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An optimal eighth-order class of three-step weighted Newton's methods and their dynamics behind the purely imaginary extraneous fixed points

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ABSTRACT

In this paper, we not only develop an optimal class of three-step eighth-order methods with higher order weight functions employed in the second and third sub-steps, but also investigate their dynamics underlying the purely imaginary extraneous fixed points. Their theoretical and computational properties are fully described along with a main theorem stating the order of convergence and the asymptotic error constant as well as extensive studies of special cases with rational weight functions. A number of numerical examples are illustrated to confirm the underlying theoretical development. Besides, to show the convergence behaviour of global character, fully explored is the dynamics of the proposed family of eighth-order methods as well as an existing competitive method with the help of illustrative basins of attraction.

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1. Introduction

Nonlinear equations of high complexity naturally arise when describing our daily-life physical phenomena such as the evolving dynamics of a spinning tennis ball, a swinging pendulum, violent whirling windstorms, turbulent fluid flow as well as unpredictable weather forecast. Since exact solutions are rarely available, we usually resort to the classical second-order Newton's method for the numerical solutions. Since Traub [40] made a pioneering work in the 1960s toward the qualitative and quantitative analyses of iterative methods locating numerical roots for nonlinear equations, many authors [7,14,17,18,21,23,26,34,37] have developed high-order multipoint methods. Petković *et al.* [33] collected and updated the state of the art of multipoint methods. A numerical scheme is said to be *optimal* according to Kung–Traub's conjecture [24] that any multipoint method without memory can attain its convergence order of at most 2^{k-1} for k functional evaluations with $k \in \mathbb{N}$. For the sake of comparison, we first introduce an existing eighth-order method in [37] presented by Equation (1).

- Sharma-Arora method (SA)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(y_n)}{2f[y_n, x_n] - f'(x_n)} = x_n - \left(\frac{1-s}{1-2s}\right) \cdot \frac{f(x_n)}{f'(x_n)},$$

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$$x_{n+1} = z_n - \frac{f[z_n, y_n]}{f[z_n, x_n]} \cdot \frac{f(z_n)}{2f[z_n, y_n] - f[z_n, x_n]} \tag{1}$$
$$= x_n - \left(\frac{1-s}{1-2s}\right)\left[1 + \frac{su(1-s)(1-u)}{(1-su)(1-2s-2u+3su)}\right] \cdot \frac{f(x_n)}{f'(x_n)},$$

where $f[r,t] = (f(r) - f(t))/(r-t)$, $s = f(y_n)/f(x_n)$ and $u = f(z_n)/f(y_n)$.

Method (1) has been found to be very competitive judging from the recent studies performed by Lee *et al.* [25] and Chun and Neta [11], which motivates us to develop a new class of efficient methods. In this paper, we shall seek a class of optimal eighth-order simple-root finders that are competitive against or comparable to method (1).

To this end, we employ an optimal three-step high-order family of iterative methods in the form of weighted Newton-like simple-root finders below:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - L_f(s) \cdot \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - K_f(s,u) \cdot \frac{f(x_n)}{f'(x_n)} = x_n - [L_f(s) + K_f(s,u)] \cdot \frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{2}$$

where s and u are given in Equation (1) and $L_f : \mathbb{C} \to \mathbb{C}$ is a weight function being analytic [1] in a neighbourhood of 0 and $K_f : \mathbb{C}^2 \to \mathbb{C}$ is a weight function being holomorphic [22,36] in a neighbourhood of $(0,0)$. Note that Equation (1) is a special case of Equation (2) with $L_f(s) = (1-s)/(1-2s)$ and $K_f(s,u) = su(1-s)^2(1-u)/\{(1-2s)(1-su)(1-2s-2u+3su)\}$. It is interesting to see that Equation (1) can be expressed by means of fifth-order rational weight function $K_f(s,u)$ without using divided differences. The forms of Equation (2) use three functional values plus a single derivative without using divided differences as used in Equation (1).

Definition 1.1 (Error equation, asymptotic error constant, order of convergence): Let $x_0, x_1, \ldots, x_n, \ldots$ be a sequence of numbers converging to α. Let $e_n = x_n - \alpha$ for $n = 0, 1, 2, \ldots$. If constants $p \geq 1$, $c \neq 0$ exist in such a way that $e_{n+1} = c\, e_n{}^p + O(e_n^{p+1})$ called the *error equation*, then p and $\eta = |c|$ are said to be the *order of convergence* and the *asymptotic error constant*, respectively. It is easy to find $c = \lim_{n\to\infty} e_{n+1}/e_n{}^p$. Some authors call c itself the asymptotic error constant.

In this paper, we aim not only to design a class of optimal eighth-order methods by fully specifying the algebraic structure of generic weight functions $L_f(s)$ and $K_f(s,u)$, but also to investigate their dynamics by means of basins of attractions [16] (to be discussed in Section 5) behind the purely imaginary extraneous fixed points [42] (to be described in Section 4) when applied to a prototype quadratic polynomial. The last sub-step of Equation (2) in the form of weighted Newton's method is clearly more convenient in dealing with extraneous fixed points that can be found directly from the roots of the weight function $L_f(s) + K_f(s,u)$.

It is of importance for us to pursue suitable parameters giving the basin of attraction with a larger region of convergence. The presence of extraneous fixed points may induce attractive, indifferent, repulsive as well as other chaotic orbits influencing the relevant dynamics of the iterative methods. Notice that the imaginary axis symmetrically divides the whole complex plane into two half planes. Since we observe the convergence behaviour in the dynamical planes through the basins of attraction in the form of a square region centred at the origin, the resulting dynamics behind the extraneous fixed points on the symmetry (imaginary) axis is expected to be less influenced by the presence of the possible periodic or chaotic attractors. Thus, in the current analysis, it would be preferable for us to choose free parameters in such a way that the extraneous fixed points should be located on the imaginary axis.

In Section 2, the main theorem regarding the convergence behaviour is described with appropriate forms of two weight functions L_f and K_f. Section 3 investigates some special cases of $K_f(s,u)$. Section 4 discusses the purely imaginary extraneous fixed points together with their stabilities and investigates their theoretical multipliers. Section 5 presents numerical experiments along with the illustration of the relevant dynamics and describes concluding remarks at the end.

2. Main theorem

We shall state in this section the main theorem with generic weight functions $L_f(s)$ and $K_f(s,u)$ employed:

Theorem 2.1: *Assume that $f:\mathbb{C}\to\mathbb{C}$ has a simple root α and is analytic in a region containing α. Let $c_j=f^{(j)}(\alpha)/j!f'(\alpha)$ for $j=2,3,\ldots$. Let x_0 be an initial guess chosen in a sufficiently small neighbourhood of α. Let $L_f:\mathbb{C}\to\mathbb{C}$ be analytic in a neighbourhood of 0. Let $L_i=(1/i!)(\mathrm{d}^i/\mathrm{d}s^i)L_f(s)|_{(s=0)}$ for $0\le i\le 7$. Let $K_f:\mathbb{C}^2\to\mathbb{C}$ be holomorphic in a neighbourhood of $(0,0)$. Let $K_{ij}=(1/i!j!)(\partial^{i+j}/\partial s^i\partial u^j)K_f(s,u)|_{(s=0,u=0)}$ for $0\le i\le 7$ and $0\le j\le 3$. If $L_0=1,\ L_1=1,\ L_2=2,\ K_{00}=0,\ K_{10}=0,\ K_{20}=0,\ K_{30}=0,\ K_{40}=0,\ K_{50}=0,\ K_{60}=0,\ K_{01}=0,\ K_{02}=0,\ K_{03}=0,\ K_{11}=1,\ K_{21}=2,\ K_{12}=1,\ K_{22}=4,\ K_{31}=1+L_3,\ K_{41}=-4+2L_3+L_4$ are satisfied, then iterative scheme (2) defines a family of eighth-order methods satisfying the error equation below: for $n=0,1,2,\ldots,$*

$$e_{n+1}=c_2[-c_2c_3c_4+c_3^3(K_{13}-1)-c_2^3c_4(L_3-5)+c_2^2c_3^2\phi_1+c_2^4c_3\phi_2+c_2^6\phi_3]e_n^8+O(e_n^9), \tag{3}$$

where $\phi_1=24-K_{32}+3K_{13}(L_3-5)-2L_3$, $\phi_2=K_{51}-2K_{32}(L_3-5)+3K_{13}(L_3-5)^2-(L_3-33)L_3-2(70+L_4)-L_5$ *and* $\phi_3=-5K_{51}-K_{70}-K_{32}(L_3-5)^2+K_{13}(L_3-5)^3+L_3(K_{51}+9L_3-2L_4-L_5)+5(45-18L_3+2L_4+L_5)$.

Proof: The Taylor series expansion of $f(x_n)$ about α up to eighth-order terms with $f(\alpha)=0$ leads us to the following:

$$f(x_n)=f'(\alpha)\{e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+c_5e_n^5+c_6e_n^6+c_7e_n^7+c_8e_n^8+O(e_n^9)\}. \tag{4}$$

It follows that

$$f'(x_n)=f'(\alpha)\{1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+6c_6e_n^5+7c_7e_n^6+8c_8e_n^7+O(e_n^8)\}. \tag{5}$$

For simplicity, we will denote e_n by e from now on. Symbolic computation of Mathematica [43] yields:

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}=\alpha+c_2\,\mathrm{e}^2-2(c_2^2-c_3)\,\mathrm{e}^3+Y_4\,\mathrm{e}^4+Y_5\,\mathrm{e}^5+Y_6e_n^6+Y_7e_n^7+Y_8e_n^8+O(\mathrm{e}^9), \tag{6}$$

where $Y_4=4c_2^3-7c_2c_3+3c_4$, $Y_5=-2(4c_2^4-10c_2^2c_3+3c_3^2+5c_2c_4-2c_5)$, $Y_6=16c_2^5-52c_2^3c_3+33c_2c_3^2+28c_2^2c_4-17c_3c_4-13c_2c_5+5c_6$, $Y_7=-2[16c_2^6-64c_2^4c_3-9c_3^3+36c_2^3c_4+6c_4^2+9c_2^2(7c_3^2-2c_5)+11c_3c_5+c_2(-46c_3c_4+8c_6)-3c_7]$ and $Y_8=64c_2^7-304c_2^5c_3+176c_2^4c_4+75c_2^2c_3^2c_4+c_2^3(408c_3^2-92c_5)-31c_4c_5-27c_3c_6+c_2^2(-348c_3c_4+44c_6)+c_2(-135c_3^3+64c_4^2+118c_3c_5-19c_7)+7c_8$.

In view of the fact that $f(y_n) = f(x_n)|_{e_n \to (y_n - \alpha)}$, we obtain

$$f(y_n) = f'(\alpha)[c_2\,\mathrm{e}^2 - 2(c_2^2 - c_3)\,\mathrm{e}^3 + D_4\,\mathrm{e}^4 + \Sigma_{i=5}^{8} D_i\,\mathrm{e}^i + O(\mathrm{e}^9)], \tag{7}$$

where $D_4 = (5c_2^3 - 7c_2c_3 + 3c_4), D_i = D_i(c_2, c_3, \ldots, c_8)$ for $5 \le i \le 8$. Hence, we have

$$s = \frac{f(y_n)}{f(x_n)} = c_2 e + (-3c_2^2 + 2c_3)\,\mathrm{e}^2 - 4(8c_2^3 - 10c_2c_3 + 3c_4)\,\mathrm{e}^3 + \Sigma_{i=4}^{7} E_i\,\mathrm{e}^i + O(\mathrm{e}^8), \tag{8}$$

where $E_i = E_i(c_2, c_3, \ldots, c_8)$ for $4 \le i \le 7$.

Noting that $s = O(e)$ and $f(x_n)/f'(x_n) = O(e)$, we need a Taylor expansion of $L_f(s)$ about 0 up to seventh-order terms:

$$L_f(s) = L_0 + L_1 s + L_2 s^2 + L_3 s^3 + L_4 s^4 + L_5 s^5 + L_6 s^6 + L_7 s^7 + O(\mathrm{e}^8), \tag{9}$$

where $L_j = (\mathrm{d}^j/\mathrm{d}s^j)L_f(s)$ for $0 \le j \le 7$.

Thus, we find

$$z_n = x_n - L_f(s) \cdot \frac{f(x_n)}{f'(x_n)} = \alpha + (1 - L_0)e + c_2(1 - L_1)\,\mathrm{e}^2$$
$$+ [c_2^2(-2L_0 + 4L_1 - L_2) + 2c_3(L_0 - L_1)]\,\mathrm{e}^3 + \Sigma_{i=4}^{8} W_i\,\mathrm{e}^i + O(\mathrm{e}^9),$$

where $W_i = W_i(c_2, c_3, \ldots, c_8, L_0, \ldots, L_7)$ for $4 \le i \le 8$. By taking

$$L_0 = 1, L_1 = 1, L_2 = 2, \tag{10}$$

we further obtain

$$z_n = \alpha - c_2[c_2^2(L_3 - 5) + c_3]\,\mathrm{e}^4 + \Sigma_{i=5}^{8} W_i\,\mathrm{e}^i + O(\mathrm{e}^9). \tag{11}$$

In view of the fact that $f(z_n) = f(x_n)|_{e_n \to (z_n - \alpha)}$, we obtain

$$f(z_n) = f'(\alpha)[-c_2[c_2^2(L_3 - 5) + c_3]\,\mathrm{e}^4 + \Sigma_{i=5}^{8} F_i\,\mathrm{e}^i + O(\mathrm{e}^9)], \tag{12}$$

where $F_i = F_i(c_2, c_3, \ldots, c_8, L_3, \ldots, L_7)$ for $5 \le i \le 8$. Hence, we have

$$u = \frac{f(z_n)}{f(y_n)} = [-c_3 - c_2^2(L_3 - 5)]\,\mathrm{e}^2 + [-2c_4 - 4c_2c_3(L_3 - 5) + c_2^3(8L_3 - L_4 - 26)]\,\mathrm{e}^3$$
$$+ \Sigma_{i=4}^{8} G_i\,\mathrm{e}^i + O(\mathrm{e}^9), \tag{13}$$

where $G_i = G_i(c_2, c_3, \ldots, c_8, L_3, \ldots, L_7)$ for $4 \le i \le 8$.

Noting that $f(x_n) = O(e), s = O(e), u = O(\mathrm{e}^2)$, $f(y_n) = O(\mathrm{e}^2)$ and $f(z_n) = O(\mathrm{e}^4)$, the Taylor expansion of $K_f(s, u)$ about $(0, 0)$ up to seventh-order terms in s and third-order terms in u yields after retaining up to seventh-order terms with $K_{71} = 0$, $K_{72} = 0$, $K_{73} = 0$, $K_{61} = 0$, $K_{62} = 0$, $K_{63} = 0$, $K_{52} = 0$, $K_{53} = 0$, $K_{43} = 0$, $K_{42} = 0$, $K_{33} = 0$, $K_{23} = 0$:

$$K_f(s, u) = K_{00} + K_{01}u + K_{02}u^2 + K_{03}u^3 + s(K_{10} + K_{11}u + K_{12}u^2 + K_{13}u^3)$$
$$+ s^2(K_{20} + K_{21}u + K_{22}u^2) + s^3(K_{30} + K_{31}u + K_{32}u^2)$$
$$+ s^4(K_{40} + K_{41}u) + s^5(K_{50} + K_{51}u) + K_{60}s^6 + K_{70}s^7 + O(\mathrm{e}^8). \tag{14}$$

By direct substitution of $z_n, f(x_n), f(y_n), f(z_n), f'(x_n)$ and $K_f(s, u)$ in Equation (2), we find

$$x_{n+1} = x_n - [L_f(s) + K_f(s, u)] \cdot \frac{f(x_n)}{f'(x_n)} = \alpha - K_{00}e + c_2(K_{00} - K_{10})\,\mathrm{e}^2$$
$$+ [c_3(2K_{00} + K_{01} - 2K_{10}) - c_2^2(2K_{00} + 5K_{01} - 4K_{10} + K_{20} - K_{01}L_3)]\,\mathrm{e}^3$$
$$+ \Sigma_{i=4}^{8} \Gamma_i\,\mathrm{e}^i + O(\mathrm{e}^9), \tag{15}$$

where $\Gamma_i = \Gamma_i(c_2, c_3, \ldots, c_8, L_3, \ldots, L_7, K_{j\ell})$, for $4 \le i \le 8$, $0 \le j \le 7$ and $0 \le \ell \le 3$.

