerre methods of order three as well as two others of the same order and show that Euler–Cauchy's method is best. We discuss the conjugacy maps and the effect of the extraneous roots on the basins of attraction.

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1. Introduction

There is a vast literature on the solution of nonlinear equations and nonlinear systems, see for example Ostrowski [1], Traub [2], Neta [3] and the recent book by Petković et al. [4] and references therein. Most of the algorithms are for finding a simple root of a nonlinear equation $f(x) = 0$, i.e., for a root α we have $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. In this paper we are interested in the case that α is a root of multiplicity $m > 1$. There are very few methods for multiple roots when the multiplicity is known. The first one is due to Schröder [5] and it is also referred to as modified Newton,

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} \tag{1}$$

The method is based on Newton's method for the function $G(x) = \sqrt[m]{f(x)}$ which obviously has a simple root at α, the multiple root with multiplicity m of $f(x)$.

Another method based on the same G is Laguerre's method

$$x_{n+1} = x_n - \frac{\lambda\frac{f(x_n)}{f'(x_n)}}{1 + sgn(\lambda - m)\sqrt{\left(\frac{\lambda-m}{m}\right)\left[(\lambda-1) - \lambda\frac{f(x_n)f''(x_n)}{f'(x_n)^2}\right]}} \tag{2}$$

where $\lambda\ (\neq 0, m)$ is a real parameter. When $f(x)$ is a polynomial of degree n, this method with $\lambda = n$ is the ordinary Laguerre method for multiple roots, see Bodewig [6]. This method converges cubically. Some special cases are:

- Euler–Cauchy for $\lambda = 2m$

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0096-3003/$ - see front matter Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.amc.2013.05.002

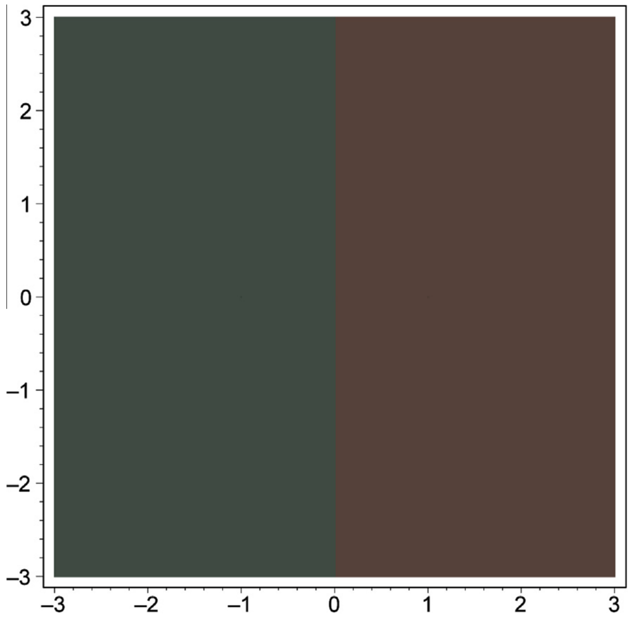

Fig. 1. Euler–Cauchy's method for the roots of the polynomial $(z^2-1)^2$.

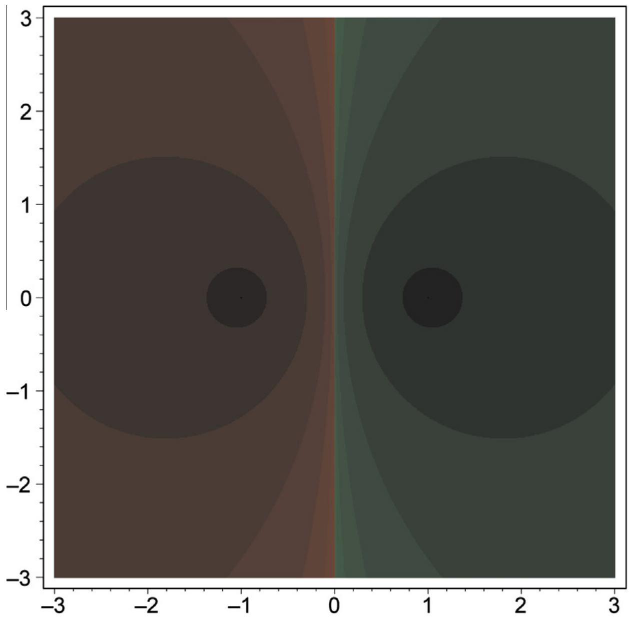

Fig. 2. Halley's method for the roots of the polynomial $(z^2-1)^2$.

$$x_{n+1} = x_n - \frac{2m\frac{f(x_n)}{f'(x_n)}}{1+\sqrt{(2m-1)-2m\frac{f(x_n)f''(x_n)}{f'(x_n)^2}}}. \tag{3}$$

- Halley for $\lambda \to 0$ after rationalization

$$x_{n+1} = x_n - \frac{\frac{f(x_n)}{f'(x_n)}}{\frac{m+1}{2m} - \frac{f(x_n)f''(x_n)}{2f'(x_n)^2}}. \tag{4}$$

- Ostrowski for $\lambda \to \infty$

$$x_{n+1} = x_n - \frac{\sqrt{m}\frac{f(x_n)}{f'(x_n)}}{\sqrt{1-\frac{f(x_n)f''(x_n)}{f'(x_n)^2}}}. \tag{5}$$

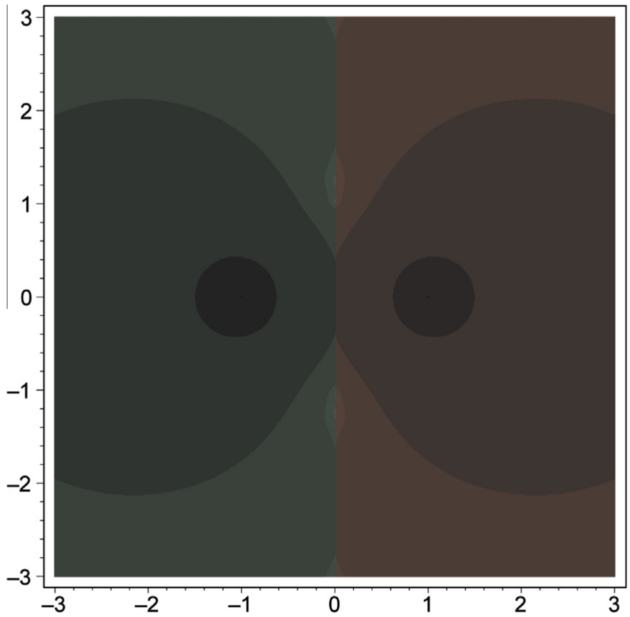

Fig. 3. Ostrowski's method for the roots of the polynomial $(z^2-1)^2$.