By taking

$$K_{00} = K_{10} = K_{01} = K_{20} = 0 \tag{16}$$

from Equation (15) along with $\Gamma_4 = 0$, we immediately obtain

$$-1 + K_{11} = 0, \quad -5 + K_{30} - K_{11}(L_3 - 5) + L_3 = 0,$$

from which we obtain

$$K_{11} = 1, \quad K_{30} = 0. \tag{17}$$

Continuing in this manner at the ith stage with $4 \le i \le 7$, $\Gamma_i = 0$ and solve $\Gamma_i = 0$ for remaining $K_{j\ell}$ to find:

$$\begin{aligned} &K_{02} = 0, \quad K_{21} = 2, \quad K_{40} = 0, \quad K_{12} = 1, \quad K_{31} = 1 + L_3, \quad K_{50} = 0, \\ &K_{03} = 0, \quad K_{22} = 4, \quad K_{41} = -4 + 2L_3 + L_4, \quad K_{60} = 0. \end{aligned} \tag{18}$$

By substituting these values of $K_{j\ell}$ into Γ_8, we eventually find

$$\Gamma_8 = c_2[-c_2c_3c_4 + c_3^3(K_{13} - 1) - c_2^3c_4(L_3 - 5) + c_2^2c_3^2\phi_1 + c_2^4c_3\phi_2 + c_2^6\phi_3], \tag{19}$$

with ϕ_1, ϕ_2 and ϕ_3 as described in Equation (3). This completes the proof. ■

3. Special cases of weight functions

As a result of Theorem 2.1, we easily find $L_f(s)$ and $K_f(s,u)$ in the form of Taylor polynomials as follows:

$$\begin{aligned} L_f(s) &= 1 + 1s + 2s^2 + L_3s^3 + L_4s^4 + L_5s^5 + L_6s^6 + L_7s^7 + O(e^8), \\ K_f(s,u) &= su[1 + u + K_{13}u^2 + 2s(1 + 2u) + s^2(1 + L_3 + K_{32}u) + s^3(-4 + 2L_3 + L_4) + K_{51}s^4] \\ &\quad + K_{70}s^7 + O(e^8), \end{aligned} \tag{20}$$

where $L_3, L_4, L_5, L_6, L_7, K_{13}, K_{32}, K_{51}$ and K_{70} may be free parameters.

Although various forms of weight functions $L_f(s)$ and $K_f(s,u)$ are applicable, either weight function L_f or K_f is of polynomial type has empirically shown poor convergence as seen in the existing studies by [9,19]. Taking into account the fact that $s = O(e)$, $u = O(e^2)$ and $f(x_n)/f'(x_n) = O(e)$, we shall establish eighth-order convergence by restricting ourselves to considering $L_f(s)$ as a family of second-order univariate rational functions and $K_f(s,u)$ as a family of fifth-order bivariate rational functions with real coefficients in the form below:

$$\begin{aligned} L_f(s) &= \frac{a_0 + a_1s + a_2s^2}{1 + b_1s + b_2s^2}, \\ K_f(s,u) &= \frac{\sum_{i=0}^{5} c_is^i + u(\sum_{i=0}^{4} A_is^i) + u^2(A_5 + A_6s + A_7s^2 + A_8s^3) + u^3(A_9 + A_{10}s)}{1 + \sum_{i=1}^{5} d_is^i + u(\sum_{i=0}^{4} B_is^i) + u^2(B_5 + B_6s + B_7s^2 + B_8s^3) + u^3(B_9 + B_{10}s)}, \end{aligned} \tag{21}$$

where $A_i, B_i, a_i, b_i, c_i, d_i \in \mathbb{R}$ are to be determined for optimal eighth-order convergence.

By Theorem 2.1, we let Equation (21) satisfy the constraints (10) and (16)–(18) which give us the coefficients:

$$\begin{aligned}
&c_0 = 0,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ c_4 = 0,\ c_5 = 0,\\
&A_0 = 0,\ A_1 = 1,\ A_2 = 2 + d_1,\ A_3 = 5 - 2a_2 + b_2 + 2d_1 + d_2,\\
&A_4 = 12 + 2a_2^2 + b_2^2 + 5d_1 + b_2(6 + d_1) - a_2(3b_2 + 2(6 + d_1)) + 2d_2 + d_3,\ A_5 = 0,\ A_9 = 0,\\
&B_0 = -1 + A_6,\ B_1 = -2 - 2A_6 + A_7 - d_1.
\end{aligned} \tag{22}$$

As a result, the reduced form of the desired weight functions is found to be:

$$\begin{aligned}
L_f(s) &= \frac{1 + (a_2 - b_2 - 1)s + a_2 s^2}{1 + (a_2 - b_2 - 2)s + b_2 s^2},\\
K_f(s,u) &= \frac{su[1 + (2 + d_1)s + A_3 s^2 + A_4 s^3 + u(A_6 + A_7 s + A_8 s^2) + A_{10}u^2]}{1 + \sum_{i=1}^{5} d_i s^i + u(\sum_{i=0}^{4} B_i s^i) + u^2(B_5 + B_6 s + B_7 s^2 + B_8 s^3) + u^3(B_9 + B_{10}s)},
\end{aligned} \tag{23}$$

where $A_6, A_7, A_8, A_{10}, B_i (2 \le i \le 10), a_2, b_2, d_i (1 \le i \le 5) \in \mathbb{R}$ are free parameters.

We first observe that weight function $K_f(s,u)$ reduces to Equation (1) studied by Sharma-Arora [37], if given a choice of parameters listed below:

$$\begin{aligned}
&A_{10} = B_{10} = B_9 = B_4 = B_5 = a_2 = b_2 = d_3 = d_4 = d_5 = 0,\\
&A_6 = -1, \quad A_7 = 2, \quad A_8 = -1, \quad B_2 = -2,\\
&B_3 = -4, \quad B_6 = 2, \quad B_7 = -7, \quad B_8 = 6, \quad d_1 = -4, \quad d_2 = 4.
\end{aligned} \tag{24}$$

For simplified analysis along with a close inspection of Equation (22), we preferably select $d_4 = d_5 = A_{10} = B_{10} = 0$ and will finally deal with a shortened form of $K_f(s,u)$ as follows:

$$\begin{aligned}
L_f(s) &= \frac{1 + (a_2 - b_2 - 1)s + a_2 s^2}{1 + (a_2 - b_2 - 2)s + b_2 s^2},\\
K_f(s,u) &= \frac{su[1 + (2 + d_1)s + A_3 s^2 + A_4 s^3 + u(A_6 + A_7 s + A_8 s^2)]}{1 + d_1 s + d_2 s^2 + d_3 s^3 + u(\sum_{i=0}^{4} B_i s^i) + u^2(B_5 + B_6 s + B_7 s^2 + B_8 s^3) + B_9 u^3},
\end{aligned} \tag{25}$$

where $A_6, A_7, A_8, B_i (2 \le i \le 9), a_2, b_2, d_1, d_2, d_3 \in \mathbb{R}$ are free parameters with A_3, A_4, B_0, B_1 given by Equation (22).

Since $s = O(e)$ and $u = O(e^2)$, we find $K_f(s,u) = O(e^7)$ from Equation (25), according to which the last sub-step iterative scheme of Equation (2) should give rise to an optimal convergence order of eight with a suitable choice of parameters.

For easier analysis, we further take $a_2 = b_2 = B_9 = 0$ leading to simplified rational weight functions with first-order $L_f(s)$ and fifth-order $K_f(s,u)$ below:

$$\begin{aligned}
L_f(s) &= \frac{1 - s}{1 - 2s},\\
K_f(s,u) &= \frac{su[1 + (2 + d_1)s + A_3 s^2 + A_4 s^3 + u(A_6 + A_7 s + A_8 s^2)]}{1 + d_1 s + d_2 s^2 + d_3 s^3 + u(\sum_{i=0}^{4} B_i s^i) + u^2(B_5 + B_6 s + B_7 s^2 + B_8 s^3)},
\end{aligned} \tag{26}$$

where $A_6, A_7, A_8, B_2, B_3, B_4, B_5, B_6, B_7, B_8, d_1, d_2, d_3 \in \mathbb{R}$ are 13 free parameters.

Although numerous cases of weight functions satisfying Theorem 2.1 can be constructed, we are especially interested in special cases for which all of the extraneous fixed points (to be discussed in

Section 4) of the proposed scheme (2) are purely imaginary. From Equation (35) of Section 4, we desire the governing equation of the extraneous fixed points to take the form of

$$H(z) = \frac{1}{2(1+t)} \cdot \frac{G(t)}{\Omega(t)}, \quad t = z^2, \tag{27}$$

where $G(t) = t^{\gamma_1}(1+t)^{\gamma_2}(1+3t)^{\gamma_3} \cdot g(t)$ and $\Omega(t) = t^{\sigma_1}(1+t)^{\sigma_2}(1+3t)^{\sigma_3} \cdot w(t)$ for $\gamma_1, \gamma_2, \gamma_3, \sigma_1, \sigma_2, \sigma_3 \in \mathbb{N}$. In addition, $g(t)$ and $w(t)$ are polynomials of degree at most 3 and 4, respectively, with $\gamma_1 + \gamma_2 + \gamma_3 = 6$, and $\sigma_1 + \sigma_2 + \sigma_3 = 4$. Observe that $G(t)$ and $\Omega(t)$ have common factors, which further simplifies the resulting expressions of $H(z)$. The remaining task is again for us to determine appropriate parameters of weight functions in such a way that all the roots of $H(z)$ should be located on the imaginary axis of the complex plane.

In Section 4, we shall give an extensive investigation with an appropriate selection of free parameters leading us to purely imaginary extraneous fixed points. To this end, we will seek feasible relationships among the free parameters by imposing some constraints on simplifying the numerator of the resulting expression $G(t)$ to be described in Equation (36). The following cases are of our main interest whose values of $(\gamma_1, \gamma_2, \gamma_3)$, $(\sigma_1, \sigma_2, \sigma_3)$ and 10 parameters $A_6, A_7, A_8, B_2, B_3, B_4, B_5, B_6, d_1, d_2$ for each case are discussed in Section 4.

Case AX: $(\gamma_1, \gamma_2, \gamma_3) = (1,2,3)$, $(\sigma_1, \sigma_2, \sigma_3) = (1,1,2)$, $\lambda = 4 - A_8 + 2d_3$,

$$\begin{aligned}
&A_6 = \frac{13 + d_3 - 3\lambda}{2}, \quad A_7 = -13 - 2d_3 + 3\lambda, \quad A_8 = 4 - \lambda + 2d_3,\\
&B_2 = \frac{116 - B_7 - B_8 + 24d_3 - 25\lambda}{4}, \quad B_3 = -\frac{B_8}{4} - 4d_3 - \frac{\lambda}{2},\\
&B_4 = 0, \quad B_5 = \frac{-4 + B_7 + B_8 + \lambda}{4}, \quad B_6 = \frac{8 - 4B_7 - 3B_8 - 2\lambda}{4},\\
&d_1 = 1 - \lambda, \quad d_2 = \frac{-17 - d_3 + 5\lambda}{2}.
\end{aligned}$$

Case AY: $(\gamma_1, \gamma_2, \gamma_3) = (1,2,3)$, $(\sigma_1, \sigma_2, \sigma_3) = (1,2,1)$, $\lambda = 56 - 12B_2 - 3B_7 - 3B_8 + 72d_3$,

$$\begin{aligned}
&A_6 = \frac{-5 + 2d_3}{4} + \frac{\lambda}{8}, \quad A_7 = \frac{5 - 4d_3}{2} - \frac{\lambda}{4}, \quad A_8 = 2d_3,\\
&B_2 = \frac{2 - B_7 - B_8 + 24d_3}{4} + \frac{\lambda}{4}, \quad B_3 = \frac{-50 - B_8 - 16d_3}{4} + \frac{3\lambda}{4}, \quad B_4 = 0,\\
&B_5 = \frac{B_7}{4} + \frac{B_8}{4}, \quad B_6 = -B_7 - \frac{3B_8}{4}, \quad d_1 = -\frac{13}{2} + \frac{\lambda}{4}, \quad d_2 = \frac{41 - 2d_3}{4} - \frac{5\lambda}{8}.
\end{aligned}$$

Case AZ: $(\gamma_1, \gamma_2, \gamma_3) = (1,2,3)$, $(\sigma_1, \sigma_2, \sigma_3) = (2,1,1)$, $\lambda = 3(16 - 9A_8 + 18d_3)$,

$$\begin{aligned}
&A_6 = \frac{181 + 27d_3}{54} - \frac{17\lambda}{324}, \quad A_7 = \frac{-181 - 54d_3}{27} + \frac{17\lambda}{162}, \quad A_8 = \frac{16 + 18d_3}{9} - \frac{\lambda}{27},\\
&B_2 = \frac{1672 - 27B_7 - 27B_8 + 648d_3}{108} - \frac{73\lambda}{324},\\
&B_3 = -\frac{B_8 + 16d_3}{4}, \quad B_4 = 0, \quad B_5 = \frac{-16 + 9B_7 + 9B_8}{36} + \frac{\lambda}{108},\\
&B_6 = \frac{32 - 36B_7 - 27B_8}{36} - \frac{\lambda}{54}, \quad d_1 = -\frac{23}{27} - \frac{5\lambda}{162}, \quad d_2 = \frac{-209 - 27d_3}{54} + \frac{25\lambda}{324}.
\end{aligned}$$

Case BX: $(\gamma_1,\gamma_2,\gamma_3)=(2,3,1),(\sigma_1,\sigma_2,\sigma_3)=(1,1,2),\lambda=-A_8+2d_3$,

$$A_6=\frac{d_3}{2}-\frac{\lambda}{2},\quad A_7=-1-2d_3+2\lambda,\quad A_8=-\lambda+2d_3,$$
$$B_2=\frac{18-B_7-B_8+24d_3}{4}-\frac{19\lambda}{4},\quad B_3=\frac{-B_8-16d_3}{4}-\frac{\lambda}{2},\quad B_4=0,$$
$$B_5=\frac{2+B_7+B_8}{4}-\frac{\lambda}{4},\quad B_6=\frac{-4-4B_7-3B_8}{4}+\frac{\lambda}{2},\quad d_1=-2-\lambda,\quad d_2=\frac{-2-d_3}{2}+\frac{5\lambda}{2}.$$