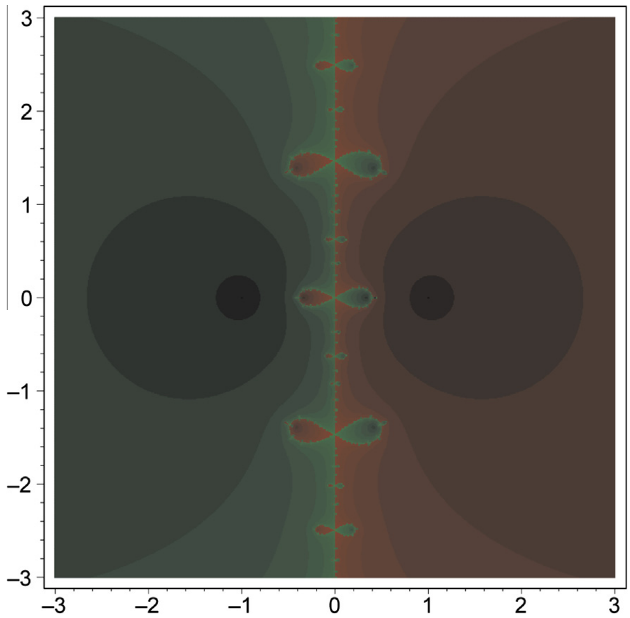

Fig. 4. Euler–Chebyshev's method for the roots of the polynomial $(z^2-1)^2$.

- Hansen–Patrick family [7] for $\lambda = m(1/\nu+1)$

$$x_{n+1} = x_n - \frac{m(\nu+1)\frac{f(x_n)}{f'(x_n)}}{\nu + \sqrt{(m(\nu+1)-\nu) - m(\nu+1)\frac{f(x_n)f''(x_n)}{f'(x_n)^2}}}. \tag{6}$$

Petković et al. [8] have shown the equivalence between Laguerre family (2) and Hansen–Patrick family (6). When $\lambda \to m$ the method becomes second order given by (1). Two other cubically convergent methods that sometimes mistaken as members of Laguerre's family are: Euler–Chebyshev [2] given by

$$x_{n+1} = x_n - \left(\frac{m(3-m)}{2} + \frac{m^2}{2}\frac{f(x_n)}{f'(x_n)}\frac{f''(x_n)}{f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{7}$$

and Osada's method [9] given by

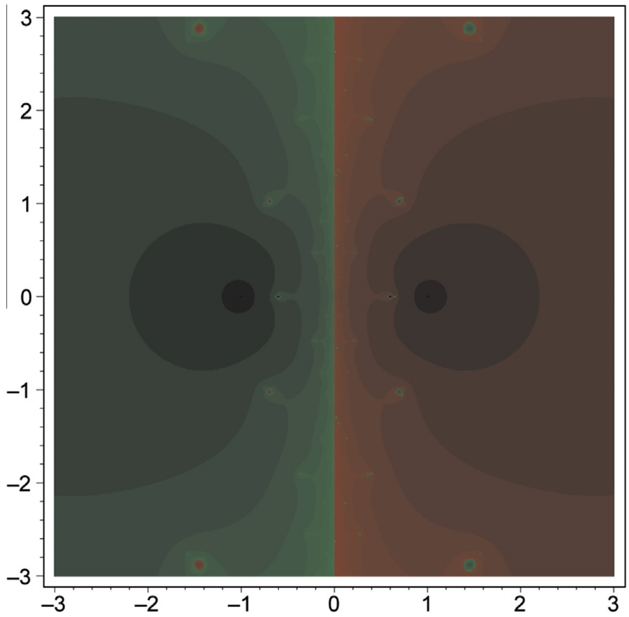

Fig. 5. Osada's method for the roots of the polynomial $(z^2-1)^2$.

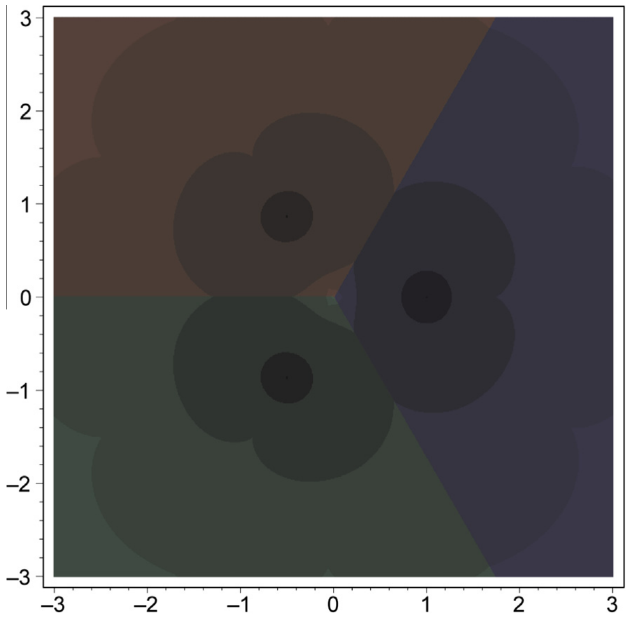

Fig. 6. Euler–Cauchy's method for the roots of the polynomial $(z^3-1)^2$.

$$x_{n+1}=x_n-\frac{1}{2}m(m+1)\frac{f(x_n)}{f'(x_n)}+\frac{(m-1)^2}{2}\frac{f'(x_n)}{f''(x_n)}. \qquad (8)$$

Other variations on Chebyshev's method are given by [10].

Osada [11] has shown that the error for Laguerre's method (2) is given by

$$e_{n+1}=K_3(m,\lambda)e_n^3+O(e_n^4), \qquad (9)$$

where the asymptotic error constant, $K_3(m,\lambda)$ is given by

$$K_3(m,\lambda)=A_1(m,\lambda)\left(\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}\right)^2-A_2(m)\frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}, \qquad (10)$$

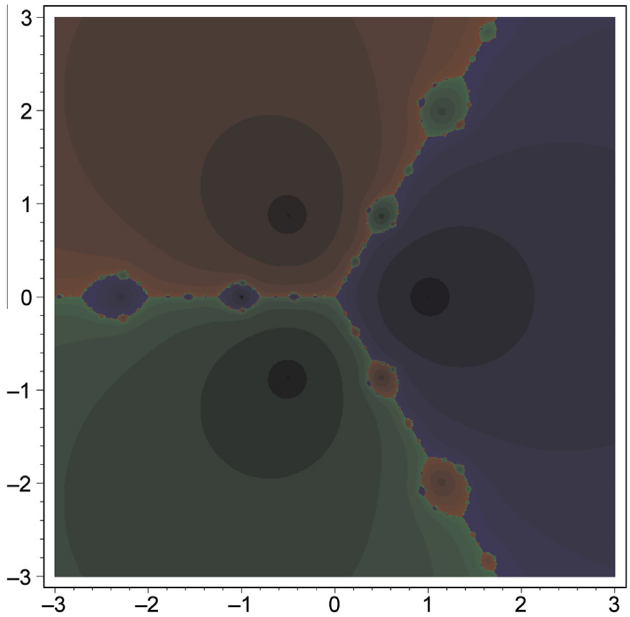

Fig. 7. Halley's method for the roots of the polynomial $(z^3 - 1)^2$.

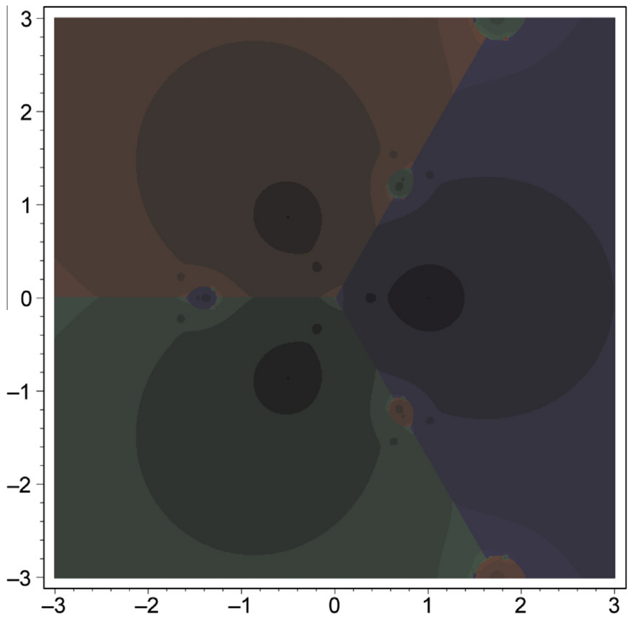

Fig. 8. Ostrowski's method for the roots of the polynomial $(z^3 - 1)^2$.

with

$$A_1(m,\lambda) = \frac{1}{2m(m+1)^2}\left(1 - \frac{1}{\lambda - m}\right),$$

$$A_2(m) = \frac{1}{m(m+1)(m+2)}.$$

For Euler–Cauchy, the asymptotic error constant is

$$K_3(m,2m) = \frac{m-1}{2m^2(m+1)^2}\left(\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}\right)^2 - A_2(m)\frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \qquad (11)$$

For Halley's method ($\lambda \to 0$) the asymptotic error constant is

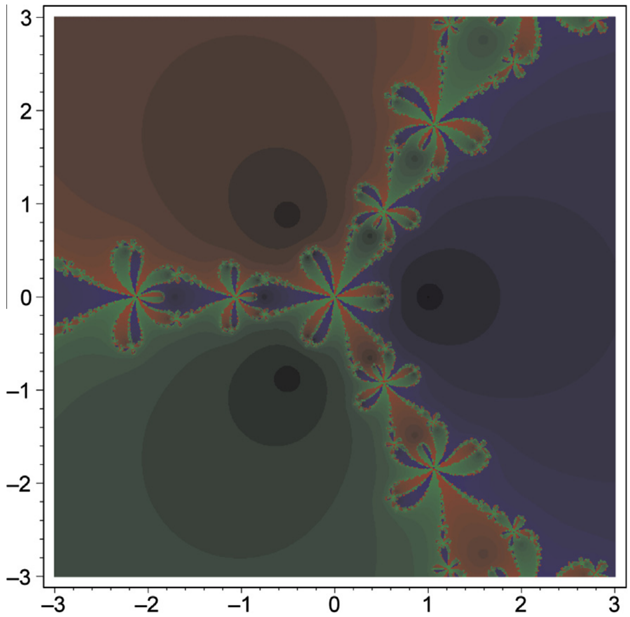

Fig. 9. Euler–Chebyshev's method for the roots of the polynomial $(z^3-1)^2$.

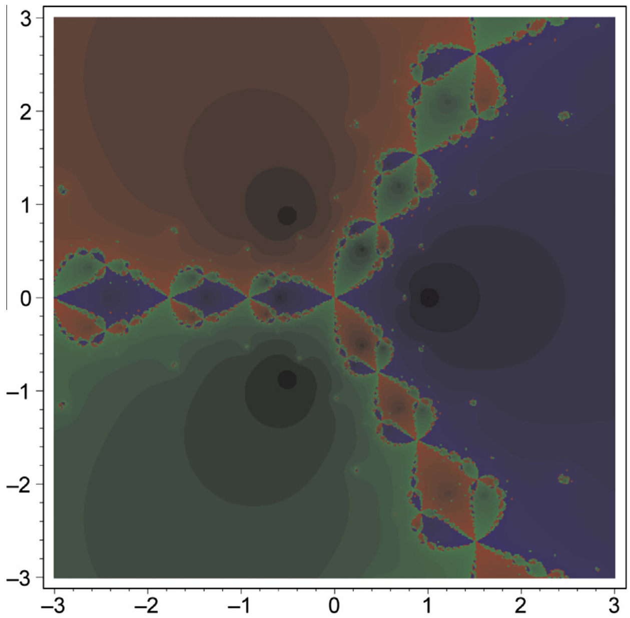

Fig. 10. Osada's method for the roots of the polynomial $(z^3-1)^2$.

$$K_3(m,\lambda\to 0)=\frac{1}{2m^2(m+1)}\left(\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}\right)^2-A_2(m)\frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \tag{12}$$

For Ostrowski's method ($\lambda\to\infty$) the asymptotic error constant is given by [1]

$$K_3(m,\lambda\to\infty)=\frac{1}{2m(m+1)^2}\left(\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}\right)^2-A_2(m)\frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \tag{13}$$

The asymptotic error constant for Euler–Chebyshev's method (see [2]) is

$$K_3=\frac{m+3}{2m^2(m+1)^2}\left(\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}\right)^2-\frac{1}{m(m+1)(m+2)}\frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \tag{14}$$

The asymptotic error constant for Osada's method [9] is

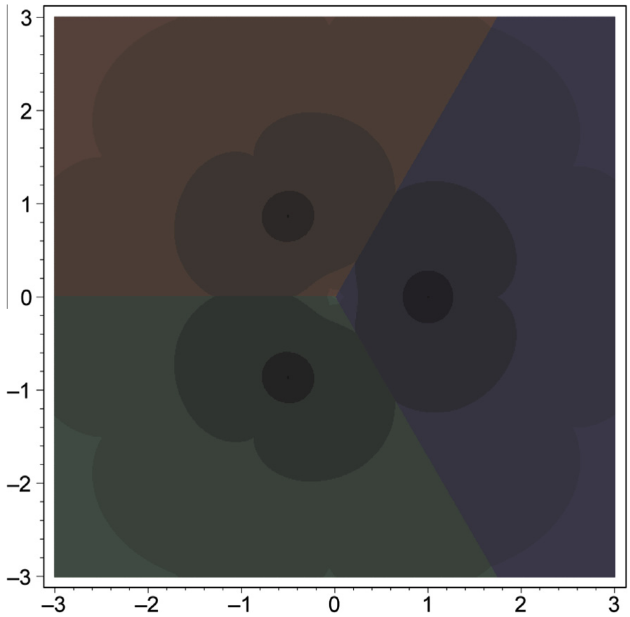

Fig. 11. Euler–Cauchy's method for the roots of the polynomial $(z^3 - 1)^4$.