Case BY: $(\gamma_1,\gamma_2,\gamma_3)=(2,3,1),(\sigma_1,\sigma_2,\sigma_3)=(1,2,1),\lambda=-8-4B_2-B_7-B_8+24d_3$,

$$A_6=\frac{-12+12d_3-\lambda}{24},\quad A_7=2-2d_3+\frac{\lambda}{12},\quad A_8=2(-1+d_3),$$
$$B_2=\frac{-8-B_7-B_8+24d_3-\lambda}{4},\quad B_3=-\frac{B_8}{4}-4d_3-\frac{\lambda}{12},$$
$$B_4=0,\quad B_5=\frac{B_7+B_8}{4},\quad B_6=-B_7-\frac{3B_8}{4},\quad d_1=-3-\frac{\lambda}{12},\quad d_2=\frac{36-12d_3+5\lambda}{24}.$$

Case BZ: $(\gamma_1,\gamma_2,\gamma_3)=(2,3,1),(\sigma_1,\sigma_2,\sigma_3)=(2,1,1),\lambda=2-7A_8+14d_3$,

$$A_6=\frac{2+14d_3-\lambda}{28},\quad A_7=-\frac{10}{7}-2d_3+\frac{3\lambda}{14},\quad A_8=\frac{2+14d_3-\lambda}{7},$$
$$B_2=\frac{152-7B_7-7B_8+168d_3-13\lambda}{28},\quad B_3=-\frac{B_8}{4}-4d_3,$$
$$B_4=0,\quad B_5=\frac{16+7B_7+7B_8-\lambda}{28},\quad B_6=-\frac{8}{7}-B_7-\frac{3B_8}{4}+\frac{\lambda}{14},$$
$$d_1=\frac{-26-\lambda}{14},\quad d_2=\frac{-38-14d_3+5\lambda}{28}.$$

Case CX: $(\gamma_1,\gamma_2,\gamma_3)=(1,3,2),(\sigma_1,\sigma_2,\sigma_3)=(1,1,2),\lambda=-8-7A_8+14d_3$,

$$A_6=\frac{-8+7d_3-\lambda}{14},\quad A_7=\frac{8-14d_3+\lambda}{7},\quad A_8=\frac{-8+14d_3-\lambda}{7},$$
$$B_2=\frac{-16-7B_7-7B_8+168d_3-9\lambda}{28},\quad B_3=-\frac{B_8}{4}-4d_3+\frac{\lambda}{2},$$
$$B_4=0,\quad B_5=\frac{8+7B_7+7B_8+\lambda}{28},\quad B_6=-B_7+\frac{-21B_8-2(8+\lambda)}{28},$$
$$d_1=\frac{-20+\lambda}{7},\quad d_2=\frac{16-7d_3-5\lambda}{14}.$$

Case CY: $(\gamma_1,\gamma_2,\gamma_3)=(1,3,2),(\sigma_1,\sigma_2,\sigma_3)=(1,2,1),\lambda=56-4B_2-B_7-B_8+24d_3$,

$$A_6=2+\frac{d_3}{2}-\frac{\lambda}{24},\quad A_7=-4-2d_3+\frac{\lambda}{12},\quad A_8=2d_3,$$
$$B_2=\frac{56-B_7-B_8+24d_3-\lambda}{4},\quad B_3=-\frac{B_8}{4}-4d_3-\frac{\lambda}{12},$$
$$B_4=0,\quad B_5=\frac{B_7+B_8}{4},\quad B_6=-B_7-\frac{3B_8}{4},\quad d_1=-\frac{\lambda}{12},\quad d_2=-6-\frac{d_3}{2}+\frac{5\lambda}{24}.$$

Case CZ: $(\gamma_1, \gamma_2, \gamma_3) = (1,3,2), (\sigma_1, \sigma_2, \sigma_3) = (2,1,1), \lambda = -40 - 49A_8 + 98d_3$,

$$A_6 = \frac{32 + 98d_3 - 9\lambda}{196}, \quad A_7 = -2d_3 + \frac{-32 + 9\lambda}{98}, \quad A_8 = \frac{-40 + 98d_3 - \lambda}{49},$$
$$B_2 = \frac{704 - 49B_7 - 49B_8 + 1176d_3 - 51\lambda}{196}, \quad B_3 = -\frac{B_8}{4} - 4d_3,$$
$$B_4 = 0, \quad B_5 = \frac{40 + 49B_7 + 49B_8 + \lambda}{196}, \quad B_6 = -\frac{20}{49} - B_7 - \frac{3B_8}{4} - \frac{\lambda}{98},$$
$$d_1 = -\frac{5(40 + \lambda)}{98}, \quad d_2 = \frac{-176 - 98d_3 + 25\lambda}{196}.$$

Case DX: $(\gamma_1, \gamma_2, \gamma_3) = (3,2,1), (\sigma_1, \sigma_2, \sigma_3) = (1,1,2), \lambda = -A_8 + 2d_3$,

$$A_6 = \frac{d_3 - 9\lambda}{2}, \quad A_7 = -1 - 2d_3 + 6\lambda, \quad A_8 = 2d_3 - \lambda,$$
$$B_2 = \frac{18 - B_7 - B_8 + 24d_3 - 43\lambda}{4}, \quad B_3 = -\frac{B_8}{4} - 4d_3 - \frac{\lambda}{2}, \quad B_4 = 0,$$
$$B_5 = \frac{2 + B_7 + B_8 + 7\lambda}{4}, \quad B_6 = -1 - B_7 - \frac{3B_8}{4} - \frac{7\lambda}{2}, \quad d_1 = -2 - \lambda, \quad d_2 = \frac{-2 - d_3 + 5\lambda}{2}.$$

Case DY: $(\gamma_1, \gamma_2, \gamma_3) = (3,2,1), (\sigma_1, \sigma_2, \sigma_3) = (1,2,1), \lambda = A_8 - 2(1 + d_3)$,

$$A_6 = \frac{60 + 12d_3 + 17\lambda}{24}, \quad A_7 = -6 - 2d_3 - \frac{23\lambda}{12}, \quad A_8 = 2 + 2d_3 + \lambda,$$
$$B_2 = \frac{56 - B_7 - B_8 + 24d_3 + 15\lambda}{4}, \quad B_3 = -\frac{B_8}{4} - 4d_3 - \frac{\lambda}{12},$$
$$B_4 = 0, \quad B_5 = \frac{B_7 + B_8}{4}, \quad B_6 = -B_7 - \frac{3B_8}{4}, \quad d_1 = -1 + \frac{5\lambda}{12}, \quad d_2 = \frac{-84 - 12d_3 - 25\lambda}{24}.$$

Case DZ: $(\gamma_1, \gamma_2, \gamma_3) = (3,2,1), (\sigma_1, \sigma_2, \sigma_3) = (2,1,1), \lambda = 2 - A_8 + 2d_3$,

$$A_6 = \frac{10 + 2d_3 - 5\lambda}{4}, \quad A_7 = -6 - 2d_3 + \frac{5\lambda}{2}, \quad A_8 = 2 + 2d_3 - \lambda,$$
$$B_2 = \frac{56 - B_7 - B_8 + 24d_3 - 19\lambda}{4}, \quad B_3 = -\frac{B_8}{4} - 4d_3, \quad B_4 = 0,$$
$$B_5 = \frac{B_7 + B_8 + \lambda}{4}, \quad B_6 = \frac{-4B_7 - 3B_8 - 2\lambda}{4}, \quad d_1 = -1 - \frac{\lambda}{2}, \quad d_2 = \frac{-14 - 2d_3 + 5\lambda}{4}.$$

Case EX: $(\gamma_1, \gamma_2, \gamma_3) = (2,2,2), (\sigma_1, \sigma_2, \sigma_3) = (1,1,2), \lambda = -A_8 + 2d_3$,

$$A_6 = \frac{2 + d_3 - 3\lambda}{2}, \quad A_7 = -2 - 2d_3 + 3\lambda, \quad A_8 = 2d_3 - \lambda,$$
$$B_2 = \frac{24 - B_7 - B_8 + 24d_3 - 25\lambda}{4}, \quad B_3 = -\frac{B_8}{4} - 4d_3 - \frac{\lambda}{2}, \quad B_4 = 0,$$
$$B_5 = \frac{B_7 + B_8 + \lambda}{4}, \quad B_6 = \frac{-4B_7 - 3B_8 - 2\lambda}{4}, \quad d_1 = -2 - \lambda, \quad d_2 = \frac{-2 - d_3 + 5\lambda}{2}.$$

Case EY: $(\gamma_1, \gamma_2, \gamma_3) = (2, 2, 2), (\sigma_1, \sigma_2, \sigma_3) = (1, 2, 1), \lambda = 24 - 4B_2 - B_7 - B_8 + 24d_3$,

$$A_6 = 1 + \frac{d_3}{2} - \frac{\lambda}{24}, \quad A_7 = -2 - 2d_3 + \frac{\lambda}{12}, \quad A_8 = 2d_3,$$
$$B_2 = \frac{24 - B_7 - B_8 + 24d_3 - \lambda}{4}, \quad B_3 = -\frac{B_8}{4} - 4d_3 - \frac{\lambda}{12}, \quad B_4 = 0,$$
$$B_5 = \frac{B_7 + B_8}{4}, \quad B_6 = -B_7 - \frac{3B_8}{4}, \quad d_1 = -2 - \frac{\lambda}{12}, \quad d_2 = -1 - \frac{d_3}{2} + \frac{5\lambda}{24}.$$

Case EZ: $(\gamma_1, \gamma_2, \gamma_3) = (2, 2, 2), (\sigma_1, \sigma_2, \sigma_3) = (2, 1, 1), \lambda = 8 - A_8 + 2d_3$,

$$A_6 = \frac{44 + 2d_3 - 5\lambda}{4}, \quad A_7 = -22 - 2d_3 + \frac{5\lambda}{2}, \quad A_8 = 8 + 2d_3 - \lambda,$$
$$B_2 = \frac{176 - B_7 - B_8 + 24d_3 - 19\lambda}{4}, \quad B_3 = -\frac{B_8}{4} - 4d_3,$$
$$B_4 = 0, \quad B_5 = \frac{-8 + B_7 + B_8 + \lambda}{4}, \quad B_6 = 4 - B_7 - \frac{3B_8}{4} - \frac{\lambda}{2},$$
$$d_1 = 2 - \frac{\lambda}{2}, \quad d_2 = \frac{-44 - 2d_3 + 5\lambda}{4}.$$

Case FX: $(\gamma_1, \gamma_2, \gamma_3) = (1, 4, 1), (\sigma_1, \sigma_2, \sigma_3) = (1, 1, 2), \lambda = 64 + 25A_8 - 50d_3$,

$$A_6 = \frac{-64 + 25d_3 + \lambda}{50}, \quad A_7 = \frac{19}{5} - 2d_3 - \frac{2\lambda}{35}, \quad A_8 = \frac{-64 + 50d_3 + \lambda}{25},$$
$$B_2 = -\frac{343}{50} - \frac{B_7}{4} - \frac{B_8}{4} + 6d_3 + \frac{93\lambda}{700},$$
$$B_3 = -\frac{B_8}{4} - 4d_3 - \frac{\lambda}{14}, \quad B_4 = 0, \quad B_5 = \frac{14 + 175B_7 + 175B_8 - \lambda}{700},$$
$$B_6 = -\frac{1}{25} - B_7 - \frac{3B_8}{4} + \frac{\lambda}{350}, \quad d_1 = \frac{-686 - \lambda}{175}, \quad d_2 = \frac{266 - 35d_3 + \lambda}{70}.$$

Case FY: $(\gamma_1, \gamma_2, \gamma_3) = (1, 4, 1), (\sigma_1, \sigma_2, \sigma_3) = (1, 2, 1), \lambda = -18 - 7A_8 + 14d_3$,

$$A_6 = \frac{-68 + 28d_3 - 3\lambda}{56}, \quad A_7 = \frac{26}{7} - 2d_3 + \frac{5\lambda}{28}, \quad A_8 = -\frac{18}{7} + 2d_3 - \frac{\lambda}{7},$$
$$B_2 = \frac{-184 - 7B_7 - 7B_8 + 168d_3 - 11\lambda}{28},$$
$$B_3 = \frac{-B_8 - 16d_3 + \lambda}{4}, \quad B_4 = 0, \quad B_5 = \frac{B_7 + B_8}{4}, \quad B_6 = -B_7 - \frac{3B_8}{4},$$
$$d_1 = \frac{-108 + \lambda}{28}, \quad d_2 = \frac{204 - 28d_3 - 5\lambda}{56}.$$

Case FZ: $(\gamma_1, \gamma_2, \gamma_3) = (1, 4, 1), (\sigma_1, \sigma_2, \sigma_3) = (2, 1, 1), \lambda = -338 - 119A_8 + 238d_3$,

$$A_6 = \frac{158 + 238d_3 + 23\lambda}{476}, \quad A_7 = \frac{404 - 476d_3 - 15\lambda}{238}, \quad A_8 = \frac{-338 + 238d_3 - \lambda}{119},$$
$$B_2 = \frac{26}{119} - \frac{B_7}{4} - \frac{B_8}{4} + 6d_3 + \frac{101\lambda}{476},$$
$$B_3 = -\frac{B_8}{4} - 4d_3, \quad B_4 = 0, \quad B_5 = \frac{-32 + 17B_7 + 17B_8 - \lambda}{68}, \quad B_6 = \frac{64 - 68B_7 - 51B_8 + 2\lambda}{68},$$
$$d_1 = \frac{-566 + 11\lambda}{238}, \quad d_2 = \frac{-26 - 238d_3 - 55\lambda}{476}.$$

4. Extraneous fixed points and their dynamics

We in this section will devote ourselves to investigating the extraneous fixed points [42] of iterative map (2) and relevant dynamics associated with their basins of attraction. The dynamics underlying basins of attraction was initiated by Stewart [39] and followed by works of Amat *et al.*, e.g. [2,3], Andreu *et al.* [4], Argyros-Magreñan [5], Chun *et al.* [12], Chicharro *et al.* [8], Chun-Neta [10], Cordero *et al.* [15], Geum *et al.* [19,20], Magreñan [27,28], Neta *et al.* [30–32] and Scott *et al.* [35].

We usually locate a zero α of a nonlinear equation $f(x) = 0$ by means of a fixed point ξ of iterative methods of the form

$$x_{n+1} = R_f(x_n), \quad n = 0, 1, \ldots, \tag{28}$$

where R_f is the iteration function under consideration. In general, R_f might possess other fixed points $\xi \neq \alpha$. Such fixed points are called the *extraneous fixed points* of the iteration function R_f. It is well known that extraneous fixed points may result in attractive, indifferent or repulsive cycles as well as other periodic orbits influencing the dynamics underlying the basins of attraction. Exploration of such dynamics as well as discovery of its complicated behaviour gives us a valuable motivation of the current analysis. In connection with proposed family of methods (2), we obtain a more specific form of iterative map (28) as follows:

$$x_{n+1} = R_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n), \tag{29}$$

where $H_f(x_n) = L_f(s) + K_f(s, u)$ can be regarded as a weight function of the classical Newton's method. It is obvious that α is a fixed point of R_f. The points $\xi \neq \alpha$ for which $H_f(\xi) = 0$ are extraneous fixed points of R_f.