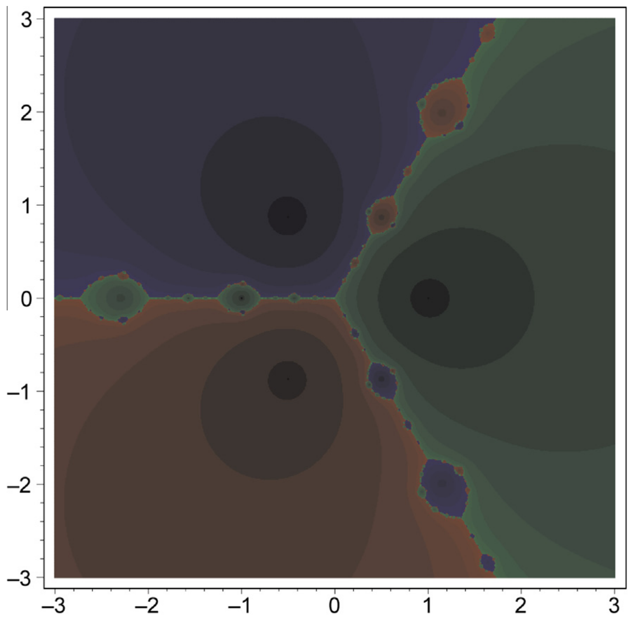

Fig. 12. Halley's method for the roots of the polynomial $(z^3 - 1)^4$.

$$K_3 = \frac{(m+1)^2}{2m^2(m-1)}\left(\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}\right)^2 - \frac{1}{m}\frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \tag{15}$$

If we define the efficiency index of a method of order p as

$$I = p^{1/d}, \tag{16}$$

where d is the number of function- (and derivative-) evaluation per step then all these methods have the same efficiency of $3^{1/3} = 1.442$. There is no indication which method is superior by looking at the error constants. In the next sections we will discuss basins of attraction and conjugacy maps for the polynomial $((z-a)(z-b))^m$ which is the generalization of a quadratic polynomial to the case of multiple roots.

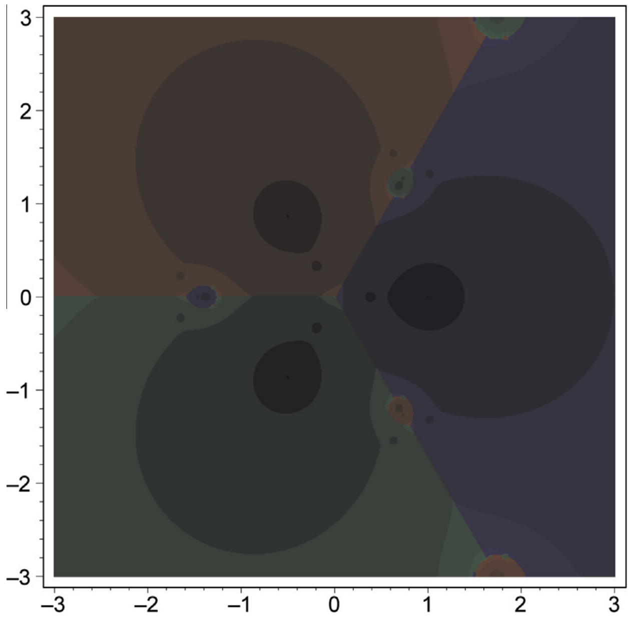

Fig. 13. Ostrowski's method for the roots of the polynomial $(z^3 - 1)^4$.

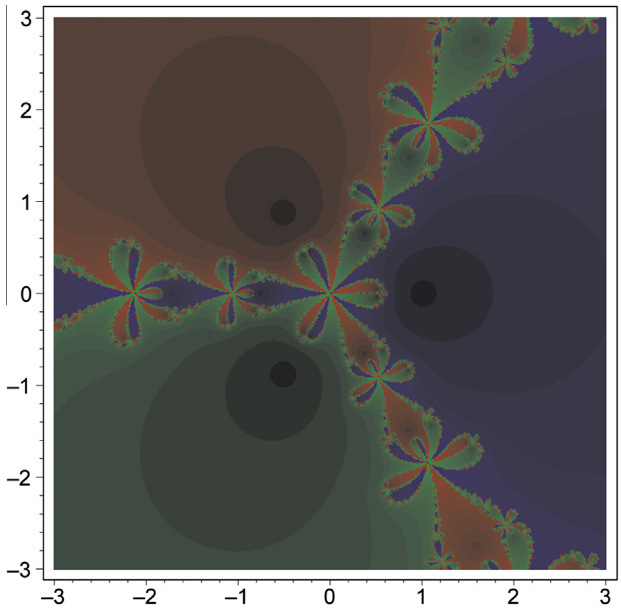

Fig. 14. Euler–Chebyshev's method for the roots of the polynomial $(z^3 - 1)^4$.

2. Corresponding conjugacy maps for quadratic polynomials

Given two maps f and g from the Riemann sphere into itself, an analytic conjugacy between the two maps is a diffemorphism h from the Riemann sphere onto itself such that $h \circ f = g \circ h$. Here we consider only quadratic polynomials raised to mth power.

Theorem 2.1 (Euler–Cauchy's method (3)). *For a rational map $R_p(z)$ arising from Euler–Cauchy's method applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b, R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = z(z-1)\left[1 + sgn(z^2 - 1)\right].$$

Proof. Let $p(z) = ((z-a)(z-b))^m$, $a \neq b$ and let M be the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \frac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

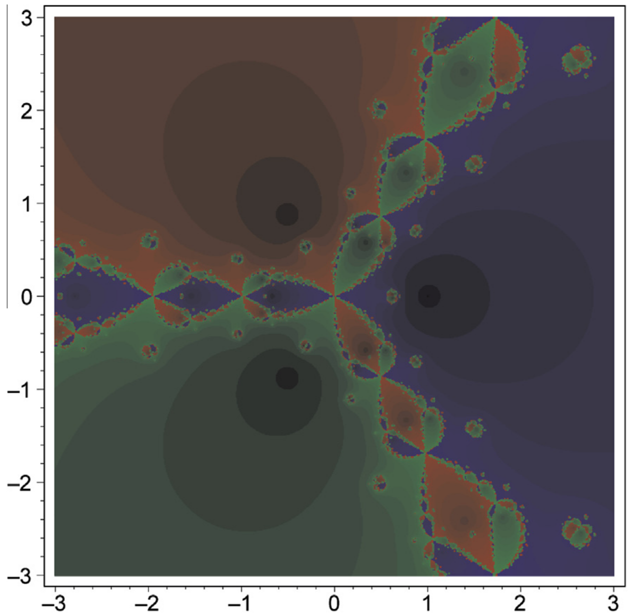

Fig. 15. Osada's method for the roots of the polynomial $(z^3-1)^4$.

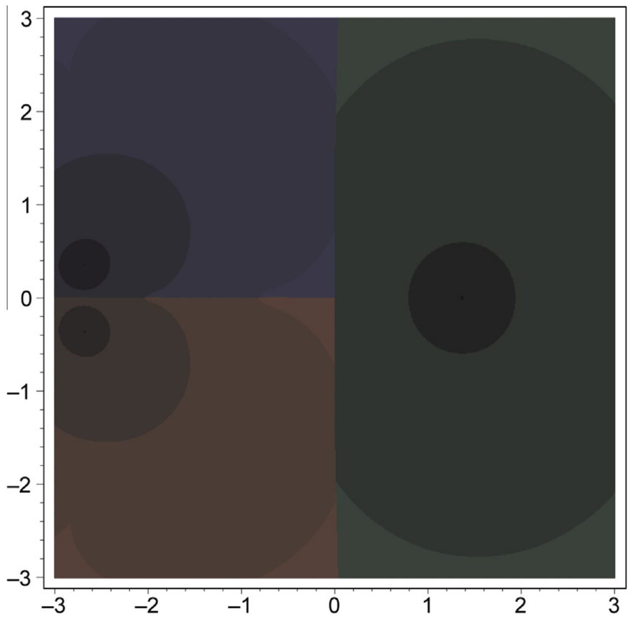

Fig. 16. Euler–Cauchy's method for the roots of the polynomial $(z^3+4z^2-10)^3$.

$$S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p\left(\frac{ub-a}{u-1}\right) = u(u-1)\left[1+sgn(u^2-1)\right]. \qquad \square$$

Theorem 2.2 (Halley's method (4)). *For a rational map $R_p(z)$ arising from Halley's method applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b, R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = z^3.$$

Proof. Let $p(z) = ((z-a)(z-b))^m$, $a \neq b$ and let M be the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \frac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p\left(\frac{ub-a}{u-1}\right) = u^3. \qquad \square$$

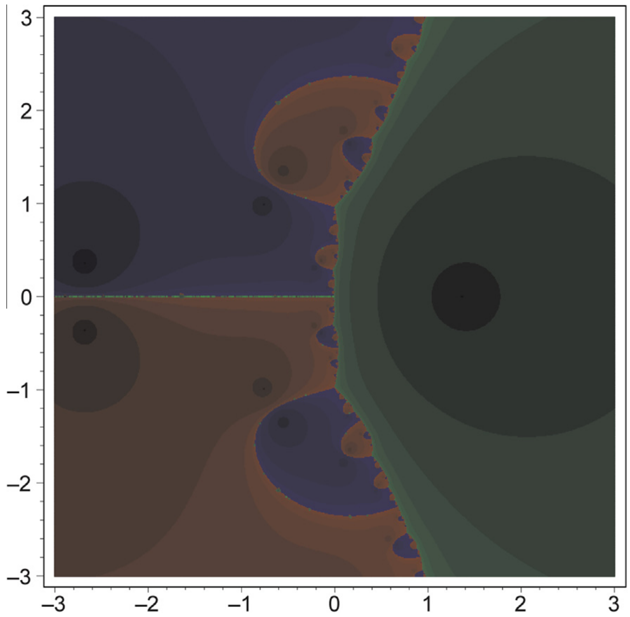

Fig. 17. Halley's method for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$.

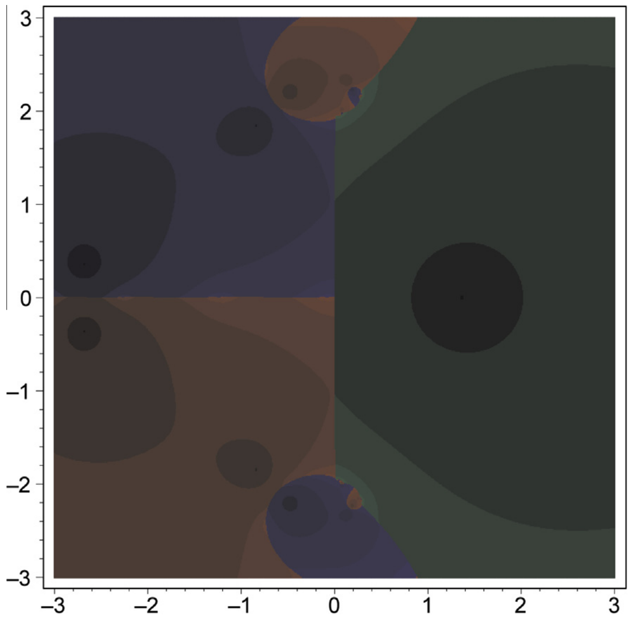

Fig. 18. Ostrowski's method for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$.

Theorem 2.3 (Ostrowski's method (5)). *For a rational map $R_p(z)$ arising from Ostrowski's method applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b, R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = z\left[sgn(z+1)\sqrt{z^2+1} - 1\right].$$

Proof. Let $p(z) = ((z-a)(z-b))^m$, $a \neq b$ and let M be the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \frac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p\left(\frac{ub-a}{u-1}\right) = u\left[sgn(u+1)\sqrt{u^2+1} - 1\right]. \qquad \square$$

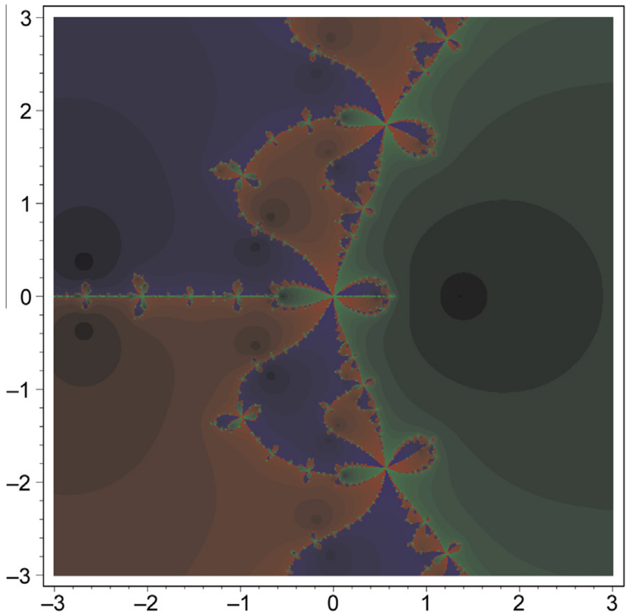

Fig. 19. Euler–Chebyshev's method for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$.

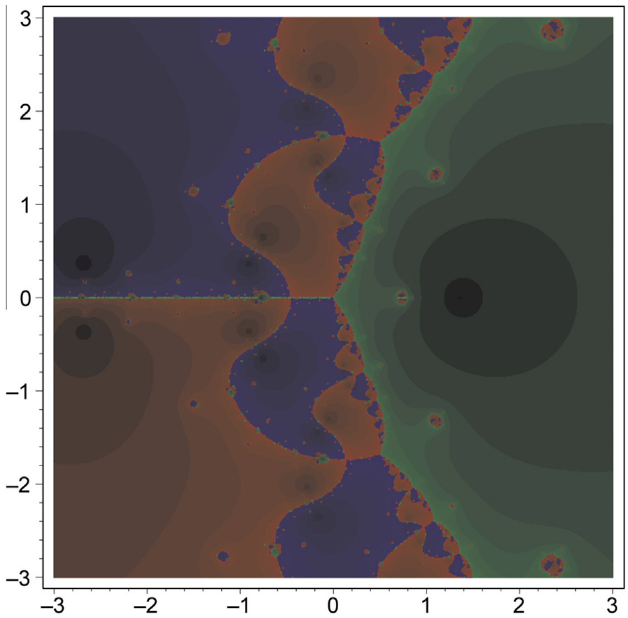

Fig. 20. Osada's method for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$.