For ease of analysis of the relevant dynamics, we restrict ourselves to considering only combinations of weight functions $L_f(s)$ and $K_f(s, u)$ in the form of univariate and bivariate rational functions as described by Equation (21). A special attention will be paid to some selected cases to be shown later in this section in order to pursue further properties of their extraneous fixed points and relevant dynamics associated with their basins of attraction. The existence of such extraneous fixed points would affect the global iteration dynamics, which was demonstrated for simple zeros via König functions and Schröder functions [42] applied to a family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$ according to the joint work of Vrscay and Gilbert [42] published in 1988. Especially, the presence of attractive cycles induced by the extraneous fixed points of R_f may alter the basins of attraction due to the trapped sequence $\{x_n\}$. Even in the case of repulsive or indifferent fixed points, an initial value x_0 chosen near a desired root may converge to another unwanted remote root. Indeed, these aspects of the Schröder functions were observed in an application to the same family of functions $\{f_k(x) = x^k - 1,\ k \geq 2\}$.

For simplified dynamics related to the extraneous fixed points underlying the basins of attraction for iterative maps (29), we first choose a simple quadratic polynomial from the family of functions $\{f_k(x) = x^k - 1,\ k \geq 2\}$. By closely following the works of Chun *et al.* [9,13] and Neta *et al.* [29,30,32], we then construct $H_f(x_n) = L_f(s) + K_f(s, u)$ in Equation (29). We now apply a prototype quadratic polynomial $f(z) = (z^2 - 1)$ to $H_f(x_n)$ and construct $H(z)$, with a change of a variable $t = z^2$, in the form of

$$H(z) = \frac{\mathcal{N}(t)}{\mathcal{D}(t)}, \tag{30}$$

where both $\mathcal{D}(t)$ and $\mathcal{N}(t)$ are polynomial functions of t with no common factors. Since H is a rational function, it would be preferable for us to deal with the underlying dynamics of iterative map (29) on the Riemann sphere [6] where points '0 (zero)' and '∞' can be treated as the desired extraneous fixed points. If such points arise, we are interested in only the finite extraneous fixed point 0 under which the relevant dynamics can be described in a region containing the origin by investigating the attractor basins associated with iterative map (29).

Indeed, the extraneous fixed points ξ of R_f in Equation (29) can be found from the roots t of $H(z)$ with $z = t^{1/2}$ via relation below:

$$\xi = \begin{cases} t^{1/2} & \text{if } t \neq 0, \\ 0(\text{double root}) & \text{if } t = 0. \end{cases} \tag{31}$$

4.1. Purely imaginary extraneous fixed points

We now pay a special attention to the dynamics underlying purely imaginary extraneous fixed points of iterative map (29). One should be aware that the boundary of two basins of attraction of two roots for the prototype quadratic polynomial $f(z) = z^2 - 1$ is the imaginary axis of the complex plane. Hence, it is worth to explore how the extraneous fixed points on the imaginary axis influence the dynamical behaviour of iterative map (29). It is our important task to find a possible combination of L_f and K_f leading to purely imaginary extraneous fixed points, whose investigation was done by Chun *et al.* [13]. As a preliminary task, we first describe the following lemma regarding the negative real roots of a quadratic equation, which would play a role in determining the desired purely imaginary extraneous fixed points in connection with the prototype quadratic polynomial $f(z) = z^2 - 1$.

As a preliminary task, we first describe the following lemma regarding the negative real roots of a cubic equation for later use in characterizing the cubic $g(t)$ described by Equation (39).

Lemma 4.1: *Let $q(x) = ax^3 + bx^2 + cx + d$ be a cubic equation with real coefficients $a \neq 0$, b,c,d satisfying $\mathcal{D} \geq 0$, where $\mathcal{D} = 18abcd + b^2c^2 - 4b^3d - a(4c^3 + 27ad^2)$. Let r_1, r_2 and r_3 be the three roots of $q(x) = 0$. Then all three roots $r_1 < 0$, $r_2 < 0$ and $r_3 < 0$ hold if and only if all four coefficients a,b,c,d have the same sign.*

Proof: In view of the elementary theory of a cubic equation [38,41], the hypothesis $\mathcal{D} \geq 0$ guarantees that all the roots of $q(x) = 0$ are real. Suppose that $r_1 < 0$, $r_2 < 0$ and $r_3 < 0$. Then via Vieta's formula we find $-b/a = r_1 + r_2 + r_3 < 0$ and $c/a = r_1r_2 + r_2r_3 + r_3r_1 > 0$ and $-d/a = r_1r_2r_3 < 0$. We easily obtain $ab > 0$, $ac > 0$ and $ad > 0$. Hence, all four coefficients a,b,c,d have the same sign. Conversely, we first suppose that all four coefficients a, b, c, d have the same sign, yielding $ab > 0$, $ac > 0$ and $ad > 0$. Then Vieta's formula again yields three relations $-b/a = r_1 + r_2 + r_3 < 0$, $c/a = r_1r_2 + r_2r_3 + r_3r_1 > 0$ and $-d/a = r_1r_2r_3 < 0$. Substituting $r_3 = (1/r_1r_2)(-d/a)$ from the last relation, we have the two remaining relations below:

$$\begin{aligned} r_1 + r_2 + \frac{1}{r_1r_2}\left(-\frac{d}{a}\right) < 0, \\ r_1r_2 + \frac{r_1 + r_2}{r_1r_2}\left(-\frac{d}{a}\right) > 0. \end{aligned} \tag{32}$$

If $r_1 + r_2 > 0$ held true, then the first relation of Equation (32) multiplied by the negative real number '$-(r_1 + r_2)$' would give $-(r_1 + r_2)^2 - ((r_1 + r_2)/r_1r_2)(-d/a) > 0$. Adding this to the second relation of Equation (32), we obtain $-(r_1 + r_2)^2 + r_1r_2 > 0$. Adding $(r_1 + r_2)^2 \geq 4r_1r_2$, we find $r_1r_2 \leq 0$ giving $r_1 + r_2 + (1/r_1r_2)(-d/a) > 0$, which contradicts the first relation of Equation (32). Hence, $r_1 + r_2 \leq 0$ must hold. If $r_1 + r_2 = 0$ held true, then it would give $r_1r_2 = -r_1^2 > 0$ contradictory to the fact that $-r_1^2 \leq 0$. Therefore, we must have $r_1 + r_2 < 0$, which yields $r_1r_2 > 0$ from the second relation of Equation (32). Consequently, $r_1 < 0$ and $r_2 < 0$. Furthermore, $r_3 = (1/r_1r_2)(-d/a) < 0$. This completes the proof. ■

Remark 4.1: The proof of the converse of the above theorem can be made alternatively by the use of Descartes' Rule of Signs [41]. The number of sign variations in the sequence of coefficients of $q(x)$ is found to be exactly zero. By virtue of Descartes' Rule of Signs, $q(x)$ has no positive real roots, i.e. all its roots are non-positive. Since d has the same sign as $a \neq 0$, it gives nonzero roots of $q(x)$, implying that all the roots are negative.

To begin the detailed study regarding the purely imaginary extraneous fixed points, we now employ weight function L_f in (26) applied to $f(z) = (z^2 - 1)$:

$$s = \frac{1}{4}\left(1 - \frac{1}{z^2}\right), \qquad L_f = \frac{1}{2}\left(\frac{3z^2+1}{z^2+1}\right). \tag{33}$$

Besides, we are able to express K_f in terms of z and free parameters $A_6, A_7, A_8, B_2, B_3, B_4, B_5, B_6, B_7, B_8, d_1, d_2, d_3$ with the use of

$$u = \frac{1}{4} \cdot \frac{(z^2-1)^2}{(z^2+1)^2}. \tag{34}$$

Although such lengthy expression of K_f is not explicitly shown here, the simplified second-order form of L_f will greatly reduce the complexity of K_f as well as the desired $H_f = L_f + K_f$ given by Equation (30). As a result, the explicit form of the relevant $H(z)$ given by Equation (30) becomes

$$H(z) = \frac{1}{2(1+t)} \cdot \frac{G(t; \beta_0, \beta_1, \ldots, \beta_9)}{\Omega(t; \omega_0, \omega_1, \ldots, \omega_8)}, \tag{35}$$

where $G(t; \beta_0, \beta_1, \ldots, \beta_9)$ and $\Omega(t; \omega_0, \omega_1, \ldots, \omega_8)$ are concisely denoted by $G(t)$ and $\Omega(t)$, respectively, as below:

$$G(t) = \sum_{i=0}^{9} \beta_i t^i, \tag{36}$$

with $\beta_0 = 24 + B_4 + 10d_1 + 4d_2 + 2d_3$, $\beta_1 = -112 - 2A_8 - 4B_3 - B_4 - B_8 - 46d_1 - 20d_2 - 22d_3$, $\beta_2 = 4(36 + 2A_7 + 3A_8 + 4B_2 - 2B_4 + B_7 + B_8 + 16d_1 + 20d_2 - 16d_3)$, $\beta_3 = 4(52 + 24A_6 - 26A_7 - 7A_8 + 4B_2 + 8B_3 + 4B_4 - 4B_6 - 3B_7 - 28d_1 + 92d_2 + 4d_3)t^3$, $\beta_4 = 2(224 + 320A_6 - 28A_7 + 14A_8 - 56B_2 - 16B_3 + B_4 + 32B_5 + 16B_6 - 6B_7 - 14B_8 - 830d_1 + 124d_2 + 90d_3)$, $\beta_5 = -2(-3096 + 16A_6 - 140A_7 - 8B_2 + 20B_3 + 13B_4 + 32B_5 - 40B_6 - 50B_7 - 35B_8 + 1550d_1 + 236d_2 - 42d_3)$, $\beta_6 = -4(-4764 + 256A_6 - 54A_7 + 7A_8 - 44B_2 - 16B_3 - 4B_4 + 96B_5 + 80B_6 + 45B_7 + 21B_8 + 252d_1 + 188d_2 + 44d_3)$, $\beta_7 = -4(-6076 + 120A_6 + 94A_7 - 7A_8 + 20B_2 - 2B_4 - 224B_5 - 100B_6 - 39B_7 - 14B_8 - 656d_1 + 20d_2 + 32d_3)$, $\beta_8 = 13{,}096 + 384A_6 - 168A_7 - 12A_8 - 80B_2 - 32B_3 - 11B_4 - 704B_5 - 224B_6 - 68B_7 - 20B_8 + 2594d_1 + 420d_2 + 58d_3$, $\beta_9 = 2176 + 416A_6 + 200A_7 + 2A_8 + 48B_2 + 12B_3 + 3B_4 + 192B_5 + 48B_6 + 12B_7 + 3B_8 + 634d_1 + 204d_2 + 50d_3$, and

$$\Omega(t) = \sum_{i=0}^{8} \omega_i t^i, \tag{37}$$

with $\omega_0 = B_4$, $\omega_1 = -4B_3 - 4B_4 - B_8 - 16d_3$, $\omega_2 = 16B_2 + 12B_3 + 4B_4 + 4B_7 + 7B_8 + 64d_2 - 16d_3$, $\omega_3 = 128 + 128A_6 - 64A_7 - 32B_2 - 4B_3 + 4B_4 - 16B_6 - 24B_7 - 21B_8 - 192d_1 + 128d_2 + 48d_3$, $\omega_4 = 640 + 128A_6 + 64A_7 - 16B_2 - 20B_3 - 10B_4 + 64B_5 + 80B_6 + 60B_7 + 35B_8 - 832d_1 - 64d_2 + 48d_3$, $\omega_5 = 3840 - 256A_6 + 128A_7 + 64B_2 + 20B_3 + 4B_4 - 256B_5 - 160B_6 - 80B_7 - 35B_8 - 640d_1 - 256d_2 - 48d_3$, $\omega_6 = 6912 - 256A_6 - 128A_7 - 16B_2 + 4B_3 + 4B_4 + 384B_5 + 160B_6 + 60B_7 + 21B_8 + 640d_1 - 64d_2 - 48d_3$, $\omega_7 = 4224 + 128A_6 - 64A_7 - 32B_2 - 12B_3 - 4B_4 - 256B_5 - 80B_6 - 24B_7 - 7B_8 + 832d_1 + 128d_2 + 16d_3$, $\omega_8 = 640 + 128A_6 + 64A_7 + 16B_2 + 4B_3 + B_4 + 64B_5 + 16B_6 + 4B_7 + B_8 + 192d_1 + 64d_2 + 16d_3$.

Note that the weight function $L_f(z) = \frac{1}{2}((1+3t)/(1+t))$ with $t = z^2$ contains two factors $(1+3t)$ and $(1+t)$. Hence we naturally consider a special case of $H(z)$ in the form of a simplified rational function possibly with such two factors. To this end, we construct

$$H_f = L_f + K_f = \frac{1}{2(1+t)} \frac{G(t)}{\Omega(t)}, \tag{38}$$

where $G(t)$ and $\Omega(t)$ may involve some of such factors in addition to a factor t corresponding to the origin (considered as purely imaginary) of the complex plane, as shown below:

$$\begin{aligned} G(t) &= t^{\gamma_1}(1+t)^{\gamma_2}(1+3t)^{\gamma_3} \cdot g(t) \quad \text{for } \gamma_1,\gamma_2,\gamma_3 \in \mathbb{N}, \quad \gamma_1+\gamma_2+\gamma_3 = 6, \\ \Omega(t) &= t^{\sigma_1}(1+t)^{\sigma_2}(1+3t)^{\sigma_3} \cdot w(t) \quad \text{for } \sigma_1,\sigma_2,\sigma_3 \in \mathbb{N}, \quad \sigma_1+\sigma_2+\sigma_3 = 4, \end{aligned} \tag{39}$$

where $g(t)$ and $w(t)$ are polynomials of degree at most 3 and 4, respectively. The expression of $H(z)$ in Equation (35) will be further simplified as follows:

$$H(z) = \frac{1}{2} \cdot t^{\gamma_1-\sigma_1}(1+t)^{\gamma_2-\sigma_2-1}(1+3t)^{\gamma_3-\sigma_3} \cdot \frac{g(t)}{w(t)} \quad \text{with } t = z^2. \tag{40}$$

If we further restrict with $\gamma_2 \geq 2$, then all possible combinations of $(\gamma_1,\gamma_2,\gamma_3)$ are listed by $\{(1,2,3),(2,3,1),(1,3,2),(3,2,1),(2,2,2),(1,4,1)\}$. Since all possible combinations of $(\sigma_1,\sigma_2,\sigma_3)$ are listed by $\{(1,1,2),(1,2,1),(2,1,1)\}$, we are able to construct 18 different combinations of $G(t)$ and $\Omega(t)$.

For systematic numbering of all possible 18 cases, we assign not only six letters **A, B, C, D, E, F** to the six triplets of $(\gamma_1,\gamma_2,\gamma_3)$ listed by $\{(1,2,3),(2,3,1),(1,3,2),(3,2,1),(2,2,2),(1,4,1)\}$ in order, but also three letters **X, Y, Z** to the three triplets of $(\sigma_1,\sigma_2,\sigma_3)$ listed by $\{(1,1,2),(1,2,1),(2,1,1)\}$ in order. Hence, combining two letters covers all possible 18 cases. Consequently, **Cases AX, AY, . . . , FZ** shall denote the respective cases when $(\gamma_1,\gamma_2,\gamma_3,\sigma_1,\sigma_2,\sigma_3) = (1,2,3,1,1,2)$, $(\gamma_1,\gamma_2,\gamma_3,\sigma_1,\sigma_2,\sigma_3) = (1,2,3,1,2,1)$, $\ldots$, $(\gamma_1,\gamma_2,\gamma_3,\sigma_1,\sigma_2,\sigma_3) = (1,4,1,2,1,1)$.