Theorem 2.4 (Euler–Chebyshev's method (7)). *For a rational map $R_p(z)$ arising from Euler–Chebyshev's method applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b, R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = z^3(z+2).$$

Proof. Let $p(z) = ((z-a)(z-b))^m$, $a \neq b$ and let M be the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \frac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p\left(\frac{ub-a}{u-1}\right) = u^3(u+2). \qquad \square$$

Theorem 2.5 (Osada's method (8)). *For a rational map $R_p(z)$ arising from Osada's method applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b, R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

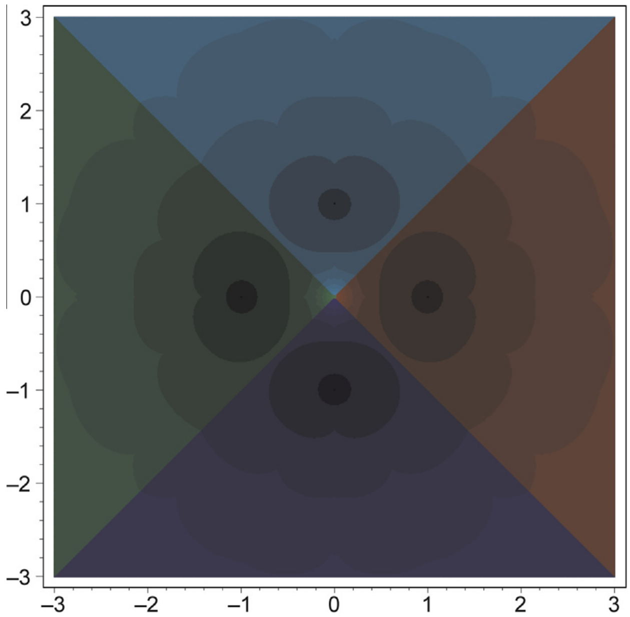

Fig. 21. Euler–Cauchy's method for the roots of the polynomial $(z^4 - 1)^5$.

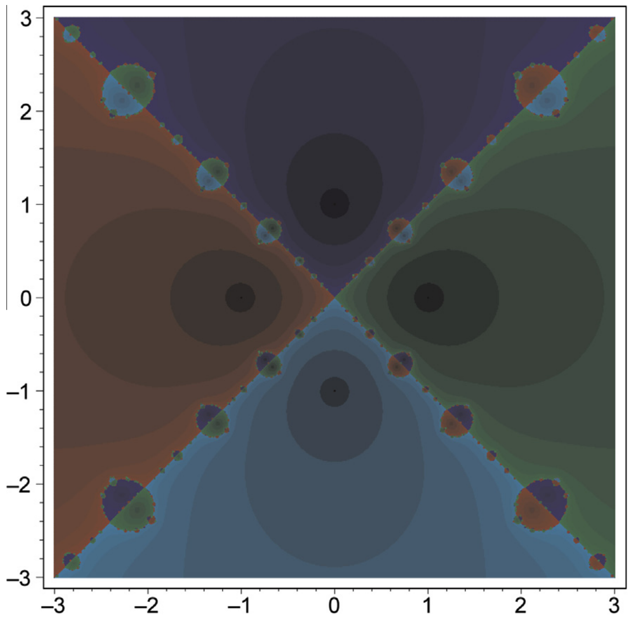

Fig. 22. Halley's method for the roots of the polynomial $(z^4 - 1)^5$.

$$S(z) = z^3[(m-1)z + 2m].$$

Proof. Let $p(z) = ((z-a)(z-b))^m$, $a \neq b$ and let M be the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \frac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p\left(\frac{ub-a}{u-1}\right) = u^3[(m-1)u + 2m]. \qquad \square$$

In the next two sections, we will analyze the basins of attraction to compare all these third order methods for multiple roots. The idea of using basins of attraction was initiated by Stewart [12] and followed by the works of Amat et al. [13–16], Scott et al. [18] and Chun et al. [17]. The only paper comparing basins of attraction for methods to obtain multiple roots is due to Neta et al. [19]. They have not considered some of the methods appearing here.

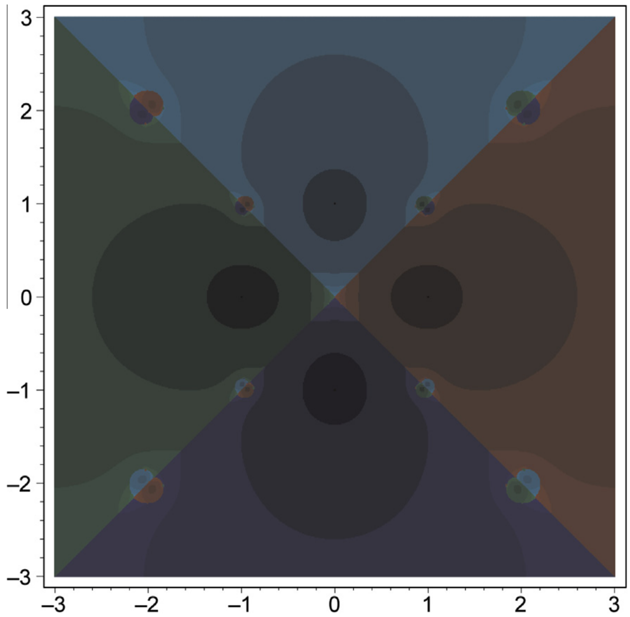

Fig. 23. Ostrowski's method for the roots of the polynomial $(z^4 - 1)^5$.

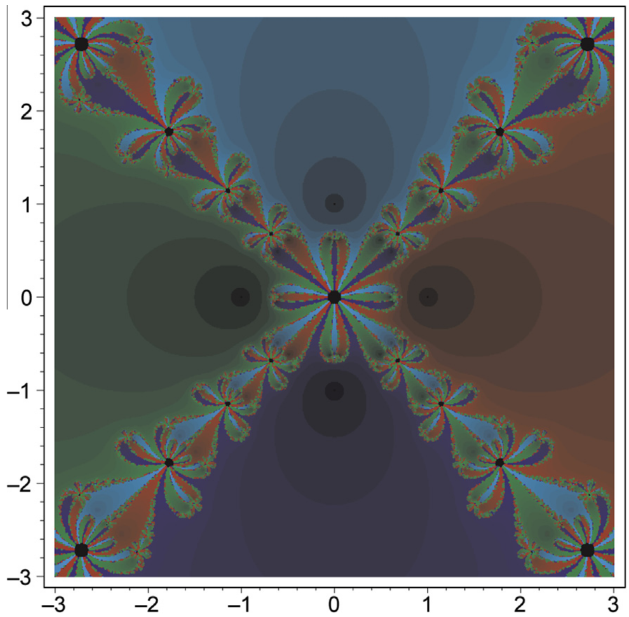

Fig. 24. Euler–Chebyshev's method for the roots of the polynomial $(z^4 - 1)^5$.

3. Extraneous fixed points

In solving a nonlinear equation iteratively we are looking for fixed points which are zeros of the given nonlinear function. Many iterative methods have fixed points that are not zeros of the function of interest. Those points are called extraneous fixed points (see [20]). Those points could be attractive which will trap an iteration sequence and give erroneous results. Even if those extraneous fixed points are repulsive or indifferent they can complicate the situation by converging to a root not close to the initial guess.

All of the methods discussed here can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n).$$

Clearly the root α of $f(x)$ is a fixed point of the method. The points $\xi \neq \alpha$ at which $H_f(\xi) = 0$ are also fixed points of the family, since the second term on the right vanishes.

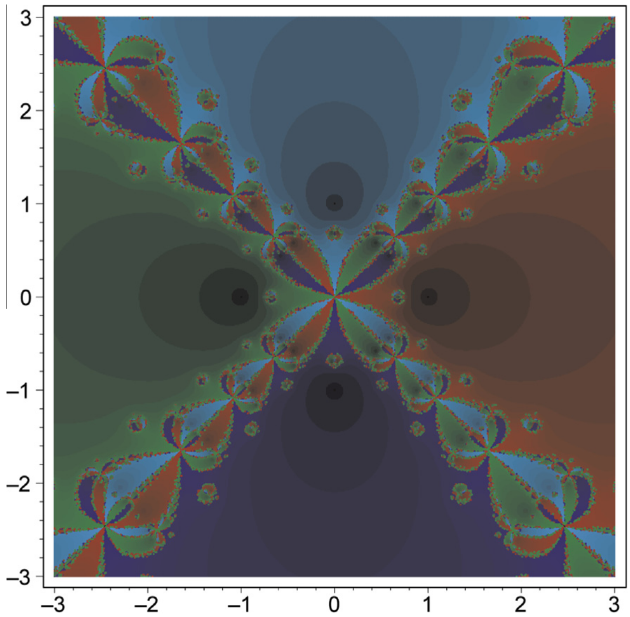

Fig. 25. Osada's method for the roots of the polynomial $(z^4-1)^5$.

It is easy to see that $H_f(x_n)$ for our methods is given in the table in Theorem 3.1.

From the table one can see that H_f does not vanish for Euler–Cauchy, Halley and Ostrowski's methods. Therefore there are no extraneous fixed points for these methods.

Theorem 3.1. *There are two extraneous fixed points for Euler–Chebyshev's method. They are the roots of*

$$\frac{f(\xi)f''(\xi)}{f'(\xi)^2}=\frac{m-3}{m}. \tag{17}$$

Method	H_f
Euler–Cauchy	$\frac{2m}{1+\sqrt{(2m-1)-2m\frac{f(x_n)f''(x_n)}{f'(x_n)^2}}}$
Halley	$\frac{1}{\frac{m+1}{2m}-\frac{f(x_n)f''(x_n)}{2f'(x_n)^2}}$
Ostrowski	$\frac{\sqrt{m}}{\sqrt{1-\frac{f(x_n)f''(x_n)}{f'(x_n)^2}}}$
Euler–Chebyshev	$\frac{m(3-m)}{2}+\frac{m^2}{2}\frac{f(x_n)}{f'(x_n)}\frac{f''(x_n)}{f'(x_n)}$
Osada	$\frac{1}{2}m(m+1)-\frac{(m-1)^2}{2}\frac{f'(x_n)}{f''(x_n)}\frac{f'(x_n)}{f(x_n)}$

Proof. The extraneous fixed points can be found by solving (17). For the polynomial $(z^2-1)^m$ this leads to the equation

$$\frac{2mz^2-z^2-1}{2mz^2}=\frac{m-3}{m}$$

for which the roots are $\xi=\pm\frac{1}{\sqrt{5}}$.

These fixed points are attractive. Vrcsay and Gilbert [20] show that if the points are attractive then the method will give erroneous results. If the points are repulsive then the method may not converge to a root near the initial guess.

The poles are at $z=0$. □

Theorem 3.2. *There are two extraneous fixed points for Osada's method. They are the roots of*

$$\frac{f(\xi)f''(\xi)}{f'(\xi)^2}=\frac{(m-1)^2}{m(m+1)}. \tag{18}$$

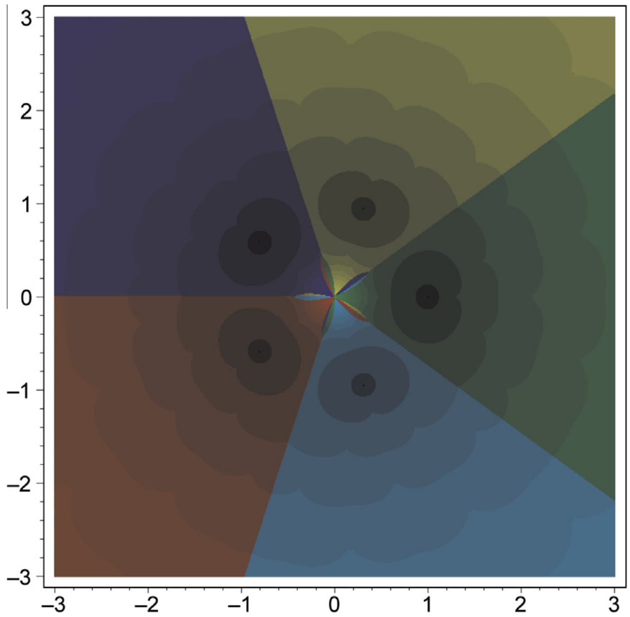

Fig. 26. Euler–Cauchy's method for the roots of the polynomial $(z^5 - 1)^3$.

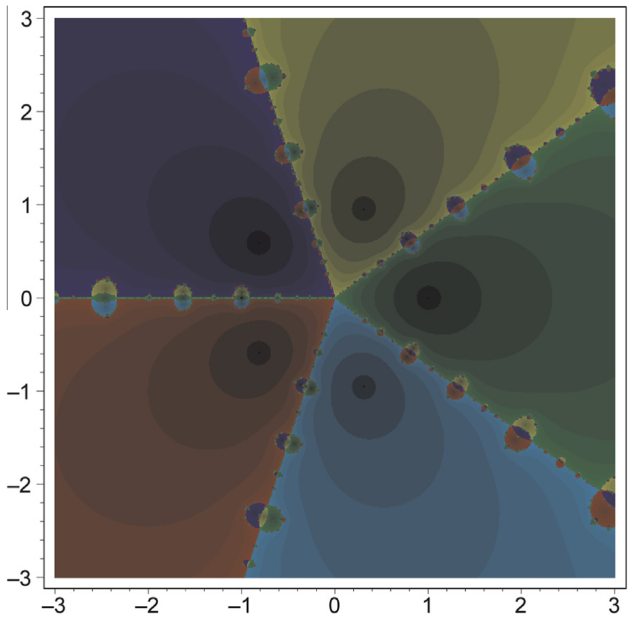

Fig. 27. Halley's method for the roots of the polynomial $(z^5 - 1)^3$.

Proof. The extraneous fixed points can be found by solving (18). For the polynomial $(z^2 - 1)^m$ this leads to the equation

$$\frac{2mz^2 - z^2 - 1}{2mz^2} = \frac{(m-1)^2}{m(m+1)},$$

for which the roots are $\xi = \pm\sqrt{\frac{m+1}{5m-3}}$.

These fixed points are repulsive for all $m > 1$.

The poles are at $z = \pm\frac{1}{\sqrt{2m-1}}$. □

4. Numerical experiments

We have used the above methods for 6 different polynomials having multiple roots with multiplicity $m = 2, 3, 4, 5$.