In order to obtain purely imaginary extraneous fixed points, we further require that all the roots of $g(t)$ should be negative. Let $g(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3$ and $w(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4$. Then, the roots of $g(t) = 0$ would contribute to the desired extraneous fixed points. In view of the fact that $\gamma_1+\gamma_2+\gamma_3 = 6$ and $\sigma_1+\sigma_2+\sigma_3 = 4$, the forms of Equation (39) would require a set of six constraints

$$\begin{aligned} 0 &= G(0) = G'(0) = \cdots G^{(\gamma_1-1)}(0) = G(-1) = G'(-1) = \cdots G^{(\gamma_2-1)}(-1) = G(-\tfrac{1}{3}) \\ &= G'(-\tfrac{1}{3}) = \cdots G^{(\gamma_3-1)}(-\tfrac{1}{3}) \end{aligned} \tag{41}$$

and additionally a set of four constraints

$$\begin{aligned} 0 &= \Omega(0) = \Omega'(0) = \cdots \Omega^{(\sigma_1-1)}(0) = \Omega(-1) = \Omega'(-1) = \cdots \Omega^{(\sigma_2-1)}(-1) = \Omega(-\tfrac{1}{3}) \\ &= \Omega'(-\tfrac{1}{3}) = \cdots \Omega^{(\sigma_3-1)}(-\tfrac{1}{3}). \end{aligned} \tag{42}$$

Since $G(-1) = -256(8B_5 + 4B_6 + 2B_7 + B_8)$, $\Omega(-1) = 128(8B_5 + 4B_6 + 2B_7 + B_8)$, we find that $G(-1) = -2\Omega(-1)$, from which $G(-1) = 0$ implies $\Omega(-1) = 0$. Consequently, the above 10 constraints reduce to 9 constraints. For the four **Cases AY, BY, CY, EY**, we can solve these nine constraints for nine parameters $A_6, A_7, A_8, B_3, B_4, B_5, B_6, d_1, d_2$ in terms of at most 4 remaining parameters B_2, B_7, B_8, d_3. For remaining 12 cases, we can solve the corresponding 9 constraints for 9 parameters $A_6, A_7, B_2, B_3, B_4, B_5, B_6, d_1, d_2$ in terms of at most 4 remaining parameters A_8, B_7, B_8, d_3. Due to the fact that $\sigma_1 \geq 1$, we immediately find that $B_4 = 0$ from the first equation $\Omega(0) = 0 = B_4$ of

Equation (42). If we substitute these nine parameters back into $G(t)$ and $\Omega(t)$ in Equation (39), the explicit forms of $g(t)$ and $w(t)$ with their coefficients in terms of at most four remaining parameters A_8(or B_2), B_7, B_8, d_3 for a given combination of $(\gamma_1, \gamma_2, \gamma_3)$ and $(\sigma_1, \sigma_2, \sigma_3)$. If a new parameter λ is conveniently introduced as an appropriate affine combination of d_3, A_8(or B_2), B_7, B_8, then all 18 **Cases AX, AY, . . . , FZ**, 10 parameters $A_6, A_7, A_8, B_2, B_3, B_4, B_5, B_6, d_1, d_2$ can be expressed in terms of four parameters d_3, B_7, B_8, λ. After a tedious algebra, the resulting parameters for all 18 cases are already described at the end of Section 3.

The following proposition is useful in the analysis of proposed family of methods (2) in both computational and dynamics aspects.

Proposition 4.2: *For each case, all coefficients of $g(t)$ and $w(t)$ can be expressed as an affine combination of λ.*

Proof: Since one proof is similar to another, it suffices to consider a typical case **AX** with $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$, $(\sigma_1, \sigma_2, \sigma_3) = (1, 1, 2)$ and $\lambda = 4 - A_8 + 2d_3$. Solving the 9 constraints for $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$ and $(\sigma_1, \sigma_2, \sigma_3) = (1, 1, 2)$, we find that $A_6 = \frac{1}{2}(1 + 3A_8 - 5d_3)$, $A_7 = -1 - 3A_8 + 4d_3$, $B_2 = \frac{1}{4}(16 + 25A_8 - B_7 - B_8 - 26d_3)$, $B_3 = -2 + A_8/2 - B_8/4 - 5d_3$, $B_4 = 0$, $B_5 = \frac{1}{4}(-A_8 + B_7 + B_8 + 2d_3)$, $B_6 = \frac{1}{4}(2A_8 - 4B_7 - 3B_8 - 4d_3)$, $d_1 = -3 + A_8 - 2d_3$, $d_2 = \frac{1}{2}(3 - 5A_8 + 9d_3)$. Substituting such nine coefficients into $G(t)$ and $\Omega(t)$, we find

$$
\begin{aligned}
g(t) &= 4[(7 + 4A_8 - 8d_3)t^3 + (35 + 8A_8 - 16d_3)t^2 - 3(-7 + 4A_8 - 8d_3)t + 1] \\
w(t) &= 2[(12 + 7A_8 - 14d_3)t^4 + t^3(88 + 28A_8 - 56d_3) - 14(A_8 - 2(4 + d_3))t^2 \\
&\quad - 20(A_8 - 2(1 + d_3))t + 4 - A_8 + 2d_3].
\end{aligned}
$$

Applying $A_8 = 4 - \lambda + 2d_3$ to the above equations yields:

$$
\begin{aligned}
g(t) &= -4[t^3(4\lambda - 23) + t^2(8\lambda - 67) - 3t(4\lambda - 9) - 1] \\
w(t) &= 2[t^4(40 - 7\lambda) - 4t^3(7\lambda - 50) + 14t^2(4 + \lambda) + 20t(\lambda - 2) + \lambda],
\end{aligned}
$$

completing the proof. ■

Remark 4.3: If we express $w(t)$ at $t = 0$ as a scalar multiple of λ, i.e. $w(0) = h\lambda, h \in \mathbb{R}$, then each of the remaining cases shows that all coefficients of $g(t)$ and $w(t)$ can be expressed as an affine combination of λ. Each selection of λ is shown at the end of Section 3. Special λ-values for interesting forms of $H(z)$ are listed in Table 1.

We are further interested in possible extraneous fixed points from the roots of the cubic equation denoted by

$$g(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3 \tag{43}$$

with $q_i = q_i(\lambda)$, $(0 \le i \le 3)$. The discriminant $\mathcal{D}$ of $g(t)$ can be expressed in terms of parameter λ. We denote a set

$$\mathbf{D} = \{\lambda \in \mathbb{R} : \mathcal{D} \ge 0\}. \tag{44}$$

We further denote a set

$$\mathbf{B} = \{\lambda \in \mathbb{R} : q_3 q_2 > 0 \text{ and } q_3 q_1 > 0 \text{ and } q_3 q_0 > 0\} \tag{45}$$

Table 1. $H(z)$ and λ for the selected values of $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$.

Case	γ	σ	$\frac{g(t)}{w(t)}$	λ	$H(z)$, $t = z^2$
AX1		$(1,1,2)$	$\frac{2[-1-3t(-9+4\lambda)+t^2(-67+8\lambda)+t^3(-23+4\lambda)]}{\lambda+20t(-2+\lambda)+14t^2(4+\lambda)+\nu_3t^3+\nu_4t^4}$,	$\frac{13}{3}$	$\frac{(1+3t)(3+75t+97t^2+17t^3)}{13+140t+350t^2+236t^3+29t^4}$
AX2			$\nu_3 = 4(50-7\lambda)$, $\nu_4 = 40-7\lambda$	4	$\frac{(1+21t+35t^2+7t^3)}{4(1+t)(1+6t+t^2)}$
AX6				5	$\frac{(1+3t)(1+33t+27t^2+3t^3)}{(5+10t+t^2)(1+10t+5t^2)}$
AY4	$(1,2,3)$	$(1,2,1)$	$\frac{2[\lambda-2+t(86+5\lambda)-5t^2(-50+\lambda)+t^3(50-\lambda)]}{\lambda+4t(16+\lambda)+\frac{2}{3}t^2(448+13\lambda)+\nu_3t^3+\nu_4t^4}$,	12	$\frac{4(1+3t)^2(1+17t+25t^2+5t^3)}{(1+t)(15+156t+578t^2+684t^3+103t^4)}$
AY5			$\nu_3 = 12(48-\lambda)$, $\nu_4 = \frac{1}{3}(256-5\lambda)$	16	$\frac{8t(1+3t)^2(2+5t+t^2)}{(1+t)(1+36t+158t^2+276t^3+41t^4)}$
AY6				20	$\frac{8t(1+3t)^2(2+5t+t^2)}{(1+t)(1+36t+158t^2+276t^3+41t^4)}$
BX1		$(1,1,2)$	$\frac{32(1+6t+t^2)(t(-4+\lambda)-\lambda)}{(1+33t+27t^2+3t^3)(t(-4+\lambda)-\lambda)}$	0	$\frac{16t(1+t)(1+6t+t^2)}{(1+3t)(1+33t+27t^2+3t^3)}$
BX6			$= \frac{32(1+6t+t^2)}{1+33t+27t^2+3t^3}$	2	$\frac{16t(1+t)(1+6t+t^2)}{(1+3t)(1+33t+27t^2+3t^3)}$
BY6	$(2,3,1)$	$(1,2,1)$	$\frac{32[t^3(28-3\lambda)+7t^2(20-\lambda)+7t(12+\lambda)+3\lambda+4]}{\nu_4t^4+\nu_3t^3+14t^2(32+\lambda)+4t(16+5\lambda)+\lambda}$, $\nu_4 = 64-7\lambda$, $\nu_3 = 28(16-\lambda)$	24	$\frac{2t(7+35t+21t^2+t^3)}{1+28t+70t^2+28t^3+t^4}$
BZ5		$(2,1,1)$	$\frac{4[t^3(228+\lambda)+t^2(1396+13\lambda)-t(13\lambda-172)-\lambda-4]}{\frac{1}{7}t^4(256+\lambda)+\frac{4}{7}t^3(608+5\lambda)+\nu_2t^2+\nu_1t-\lambda}$, $\nu_1 = 4(32-\lambda)$, $\nu_2 = 2(256+\lambda)$	16	$\frac{1+21t+35t^2+7t^3}{4(1+t)(1+6t+t^2)}$
CX4		$(1,1,2)$	$\frac{4[t^3(57+\lambda)+t^2(349+13\lambda)-t(13\lambda-43)-\lambda-1]}{-\lambda+4t(8-\lambda)+2t^2(64+\lambda)+\nu_3t^3+\nu_4t^4}$, $\nu_3 = \frac{4}{7}(152+5\lambda)$, $\nu_4 = \frac{1}{7}(64+\lambda)$	-8	$\frac{1+21t+35t^2+7t^3)}{4(1+t)(1+6t+t^2)}$
CY1	$(1,3,2)$	$(1,2,1)$	$\frac{-8[t^3(69-\lambda)+t^2(201-2\lambda)+3t(-27+\lambda)+3]}{\lambda+20t(-24+\lambda)+14t^2(48+\lambda)+\nu_3t^3+\nu_4t^4}$,	56	$\frac{(1+3t)(3+87t+89t^2+13t^3)}{2(7+80t+182t^2+104t^3+11t^4)}$
CY2			$\nu_3 = 4(600-7\lambda)$, $\nu_4 = 480-7\lambda$	40	$\frac{(1+3t)(3+39t+121t^2+29t^3)}{2(5+40t+154t^2+160t^3+25t^4)}$
CY6				60	$\frac{(1+3t)(1+33t+27t^2+3t^3)}{(5+10t+t^2)(1+10t+5t^2)}$
DX2		$(1,1,2)$	$\frac{64[t^3(2-3\lambda)+t^2(14-3\lambda)+7t(2+\lambda)-\lambda+2]}{\nu_4t^4+\nu_3t^3+2t^2(66+37\lambda)+t(4-16\lambda)+\lambda}$, $\nu_4 = (12-19\lambda)$, $\nu_3 = (108-40\lambda)$	$-\frac{2}{7}$	$\frac{64t^2(4+21t+26t^2+5t^3)}{(1+t)(1+3t)(-1+31t+357t^2+61t^3)}$
DY7	$(3,2,1)$	$(1,2,1)$	$\frac{16[3t^3(28+3\lambda)+t^2(420+37\lambda)-21t(\lambda-12)+12-25\lambda]}{\lambda-28t\lambda+t^2(384-226\lambda)+\nu_3t^3+\nu_4t^4}$	-4	$\frac{32t^2(7+14t+3t^2)}{-1+28t+322t^2+364t^3+55t^4}$
DY9			$\nu_3 = 4(576+53\lambda)$, $\nu_4 = 384+41\lambda$	0	$\frac{1+21t+35t^2+7t^3}{4(1+t)(1+6t+t^2)}$
EX2		$(1,1,2)$	$\frac{8[t^3(7-3\lambda)-7t^2(-5+\lambda)+7t(3+\lambda)+3\lambda+1]}{\nu_4t^4+\nu_3t^3+14t^2(8+\lambda)+4t(4+5\lambda)+\lambda}$	$\frac{2}{3}$	$\frac{2t(9+77t+91t^2+15t^3)}{1+44t+182t^2+140t^3+17t^4}$
EX6			$\nu_4 = 16-7\lambda$, $\nu_3 = 28(4-\lambda)$	0	$\frac{1+21t+35t^2+7t^3}{4(1+t)(1+6t+t^2)}$
EY6	$(2,2,2)$	$(1,2,1)$	$\frac{8[t^3(42-\lambda)-3t^2(-70+\lambda)+3(2+\lambda)+t(126+\lambda)]}{\lambda+4t(24+5\lambda)+2t^2(432+7\lambda)+\nu_3t^3+\nu_4t^4}$, $\nu_3 = 4(456-7\lambda)$, $\nu_4 = 288-7\lambda$	2	$\frac{8t(1+3t)(3+5t)(1+9t+2t^2)}{(1+t)(1+68t+446t^2+884t^3+137t^4)}$
EZ1		$(2,1,1)$	$\frac{4[t^3(23-2\lambda)+t^2(67-4\lambda)+3t(-9+2\lambda)+1]}{\lambda+20t(-4+\lambda)+14t^2(8+\lambda)+\nu_3t^3+\nu_4t^4}$,	$\frac{44}{5}$	$\frac{(1+3t)(5+129t+159t^2+27t^3)}{2(11+120t+294t^2+192t^3+23t^4)}$
EZ7			$\nu_3 = 400-28\lambda$, $\nu_4 = 80-7\lambda$	10	$\frac{(1+3t)(1+33t+27t^2+3t^3)}{(5+10t+t^2)(1+10t+5t^2)}$
FY4	$(1,4,1)$	$(1,2,1)$	$\frac{4[t^3(228+\lambda)+t^2(1396+13\lambda)-t(13\lambda-172)-\lambda-4]}{-\lambda+4t(32-\lambda)+2t^2(256+\lambda)+\nu_3t^3+\nu_4t^4}$,	-46	$\frac{2(1+t)(3+t)(1+18t+13t^2)}{23+156t+210t^2+108t^3+15t^4}$
FY5			$\nu_3 = \frac{4}{7}(608+5\lambda)$, $\nu_4 = \frac{1}{7}(256+\lambda)$	-32	$\frac{1+21t+35t^2+7t^3}{4(1+t)(1+6t+t^2)}$
FY6				-18	$\frac{2(1+t)(1+3t)(1+26t+5t^2)}{9+100t+238t^2+148t^3+17t^4}$

whose elements make all four coefficients q_0, q_1, q_2, q_3 have the same sign. We now use Lemma 4.1 to locate all three negative roots of $g(t) = 0$ for purely imaginary extraneous fixed points. After a lengthy algebraic process, we are able to find the desired sets **D**, **B** and **D** ∩**B** containing λ-values for which purely imaginary extraneous fixed points can be located.

One should note that extraneous fixed point zeros $\xi = 0$ (being considered as purely imaginary) may be found on the boundary of **B**. Let $\bar{\mathbf{B}}$ denote the closure of **B**. According to interesting values of $\lambda \in \mathbf{D} \cap \bar{\mathbf{B}}$, we classify the sub-cases of each case from **Cases AX, AY, . . . , FZ** by appending sequential arabic numerals such as **Cases AX1, AX2, . . . , FX2, ...**

Presented below are values of $(\gamma_1, \gamma_2, \gamma_3)$, $(\sigma_1, \sigma_2, \sigma_3)$, λ, $g(t)$, **D**, **B** and **D** ∩ **B** for each case under consideration.

Case AX: $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3), (\sigma_1, \sigma_2, \sigma_3) = (1, 1, 2), \lambda = 4 - A_8 + 2d_3$.

(1) $g(t) = 4\,[t^3(-23 + 4\lambda) + t^2(-67 + 8\lambda) - 3t(-9 + 4\lambda) - 1]$.
(2) $\mathbf{D} = \{\lambda : \lambda \le 0.938575 \text{ or } \lambda \ge 3.29184\}$, $\mathbf{B} = \{\lambda : 2.25 < \lambda < 5.75\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{\lambda : 3.29184 \le \lambda < 5.75\}$.

The seven sub-cases **AX1–AX7** are identified with $\lambda \in \{\frac{13}{3}, 4, \frac{17}{5}, \frac{116}{25}, \frac{7}{2}, 5, \frac{23}{4}\}$ in order.

Case AY: $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3), (\sigma_1, \sigma_2, \sigma_3) = (1, 2, 1), \lambda = 56 - 12B_2 - 3B_7 - 3B_8 + 72d_3$.

(1) $g(t) = 2\,[t^3(50 - \lambda) - 5t^2(-50 + \lambda) + t(86 + 5\lambda) + \lambda - 2]$.
(2) $\mathbf{D} = \{\lambda : \lambda \le 20.8093 \text{ or } \lambda \ge 50\}$, $\mathbf{B} = \{\lambda : 2 < \lambda < 50\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{\lambda : 2 < \lambda \le 20.8093\}$.

The six sub-cases **AY1–AY6** are identified with $\lambda \in \{2, 4, 8, 12, 16, 20\}$ in order.

Case AZ: $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3), (\sigma_1, \sigma_2, \sigma_3) = (2, 1, 1), \lambda = 3(16 - 9A_8 + 18d_3)$.

(1) $g(t) = \frac{4}{81}\,[t^3(1203 - 11\lambda) + t^2(4287 - 19\lambda) + t(-327 + 31\lambda) + 21 - \lambda]$.
(2) $\mathbf{D} = \{\lambda : \lambda \le -16.6790 \text{ or } \lambda \ge 17.7879\}$, $\mathbf{B} = \{\lambda : 10.5484 < \lambda < 21\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{\lambda : 17.7879 \le \lambda < 21\}$.

The two sub-cases **AZ1, AZ2** are identified with $\lambda \in \{18, 21\}$ in order.

Case BX: $(\gamma_1, \gamma_2, \gamma_3) = (2, 3, 1), (\sigma_1, \sigma_2, \sigma_3) = (1, 1, 2), \lambda = -A_8 + 2d_3$.

(1) $g(t) = -64\,(1 + 6t + t^2)(t(\lambda - 4) - \lambda)$.
(2) $\mathbf{D} = \mathbb{R}$, $\mathbf{B} = \{\lambda : 0 < \lambda < 4\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{\lambda : 0 < \lambda < 4\}$.

The eight sub-cases **BX1–BX8** are identified with $\lambda \in \{0, \frac{1}{2}, \frac{36}{38}, \frac{2}{5}, 1, 2, 3, 4\}$ in order.

Case BY: $(\gamma_1, \gamma_2, \gamma_3) = (2, 3, 1), (\sigma_1, \sigma_2, \sigma_3) = (1, 2, 1), \lambda = -8 - 4B_2 - B_7 - B_8 + 24d_3$.

(1) $g(t) = -\frac{8}{3}\,[3t^3(-28 + \lambda) + 7t^2(-60 + \lambda) - 7t(36 + \lambda) - 3(4 + \lambda)]$.
(2) $\mathbf{D} = \mathbb{R}$, $\mathbf{B} = \{\lambda : -4 < \lambda < 28\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{\lambda : -4 < \lambda < 28\}$.

The seven sub-cases **BY1–BY7** are identified with $\lambda \in \{-4, 0, 4, 6, 12, 24, 28\}$ in order.

Case BZ: $(\gamma_1, \gamma_2, \gamma_3) = (2, 3, 1), (\sigma_1, \sigma_2, \sigma_3) = (2, 1, 1), \lambda = 2 - 7A_8 + 14d_3$.

(1) $g(t) = \frac{16}{7}\,[t^3(114 - \lambda) + t^2(698 - 13\lambda) + t(86 + 13\lambda) + \lambda - 2]$.
(2) $\mathbf{D} = \{\lambda : \lambda \le 24.7402 \text{ or } \lambda \ge 83.0709\}$, $\mathbf{B} = \{\lambda : 2 < \lambda < \frac{698}{13}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{\lambda : 2 \le \lambda \le 24.7402\}$.

The six sub-cases **BZ1–BZ6** are identified with $\lambda \in \{2, \frac{20}{3}, \frac{38}{5}, \frac{152}{13}, 16, 24\}$ in order.

Case CX: $(\gamma_1, \gamma_2, \gamma_3) = (1, 3, 2), (\sigma_1, \sigma_2, \sigma_3) = (1, 1, 2), \lambda = -8 - 7A_8 + 14d_3$.

(1) $g(t) = -\frac{8}{7}\,[t^3(-57 - \lambda) + t^2(-349 - 13\lambda) + t(-43 + 13\lambda]) + \lambda + 1]$.
(2) $\mathbf{D} = \{\lambda : \lambda \le -41.5354 \text{ or } \lambda \ge -12.3701\}, \mathbf{B} = \{\lambda : -26.8462 < \lambda < -1\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{\lambda : -12.3701 \le \lambda < -1\}$.

The seven sub-cases **CX1–CX7** are identified with $\lambda \in \{-\frac{16}{9}, -12, -\frac{23}{2}, -8, -\frac{9}{2}, -\frac{7}{2}, -1\}$ in order.

Case CY: $(\gamma_1, \gamma_2, \gamma_3) = (1, 3, 2), (\sigma_1, \sigma_2, \sigma_3) = (1, 2, 1), \lambda = 56 - 4B_2 - B_7 - B_8 + 24d_3$.

(1) $g(t) = \frac{8}{3}\,[t^3(69 - \lambda) + t^2(201 - 2\lambda) + 3t(-27 + \lambda) + 3]$.
(2) $\mathbf{D} = \{\lambda : \lambda \le 11.2629 \text{ or } \lambda \ge 39.5021\}, \mathbf{B} = \{\lambda : 27 < \lambda < 69\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{\lambda : 39.5021 \le \lambda < 69\}$.

The seven sub-cases **CY1–CY7** are identified with $\lambda \in \{56, 40, 42, 48, 54, 60, 69\}$ in order.

Case CZ: $(\gamma_1, \gamma_2, \gamma_3) = (1, 3, 2), (\sigma_1, \sigma_2, \sigma_3) = (2, 1, 1),\ \lambda = -40 - 49A_8 + 98d_3$.

(1) $g(t) = \frac{8}{49}\,[t^3(607 - 13\lambda) + t^2(2683 - 15\lambda) + t(-163 + 29\lambda) - \lambda + 9]$.
(2) $\mathbf{D} = \{\lambda : \lambda \le -11.0311 \text{ or } \lambda \ge 8.37459\}, \mathbf{B} = \{\lambda : 5.62069 < \lambda < 9\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{\lambda : 8.37459 \le \lambda < 9\}$.

The two sub-cases **CZ1, CZ2** are identified with $\lambda \in \{\frac{17}{2}, 9\}$ in order.

Case DX: $(\gamma_1, \gamma_2, \gamma_3) = (3, 2, 1), (\sigma_1, \sigma_2, \sigma_3) = (1, 1, 2), \lambda = -8 - 7A_8 + 14d_3$.

(1) $g(t) = 128\,[t^3(2 - 3\lambda) + t^2(14 - 3\lambda) + 7t(2 + \lambda) - \lambda + 2]$.
(2) $\mathbf{D} = \{\lambda : \lambda \le -8.82797 \text{ or } \lambda \ge -0.367756\}, \mathbf{B} = \{\lambda : -2 < \lambda < \frac{2}{3}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{\lambda : -0.367756 \le \lambda < \frac{2}{3}\}$.

The seven sub-cases **DX1–DX7** are identified with $\lambda \in \{\frac{1}{6}, -\frac{2}{7}, \frac{2}{5}, \frac{18}{43}, -\frac{1}{3}, 0, \frac{2}{3}\}$ in order.

Case DY: $(\gamma_1, \gamma_2, \gamma_3) = (3, 2, 1), (\sigma_1, \sigma_2, \sigma_3) = (1, 2, 1),\ \lambda = A_8 - 2(1 + d_3)$.

(1) $g(t) = \frac{16}{3}\,[3t^3(28 + 3\lambda) + t^2(420 + 37\lambda) - 21t(-12 + \lambda) + 12 - 25\lambda]$.
(2) $\mathbf{D} = \{\lambda :\ \lambda \le -8.64561 \text{ or } \lambda \ge -6\},\ \mathbf{B} = \{\lambda : -\frac{28}{3} < \lambda < 0.48\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{\lambda : -\frac{28}{3} < \lambda \le -8.64561 \text{ or } -6 \le \lambda < 0.4869\}$.

The nine sub-cases **DY1–DY9** are identified with $\lambda \in \{-\frac{24}{17}, -\frac{56}{15}, -\frac{84}{25}, -\frac{28}{3}, -9, -6, -4, -2, 0\}$ in order.

Case DZ: $(\gamma_1, \gamma_2, \gamma_3) = (3, 2, 1), (\sigma_1, \sigma_2, \sigma_3) = (2, 1, 1),\ \lambda = 2 - A_8 + 2d_3$.

(1) $g(t) = 32\,[t^3(14 - 3\lambda) - 7t^2(-10 + \lambda) + 7t(6 + \lambda]) + 3\lambda + 29]$.
(2) $\mathbf{D} = \mathbb{R},\ \mathbf{B} = \{\lambda : -\frac{2}{3} < \lambda < \frac{14}{3}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{\lambda : -\frac{2}{3} < \lambda < \frac{14}{3}\}$.

The nine sub-cases **DZ1–DZ9** are identified with $\lambda \in \{\frac{12}{5}, \frac{56}{19}, \frac{14}{5}, -\frac{2}{3}, 0, 1, 2, 4, \frac{14}{3}\}$ in order.

Case EX: $(\gamma_1,\gamma_2,\gamma_3)=(2,2,2),(\sigma_1,\sigma_2,\sigma_3)=(1,1,2),\ \lambda=-A_8+2d_3$.

(1) $g(t)=16\,[t^3(7-3\lambda)-7t^2(-5+\lambda)+7t(3+\lambda)+3\lambda+1]$.
(2) $\mathbf{D}=\mathbb{R},\ \mathbf{B}=\{\lambda:-\frac{1}{3}<\lambda<\frac{7}{3}\}$.
(3) $\mathbf{D}\cap\mathbf{B}=\{\lambda:-\frac{1}{3}<\lambda<\frac{7}{3}\}$.

The eight sub-cases **EX1–EX8** are identified with $\lambda\in\{\frac{1}{6},\frac{2}{3},\frac{24}{25},\frac{2}{5},-\frac{1}{3},0,1,\frac{7}{3}\}$ in order.
Case EY: $(\gamma_1,\gamma_2,\gamma_3)=(2,2,2),(\sigma_1,\sigma_2,\sigma_3)=(1,2,1),\ \lambda=24-4B_2-B_7-B_8+24d_3$.

(1) $g(t)=\frac{8}{3}\,[t^3(42-\lambda)-3t^2(-70+\lambda)+t(126+\lambda)+3(2+\lambda)]$.
(2) $\mathbf{D}=\{\lambda:\lambda\le-8.64561$ or $\lambda\ge-6\}$, $\mathbf{B}=\{\lambda:-2<\lambda<42\}$.
(3) $\mathbf{D}\cap\mathbf{B}=\{\lambda:-2<\lambda\le 6.18678\}$.

The six sub-cases **EY1–EY6** are identified with $\lambda\in\{\frac{1}{24},\frac{24}{5},-2,0,2,6\}$ in order.
Case EZ: $(\gamma_1,\gamma_2,\gamma_3)=(2,2,2),(\sigma_1,\sigma_2,\sigma_3)=(2,1,1),\ \lambda=8-A_8+2d_3$.

(1) $g(t)=16\,[t^3(23-2\lambda)+t^2(67-4\lambda)+3t(-9+2\lambda)+1]$.
(2) $\mathbf{D}=\lambda\le 1.87715$ or $\lambda\ge 6.58369$, $\mathbf{B}=\{\lambda:\frac{9}{2}<\lambda<\frac{23}{2}\}$.
(3) $\mathbf{D}\cap\mathbf{B}=\{\lambda:6.58369\le\lambda<\frac{23}{2}\}$.

The eight sub-cases **EZ1–EZ8** are identified with $\lambda\in\{\frac{44}{5},\frac{176}{19},4,7,8,9,10,\frac{23}{2}\}$ in order.
Case FX: $(\gamma_1,\gamma_2,\gamma_3)=(1,4,1),(\sigma_1,\sigma_2,\sigma_3)=(1,1,2),\ \lambda=64+25A_8-50d_3$.

(1) $g(t)=\frac{32}{175}\,[t^3(658+3\lambda)+t^2(4382-13\lambda)+t(574+9\lambda)-14+\lambda]$.
(2) $\mathbf{D}=\lambda\le 132.648$ or $\lambda\ge 2565.89$, $\mathbf{B}=\{\lambda:14<\lambda<337.0769\}$.
(3) $\mathbf{D}\cap\mathbf{B}=\{\lambda:14<\lambda\le 132.6478\}$.

The six sub-cases **FX1–FX6** are identified with $\lambda\in\{64,\frac{133}{2},\frac{4802}{93},14,60,132\}$ in order.
Case FY: $(\gamma_1,\gamma_2,\gamma_3)=(1,4,1),(\sigma_1,\sigma_2,\sigma_3)=(1,2,1),\ \lambda=-18-7A_8+14d_3$.

(1) $g(t)=\frac{4}{7}\,[t^3(-228-\lambda)+t^2(-1396-13\lambda)+t(-172+13\lambda)+\lambda+4]$.
(2) $\mathbf{D}=\{\lambda:\lambda\le-166.1417$ or $\lambda\ge-49.4805\}$, $\mathbf{B}=\{\lambda:-107.3846<\lambda<-4\}$.
(3) $\mathbf{D}\cap\mathbf{B}=\{\lambda:-49.4805\le\lambda<-4\}$.