In our first example, we have taken the polynomial

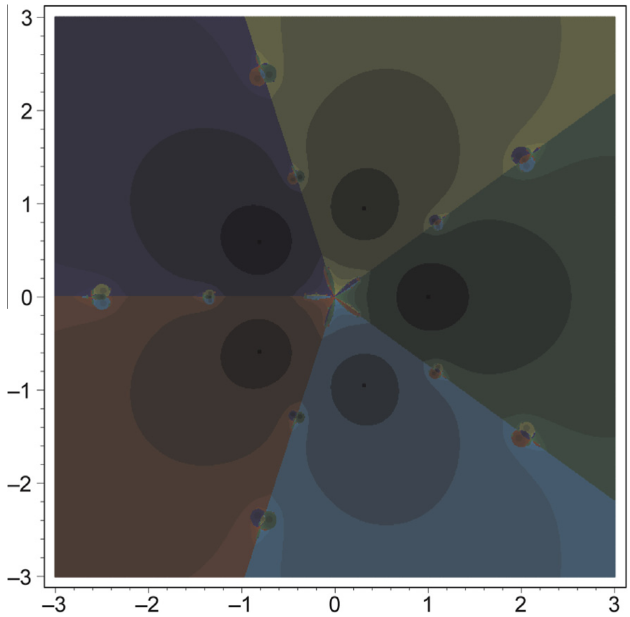

Fig. 28. Ostrowski's method for the roots of the polynomial $(z^5 - 1)^3$.

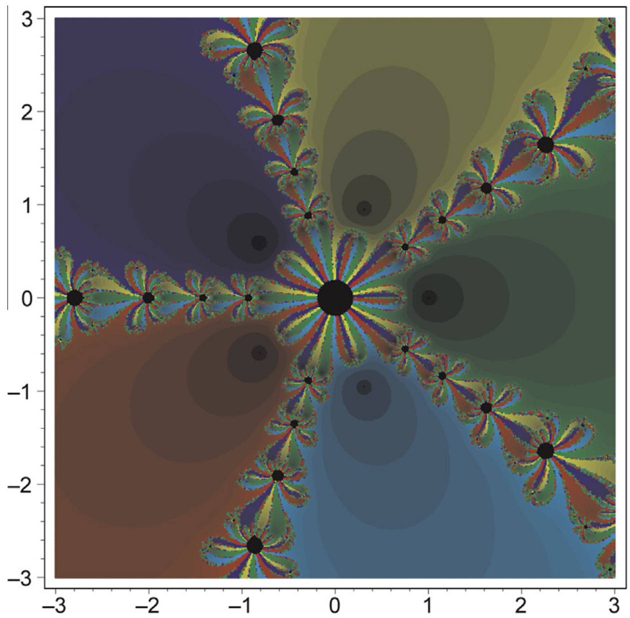

Fig. 29. Euler–Chebyshev's method for the roots of the polynomial $(z^5 - 1)^3$.

$$p_1(z) = (z^2 - 1)^2 \tag{19}$$

whose roots $z = \pm 1$ are both real and of multiplicity $m = 2$. The results are presented in Figs. 1–5. Notice that the darker the shade in each basin, the faster the convergence to the root. Euler–Cauchy's method (Fig. 1) for this example converged in 1 iteration to the closest root and in order to avoid having black points everywhere, we have used two different colors. This only happened for $(z^2 - 1)^m$. Halley's method (Fig. 2) is slightly better than Ostrowski's (Fig. 3). Euler–Chebyshev's (Fig. 4) and and Osada's method (Fig. 5) are not as good. Notice that these two methods are the only ones with extraneous fixed points and poles along the real line.

Our next example is also having double roots. The polynomial have the three roots of unity,

$$p_2(z) = (z^3 - 1)^2. \tag{20}$$

The results are presented in Figs. 6–10. Again Euler–Cauchy's (Fig. 6) and Ostrowski's (Fig. 8) methods performed better than Halley's method (Fig. 2). The Euler–Chebyshev's method (Fig. 9) was the worst and Osada's method (Fig. 10) only slightly better than that.

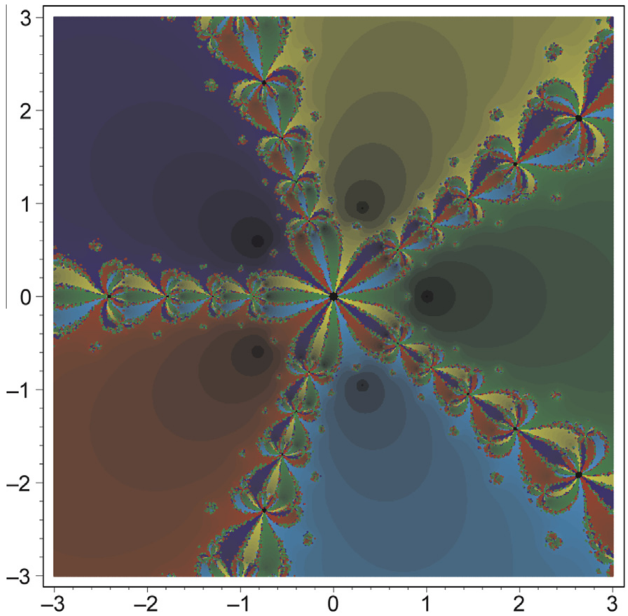

Fig. 30. Osada's method for the roots of the polynomial $(z^5-1)^3$.

The third example is a polynomial whose roots are all of multiplicity four. The roots are the three roots of unity, i.e.,

$$p_3(z) = (z^3-1)^4. \tag{21}$$

The results are presenetd in Figs. 11–15. Euler–Cauchy's method was the best followed by Ostrowski's method, Halley's method, Euler–Chebyshev's and Osada's schemes. The change in multiplicity, did not change the conclusions.

The fourth example is a polynomial whose roots are all of multiplicity three. The roots are $-2.68261500670705\pm .358259359924043i$, 1.36523001341410, i.e.,

$$p_4(z) = (z^3+4z^2-10)^3. \tag{22}$$

The results are presented in Figs. 16–20. Based on these figure, we arrive at the same conclusions as before.

In our next example we took the polynomial

$$p_5(z) = (z^4-1)^5 \tag{23}$$

where the roots are symmetrically located on the axes. In some sense this is similar to the first example, since in both cases we have an even number of roots. The results are shown in Figs. 21–25. Again we can see the best is Euler–Cauchy's method (Fig. 21) and the worst is Osada's method (Fig. 25).

In our last example we have the 5 roots of unity all with multiplicity three

$$p_6(z) = (z^5-1)^3. \tag{24}$$

The results are given in Figs. 26–30. Again we can see the best is Euler–Cauchy's method (Fig. 26) and and the worst is Osada's method (Fig. 30).

5. Conclusions

In all six examples, we find that the best is Euler–Cauchy's method and the worst are those with extraneous fixed points and poles on the real line, namely Euler–Chebyshev's and Osada's schemes. Notice that Neta et al. [19] have found that Halley's method is one of the best, but now that we have compared it to Euler–Cauchy's method, we realized that the latter is even better.

Acknowledgements

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012–0007160).

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