The nine sub-cases **FY1–FY9** are identified with $\lambda\in\{-\frac{68}{3},-\frac{104}{5},-\frac{184}{11},-46,-32,-18,-14,-10,-4\}$ in order.
Case FZ: $(\gamma_1,\gamma_2,\gamma_3)=(1,4,1),(\sigma_1,\sigma_2,\sigma_3)=(2,1,1),\ \lambda=-338-119A_8+238d_3$.

(1) $g(t)=\frac{8}{119}\,[t^3(-5426-109\lambda)+t^2(-10,618+39\lambda)+t(874+73\lambda)-3\lambda-62]$.
(2) $\mathbf{D}=\lambda\le-18.8202$ or $\lambda\ge 18.8087$, $\mathbf{B}=\{\lambda:-20.6667<\lambda<-11.9726\}$.
(3) $\mathbf{D}\cap\mathbf{B}=\{\lambda:-20.6667<\lambda\le-18.8202\}$.

The three sub-cases **FZ1–FZ3** are identified with $\lambda\in\{-\frac{62}{3},-20,-19\}$ in order.

Despite the availability of rich sub-cases considered thus far, we typically list $(\gamma_1,\gamma_2,\gamma_3)$, $(\sigma_1,\sigma_2,\sigma_3)$, $g(t)/w(t),\lambda$ and $H(z)$ in Table 1, for selected 25 sub-cases **AX1, AX2, AX6, AY4, AY5, AY6, BX1, BX6, BY6, BZ5, CX4, CY1, CY2, CY6, DX2, DY7, DY9, EX2, EX6, EY6, EZ1, EZ7**, **FY4, FY5, FY6** with simplified forms of $K_f(s,u)$. Besides, the extraneous fixed points for the selected 25 sub-cases are listed in Table 2 together with those extraneous fixed points of existing method **SA**.

In view of the analysis done so far and a close inspection of Table 1, the following remark is useful.

Remark 4.4: (i) Once λ is chosen, we have freedom to select parameters d_3, B_7, B_8. Note that $H(z)$ can be obtained without specifying parameter values of d_3, B_7, B_8 for all selected cases.

(ii) Three cases **(AX6, CY6, EZ7)** (highlighted in yellow) give the same $H(z) = (1+3t)(1+33t+27t^2+3t^3)/\{(5+10t+t^2)(1+10t+5t^2)\}$ which is also the same as **SA**, two cases **(BX1, BX6)** (highlighted in green) the same $H(z) = 16t(1+t)(1+6t+t^2)/\{(1+3t)(1+33t+27t^2+3t^3)\}$, and six cases **(AX2, BZ5, CX4, DY9, EX6, FY5)** (highlighted in cyan) the same $H(z) = (1+21t+35t^2+7t^3)/\{4(1+t)(1+6t+t^2)\}$.

4.2. Stability of extraneous fixed points

As a result of the case studies pursued thus far for $f(z) = z^2 - 1$, we include in Table 2 the desired purely imaginary extraneous fixed points in typical sub-cases. By direct computation of absolute values of multipliers $R'_f(\xi)$ for iterative map (29) with $f(z) = z^2 - 1$, we find that all of the purely imaginary extraneous fixed points ξ of H in each of the listed cases in Table 2 are found to be indifferent except for extraneous fixed point double 0. The extraneous fixed point double 0 for each of **Cases BX1, BX6, BY6, EX2, EY6** is found to be repulsive and highlighted by a framed-value. Interestingly, no case with attractive extraneous fixed points has been found. The following proposition describes the details of stabilities of the multipliers for the all the cases **AX, AY, ..., FZ**.

Proposition 4.5: *Let $\pm\xi$ be the extraneous fixed points obtained from the expression $g(t)/w(t)$ of $H(z)$ in Equation(40). Then stabilities of the possible extraneous fixed points $0, \pm i, \pm i/\sqrt{3}$ and $\pm\xi$ for the 18 cases* ***AX, AY, ..., FZ*** *are characterized by the following:*

Table 2. Extraneous fixed points ξ and their stability for selected cases.

Case	ξ	No. of ξ
AX1	±2.18932i, ±0.932983i, $\pm i/\sqrt{3}$, ±0.205661i	8
AX2	±2.07652i, ±0.797473i, ±0.228243i	6
AX6	±2.74748i, ±1.19175i, $\pm i/\sqrt{3}$, ±0.176327i	8
AY4	±2.01802i, ±0.922879i, $\pm i/\sqrt{3}$(double), ±0.275446i	10
AY5	±1.92767i, ±1.09135i, $\pm i/\sqrt{3}$(double), ±0.305019i	10
AY6	±1.73205i, ±1.37638i, $\pm i/\sqrt{3}$(double)i, ±0.32492i	10
BX1	±2.41421i, ±i, ±0.414214i, [0](double)	8
BX6	±2.41421i, ±i, ±0.414214i, [0](double)	8
BY6	±4.38129i, ±1.25396i, ±0.481575i, [0](double)	8
BZ5	±2.07652i, ±0.797473i, ±0.228243i	6
CX4	±2.07652i, ±0.797473i, ±0.228243i	6
CY1	±2.38198i, ±1.06609i, $\pm i/\sqrt{3}$, ±0.189172i	8
CY2	±1.95657i, $\pm i/\sqrt{3}$, ±0.472367i, ±0.348006i	8
CY6	±2.74748i, ±1.19175i, $\pm i/\sqrt{3}$, ±0.176327i	8
DX2	±2.06341i, ±0.809824i, ±0.535264i, 0(quadruple)	10
DY7	±2.02415i, ±0.754652i, 0(quadruple)	8
DY9	±2.07652i, ±0.797473i, ±0.228243i	6
EX2	±2.25373i, ±0.920924i, ±0.373208i, [0](double)	8
EX6	±2.07652i, ±0.797473i, ±0.228243i	6
EY6	±2.13578i, ±0.662153i, $\pm i/\sqrt{3}$(double), [0](double)	10
EZ1	±2.21963i, ±0.959862i, $\pm i/\sqrt{3}$, ±0.201983i	8
EZ7	±2.74748i, ±1.19175i, $\pm i/\sqrt{3}$, ±0.176327i	8
FY4	±1.73205i, ±1.15179i, ±i, ±0.240798i	8
FY5	±2.07652i, ±0.797473i, ±0.228243i	6
FY6	±2.27184i, ±i, $\pm i/\sqrt{3}$, ±0.196851i	8
SA	±2.74748i, ±1.19175i, $\pm i/\sqrt{3}$, ±0.176327i	8

Note: In this table, most extraneous fixed points are indifferent, while boxed-values are repulsive extraneous fixed points. Interestingly, no attractive extraneous fixed points exist for the selected cases.

(1) *The extraneous fixed points quadruple 0,* $\pm\mathrm{i}$ *(simple or double),* $\pm\mathrm{i}/\sqrt{3}$ *(simple or double) and* $\pm\xi$ *are all found to be indifferent.*
(2) *The extraneous fixed point double 0 is found to be repulsive.*

Proof: To prove (1) and (2), it should suffice to take several typical cases **AX, AY, BX, DY, FX, BY, DZ, EX, EY** as follows:

(i) Case **AX** for extraneous fixed points $\pm\mathrm{i}/\sqrt{3}$ (simple) and $\pm\xi$.

The corresponding $H(z)$ for **Case AX** found to be:

$$H(z) = \frac{(1+3t)[-1+t(27-12\lambda)+t^2(8\lambda-67)+t^3(4\lambda-23)]}{-\lambda-20t(\lambda-2)-14t^2(4+\lambda)+4t^3(7\lambda-50)+t^4(7\lambda-40)}, \tag{46}$$

where $t = z^2$ and λ is described earlier in Section 4.1. Besides, the derivative of iterative map R_f in Equation (29) is given by

$$R_f'(z) = \frac{(t-1)[-1+t(22-10\lambda)-10t^2(\lambda+2)+2t^3(9\lambda-59)+t^4(2\lambda-11)]}{2t[-\lambda-20t(\lambda-2)-14t^2(\lambda+4)+4t^3(7\lambda-50)+t^4(7\lambda-40)]}. \tag{47}$$

By direct substitution of the extraneous fixed points $z = \pm\mathrm{i}/\sqrt{3}$ (simple), i.e. $t = -\frac{1}{3}$ into $R_f'(z)$, we immediately find $R_f'(\pm\mathrm{i}/\sqrt{3}) = 1$. We now let the extraneous fixed points $\pm\xi$ satisfy

$$-1+t(27-12\lambda)+t^2(8\lambda-67)+t^3(4\lambda-23) = 0$$

with $t = \xi^2$. For brevity, we first denote the left side of the above equation by $d_\lambda(t) = -1+t(27-12\lambda)+t^2(8\lambda-67)+t^3(4\lambda-23)$. Then the second factor of the numerator of Equation (47) is given by

$$-1+t(22-10\lambda)-10t^2(\lambda+2)+2t^3(9\lambda-59)+t^4(2\lambda-11) = q_\lambda(t)\cdot d_\lambda(t)+r_\lambda(t) = r_\lambda(t),$$

where

$$q_\lambda(t) = \frac{1977-664\lambda+56\lambda^2+t(253-90\lambda+8\lambda^2)}{(23-4\lambda)^2}$$

and

$$r_\lambda(t) = -\frac{\begin{array}{c}8(-181+60\lambda-5\lambda^2+t(5186-4028\lambda+910\lambda^2-64\lambda^3)+\\ t^2(-14,381+7056\lambda-1161\lambda^2+64\lambda^3)\end{array}}{(23-4\lambda)^2}.$$

Hence, Equation (47) at this extraneous fixed points $\pm\xi$ becomes

$$R_f'(z) = \frac{(t-1)r_\lambda(t)}{2t[-\lambda-20t(\lambda-2)-14t^2(\lambda+4)+4t^3(7\lambda-50)+t^4(7\lambda-40)]}. \tag{48}$$

Since $(t-1)r_\lambda(t) - 2t[-\lambda-20t(\lambda-2)-14t^2(\lambda+4)+4t^3(7\lambda-50)+t^4(7\lambda-40)]$

$$= -\frac{2d_\lambda(t)[-4(181-60\lambda+5\lambda^2)+t(1920-655\lambda+56\lambda^2)+t^2(920-321\lambda+28\lambda^2)]}{(-23+4\lambda)^2} = 0$$

in view of the fact $d_\lambda(t) = 0$, we find $R_f'(\pm\xi) = 1$.

(ii) Case **AY** for extraneous fixed points $\pm\mathrm{i}/\sqrt{3}$ (double).

The corresponding $H(z)$ and $R_f'(z)$ for **Case AY** are found to be:

$$\begin{aligned} H(z) &= \frac{(1+3t)^2[2-\lambda+t(-86-5\lambda)+5t^2(\lambda-50)+t^3(\lambda-50)]}{-3\lambda-12t(16+\lambda)+t^2(-896-26\lambda)+36t^3(-48+\lambda)+t^4(-256+5\lambda)}, \\ R_f'(z) &= \frac{(t-1)[2-\lambda-6t(12+\lambda)-4t^2(109+4\lambda)+t^4(-62+\lambda)+22t^3(-44+\lambda)]}{2t[t^2(-896-26\lambda)+36t^3(-48+\lambda)-3\lambda-12t(16+\lambda)+t^4(-256+5\lambda)]}. \end{aligned} \tag{49}$$

By direct substitution of the extraneous fixed points $z = \pm i/\sqrt{3}$ (double), i.e. $t = -\frac{1}{3}$ into $R_f'(z)$, we immediately find $R_f'(\pm i/\sqrt{3}) = 1$.

(iii) Case **BX** for extraneous fixed points $\pm$i and 0 (double).

The corresponding $H(z)$ and $R_f'(z)$ for **Case BX** are found to be:

$$\begin{aligned} H(z) &= \frac{16t(1+t)(1+6t+t^2)}{(1+3t)(1+33t+27t^2+3t^3)}, \\ R_f'(z) &= \frac{(t-1)(7+35t+21t^2+t^3)}{(1+3t)(1+33t+27t^2+3t^3)}. \end{aligned} \tag{50}$$

By direct substitution of the extraneous fixed points $z = \pm i$ and 0 (double), i.e. $t = -1$ and $t = 0$, respectively, into $R_f'(z)$, we immediately find $R_f'(\pm i) = 1$ and $R_f'(0) = -7$, respectively.

(iv) Case **DY** for extraneous fixed point 0 (quadruple).

The corresponding $H(z)$ and $R_f'(z)$ for **Case DY** are found to be:

$$\begin{aligned} H(z) &= \frac{8t^2[12-25\lambda-21t(-12+\lambda)+t^2(420+37\lambda)+t^3(84+9\lambda)]}{(1+t)[\lambda-28t\lambda+t^2(384-226\lambda)+4t^3(576+53\lambda)+t^4(384+41\lambda)]}, \\ R_f'(z) &= \frac{(t-1)[-\lambda+t(48-73\lambda)+t^2(672+69\lambda)+t^3(48+5\lambda)]}{\lambda-28t\lambda+t^2(384-226\lambda)+4t^3(576+53\lambda)+t^4(384+41\lambda)}. \end{aligned} \tag{51}$$

By direct substitution of the extraneous fixed point 0 (quadruple), i.e. $t = 0$(double) into $R_f'(z)$, we immediately find $R_f'(0) = 1$.

(v) Case **FX** for extraneous fixed point $\pm$i (double).

The corresponding $H(z)$ and $R_f'(z)$ for **Case FX** are found to be:

$$\begin{aligned} H(z) &= \frac{8(1+t)^2[-14+t(574+9\lambda)+t^2(4382-13\lambda)+\lambda+t^3(658+3\lambda)]}{(1+3t)[25\lambda+4t(2919+4\lambda)+t^2(21,532-38\lambda)+t^3(10,612-8\lambda)+5t^4(196+\lambda)]}, \\ R_f'(z) &= \frac{(t-1)[4(\lambda-14)+t(2072+27\lambda)+4t^2(3661+\lambda)+t^3(20,132-38\lambda)+7700t^4+t^5(308+3\lambda)]}{t(1+3t)[25\lambda+4t(2919+4\lambda)+t^2(21,532-38\lambda)+t^3(10,612-8\lambda)+5t^4(196+\lambda)]}. \end{aligned} \tag{52}$$

By direct substitution of the extraneous fixed point $\pm$i (double), i.e. $t = -1$(double) into $R_f'(z)$, we immediately find $R_f'(\pm i) = 1$.

(vi) Cases **BY, DZ, EX, EY** for extraneous fixed point 0 (double).

The corresponding $H(z)$ and $R_f'(z)$ for these cases can be similarly found as obtained so far. Here, we list their respective multipliers at double 0 by means of λ as follows:

$$R_f'(0) = \begin{cases} -5-24/\lambda & \text{for } -4 < \lambda < 28, \\ -5-4/\lambda & \text{for } -2/3 < \lambda < 14/3, \\ -5-2/\lambda & \text{for } -1/3 < \lambda < 7/3, \\ -5-12/\lambda & \text{for } -2 < \lambda \le 6.18678. \end{cases} \tag{53}$$

After a close examination, we find that $|R_f'(0)| > 1$, implying the repulsiveness of these multipliers.

The stabilities of remaining cases can be similarly shown as those of the above typical cases, completing the proof. ■

Remark 4.6: Among all selected cases, no case with attractive extraneous fixed points has been found. It is interesting to observe that the extraneous fixed point double 0 is found to be repulsive, while the extraneous fixed point quadruple 0 is found to be indifferent throughout the selected cases.

In case that $f(z)$ is a generic polynomial rather than $z^2 - 1$, it would be certainly interesting to investigate the dynamics underlying the relevant extraneous fixed points. However, due to the increased algebraic complexity, we would encounter difficulties in describing the dynamics underlying the extraneous fixed points. An effective way of exploring such dynamics is to illustrate basins of attraction under iterative map (29) with $f(z)$ as a generic polynomial. We will illustrate the basins of attraction to pursue the dynamics of the iterative map R_p of the form

$$z_{n+1} = R_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)} H_p(z_n), \tag{54}$$

for a generic polynomial $p(z_n)$ and a weight function $H_p(z_n)$. Indeed, basins of attraction for the fixed points or the extraneous fixed points as well as their attracting periodic orbits would reflect complex dynamics whose illustrative description will be made for various polynomials in the latter part of Section 5.

Before closing this section, we prefix the iterative maps in Table 2 corresponding to cases **AX1, AX2, AX6, AY4, AY5, AY6, BX1, BX6, BY6, BZ5, CX4, CY1, CY2, CY6, DX2, DY7, DY9, EX2, EX6, EY6, EZ1, EZ7**, **FY4, FY5, FY6** with **W** for later use in describing the relevant dynamics. In addition, we identify map **SA** for method (1).

5. Numerical experiments and complex dynamics

In this section, we first deal with computational aspects of proposed family of methods (2) for a variety of test functions along with an existing competitive method **SA**; then we discuss the dynamics underlying extraneous fixed points based on iterative maps (54) by illustrating the relevant basins of attraction. In Section 4, we were able to find extraneous fixed points using λ-values without specifying parameters d_3, B_7, B_8. For numerical experiments in both computational and dynamical aspects, we need to provide all 10 coefficients $A_6, A_7, A_8, B_2, B_3, B_4, B_5, B_6, d_1, d_2$ of $K_f(s, u)$ for a given λ. For simplified $K_f(s, u)$, we set $d_3 = B_7 = B_8 = 0$. Table 3 shows the desired parameter values and $K_f(s, u)$ for the selected cases **AX1, AX2, AX6, AY4, AY5, AY6, BX1, BX6, BY6, BZ5, CX4, CY1, CY2, CY6, DX2, DY7, DY9, EX2, EX6, EY6, EZ1, EZ7**, **FY4, FY5, FY6**. Each case has been implemented to verify the theoretical convergence. Later on in this section, we will explore the complex dynamics with the use of illustrated basins of attraction of selected rational iterative maps **WAX1** through **WFY6** and an existing method **SA**.

A number of numerical experiments have been implemented with Mathematica programming to confirm the developed theory. Throughout these experiments, we have maintained 160 digits of minimum number of precision, via Mathematica command $\$MinPrecision = 160$, to achieve the specified accuracy. In case that α is not exact, it is replaced by a more accurate value which has more number of significant digits than the preassigned number $\$MinPrecision = 160$.

Definition 5.1 (Computational convergence order): Assume that theoretical asymptotic error constant $\eta = \lim_{n\to\infty} (|e_n|/|e_{n-1}|^p)$ and convergence order $p \geq 1$ are known. Define $p_n = \log|e_n/\eta|/\log|e_{n-1}|)$ as the computational convergence order. Note that $\lim_{n\to\infty} p_n = p$.

Remark 5.1: Note that p_n requires knowledge at two points x_n, x_{n-1}, while the usual COC (computational order of convergence) $\log(|x_n - x_{n-1}|/|x_{n-1} - x_{n-2}|)/\log(|x_{n-1} - x_{n-2}|/|x_{n-2} - x_{n-3}|)$

Table 3. Parameter values of $\lambda, A_6, A_7, A_8, B_2, B_3, B_4, B_5, B_6, d_1, d_2$ and $K_f(s,u)$ for all selected cases as well as **SA**.

Case	λ	(A_6, A_7, A_8)	$(B_2, B_3, B_4, B_5, B_6)$	(d_1, d_2)	$K_f(s,u)$
AX1	$\frac{13}{3}$	$(0, 0, -\frac{1}{3})$	$(\frac{23}{12}, -\frac{13}{6}, 0, \frac{1}{12}, -\frac{1}{6})$	$(-\frac{10}{3}, \frac{7}{3})$	$\frac{4su(3-4s-s^2(u-2))}{12-12u+u^2-2s(20-8u+u^2)+s^2(28+23u)-26s^3u}$
AX2	4	$(\frac{1}{2}, -1, 0)$	$(4, -2, 0, 0, 0)$	$(-3, \frac{3}{2})$	$\frac{su(2+u-2s(1+u)+s^2)}{2-u-2s(3+u)+s^2(3+8u)-4s^3u}$
AX6	5	$(-1, 2, -1)$	$(-\frac{9}{4}, -\frac{5}{2}, 0, \frac{1}{4}, -\frac{1}{2})$	$(-4, 4)$	$\frac{4su(1-s)^2(1-u)}{4-8u+u^2-2s(8-12u+u^2)-s^2(9u-16)-10s^3u}$
AY4	12	$(\frac{1}{3}, -\frac{2}{3}, 0)$	$(\frac{11}{3}, -3, 0, 0, 0)$	$(-\frac{10}{3}, \frac{7}{3})$	$\frac{su(3+u-2s(2+u)+2s^2)}{(1-s)(3-2u+9s^2u-s(7+2u))}$
AY5	16	$(\frac{1}{6}, -\frac{1}{3}, 0)$	$(\frac{10}{3}, -4, 0, 0, 0)$	$(-\frac{11}{3}, \frac{19}{6})$	$\frac{su(6+5s^2+u-2s(5+u))}{6-s(22-6u)-5u+s^2(19+20u)-24s^3u}$
AY6	20	$(0, 0, 0)$	$(3, -5, 0, 0, 0)$	$(-4, 4)$	$\frac{su(1-s)^2}{1+2s(u-2)-u+s^2(4+3u)-5s^3u}$
BX1	0	$(0, -1, 0)$	$(\frac{9}{2}, 0, 0, \frac{1}{2}, -1)$	$(-2, -1)$	$\frac{2su(1-su)}{2-2u+u^2-2s(2+u+u^2)+s^2(9u-2)}$
BX6	2	$(-1, 3, -2)$	$(-5, -1, 0, 0, 0)$	$(-4, 4)$	$\frac{su(1-s)(1-u+s(2u-1))}{1-2u-s(4-7u)-s^2(5u-4)-s^3u}$
BY6	24	$(-\frac{3}{2}, 4, -2)$	$(-8, -2, 0, 0, 0)$	$(-5, \frac{13}{2})$	$\frac{su(2-3u-2s(3-4u)-s^2(4u-3))}{2-5u-10s(1-2u)-s^2(16u-13)-4s^3u}$
BZ5	16	$(-\frac{1}{2}, 2, -2)$	$(-2, 0, 0, 0, 0)$	$(-3, \frac{3}{2})$	$\frac{su(2-u-2s(1-2u)-s^2(4u-1))}{2-3u-2s(3-4u)-s^2(4u-3)}$
CX4	-8	$(0, 0, 0)$	$(2, -4, 0, 0, 0)$	$(-4, 4)$	$\frac{su(1-s)^2}{(1-2s)(1-u-2s+2s^2u)}$
CY1	56	$(-\frac{1}{3}, \frac{2}{3}, 0)$	$(0, -\frac{14}{3}, 0, 0, 0)$	$(-\frac{14}{3}, \frac{17}{3})$	$\frac{su(2s-1)(2s+u-3)}{3-4u+2s(6u-7)+17s^2-14s^3u}$
CY2	40	$(\frac{1}{3}, -\frac{2}{3}, 0)$	$(4, -\frac{10}{3}, 0, 0, 0)$	$(-\frac{10}{3}, \frac{7}{3})$	$\frac{su(3+u-2s(2+u)+2s^2)}{(1-s)(3-2u-s(7+2u)+10s^2u)}$
CY6	60	$(-\frac{1}{2}, 1, 0)$	$(-1, -5, 0, 0, 0)$	$(-5, \frac{13}{2})$	$\frac{su(2-u+2s(u-3)+3s^2)}{2-3u+10s(u-1)-s^2(2u-13)-10s^3u}$
DX2	$-\frac{2}{7}$	$(\frac{9}{7}, -\frac{19}{7}, \frac{2}{7})$	$(\frac{53}{7}, \frac{1}{7}, 0, 0, 0)$	$(-\frac{12}{7}, -\frac{12}{7})$	$\frac{su(7+9u+s(2-19u)+s^2(2u-1))}{7+2u-3s(4+13u)+s^2(53u-12)+s^3u}$
DY7	-4	$(-\frac{1}{3}, \frac{5}{3}, -2)$	$(-1, \frac{1}{3}, 0, 0, 0)$	$(-\frac{8}{3}, \frac{2}{3})$	$\frac{su(3-u-s(2-5u)-s^2(6u-1))}{3-4u+s(9u-8)+s^2(2-3u)+s^3u}$
DY9	0	$(\frac{5}{2}, -6, -2)$	$(14, 0, 0, 0, 0)$	$(-1, -\frac{7}{2})$	$\frac{su(2+5u+2s(1-6u)+s^2(4u-1))}{2+3u-2s(1+12u)+7s^2(4u-1)}$
EX2	$\frac{2}{3}$	$(0, 0, -\frac{2}{3})$	$(\frac{11}{6}, -\frac{1}{3}, 0, \frac{1}{6}, -\frac{1}{3})$	$(-\frac{8}{3}, \frac{2}{3})$	$\frac{2su(3-2s-s^2(2u-1))}{6-6u+u^2-2s(8-2u+u^2)+s^2(4+11u)-2s^3u}$
EX6	0	$(1, -2, 0)$	$(6, 0, 0, 0, 0)$	$(-2, -1)$	$\frac{su(1+u-2su)}{1-2s(1+2u)+s^2(6u-1)}$
EY6	6	$(\frac{3}{4}, -\frac{3}{2}, 0)$	$(\frac{9}{2}, -\frac{1}{2}, 0, 0, 0)$	$(-\frac{5}{2}, \frac{1}{4})$	$\frac{su(4+3u-2s(1+3u)+s^2)}{4-u-10s(1+u)+s^2(1+18u)-2s^3u}$
EZ1	$\frac{44}{5}$	$(0, 0, -\frac{4}{5})$	$(\frac{11}{5}, 0, 0, \frac{1}{5}, -\frac{2}{5})$	$(-\frac{12}{5}, 0)$	$\frac{su(5-2s-s^2(4u-1))}{5-5u+u^2-2s(6-u+u^2)+11s^2u}$
EZ7	10	$(-\frac{3}{2}, 3, -2)$	$(-\frac{7}{2}, 0, 0, \frac{1}{2}, -1)$	$(-3, \frac{3}{2})$	$\frac{su(2-3u-2s(1-3u)-s^2(4u-1))}{2-5u+u^2-2s(3-7u+u^2)-s^2(7u-3)}$
FY4	-46	$(\frac{5}{4}, -\frac{9}{2}, 4)$	$(\frac{23}{2}, -\frac{23}{2}, 0, 0, 0)$	$(-\frac{11}{2}, \frac{31}{4})$	$\frac{su(4+5u-2s(7+9u)+s^2(7+16u))}{4+u-2s(11+7u)+s^2(31+46u)-46s^3u}$
FY5	-32	$(\frac{1}{2}, -2, 2)$	$(6, -8, 0, 0, 0)$	$(-5, \frac{13}{2})$	$\frac{su(2+u-2s(3+2u)+s^2(3+4u))}{2-u-10s+s^2(13+12u)-16s^3u}$
FY6	-18	$(-\frac{1}{4}, \frac{1}{2}, 0)$	$(\frac{1}{2}, -\frac{9}{2}, 0, 0, 0)$	$(-\frac{9}{2}, \frac{21}{4})$	$\frac{su(4-u+2s(u-5)+5s^2)}{4-5u+2s(7u-9)+s^2(21+2u)-18s^3u}$
SA	N/A^*	$(-1, 2, -1)$	$(-2, -4, 2, 0, 0)$	$(-4, 4)$	$\frac{su(1-u)(1-s)^2}{(1-2s)(1-su)(1-2s-2u+3su)}$

Note: For all above cases other than **SA** use $d_3 = B_7 = B_8 = 0$, while **SA** uses $d_3 = 0, B_7 = -7, B_8 = 6$.
Three cases **AX6, CY6, EZ7** (highlighted in yellow) have different forms of K_f but show identical $H(z)$ as that of **SA**.
* N/A = not available.

does require knowledge at four points $x_n, x_{n-1}, x_{n-2}, x_{n-3}$. Hence, p_n can be handled with a less number of working precision digits than the usual COC whose number of working precision digits is at least p times as large as that of p_n.

Computed values of x_n are accurate with up to \$*MinPrecision* significant digits. If α has the same accuracy of \$*MinPrecision* as that of x_n, then $e_n = x_n - \alpha$ would be nearly zero and hence computing $|e_{n+1}|/|e_n|^p$ would unfavourably break down. To clearly observe the convergence behaviour, we desire α to have more significant digits that are Φ digits higher than \$*MinPrecision*. To supply such α, a set of following Mathematica commands are used:

$$sol = FindRoot[f(x), \{x, x_0\}, PrecisionGoal \to \Phi + \$MinPrecision,$$
$$WorkingPrecision \to 2 * \$MinPrecision];$$
$$\alpha = sol[[1, 2]]$$

In this experiment, we assign $\Phi = 16$. As a result, the numbers of significant digits of x_n and α are found to be 160 and 176, respectively. Nonetheless, we list both of them with up to 15 significant digits for proper readability. The error bound $\epsilon = \frac{1}{2} \times 10^{-120}$ is assigned to satisfy $|x_n - \alpha| < \epsilon$.

Typical methods **WAX2, WBX1, WDY7** have been successfully implemented with test functions $F_1 - F_3$ below:

$$\textbf{WAX2}: F_1(x) = \cos\left(\frac{\pi}{x}\right) + x^2 - \pi, \quad \alpha \approx 1.81648572902222,$$

$$\textbf{WBX1}: F_2(x) = x - \sqrt{3}x^3 \cos\left(\frac{\pi}{x+1}\right) + \frac{1}{x^2+1} - \frac{11}{5} + 4\sqrt{3}, \quad \alpha = 2,$$

$$\textbf{WDY7}: F_3(x) = -\log\left[(x-2)^2 + \frac{19}{16}\right], \quad \alpha = 2 - \mathrm{i}\frac{\sqrt{3}}{4},$$

where $\log z (z \in \mathbb{C})$ represents a principal analytic branch such that $-\pi < \mathrm{Im}(\log z) \le \pi$.

Table 4 clearly confirms eighth-order convergence. The values of computational asymptotic error constant agree up to 10 significant digits with η. It appears that the computational convergence order well approaches 8.

Table 4. Convergence for test functions $F_1(x) - F_3(x)$ with typically selected methods **AX2, BX1, DY7**.

MT	F	n	x_n	$\lvert F(x_n)\rvert$	$\lvert x_n - \alpha\rvert$	$\lvert e_n/e_{n-1}^8\rvert$	η	p_n
		0	1.7	0.525256	0.116486			
		1	1.81648572902243	9.564×10^{-13}	2.091×10^{-13}	$6.169498388 \times 10^{-6}$	$2.749110983 \times 10^{-7}$	6.55305
WAX2	F_1	2	1.81648572902222	4.601×10^{-108}	1.006×10^{-108}	$2.749110983 \times 10^{-7}$		8.00000
		3	1.81648572902222	2.425×10^{-173}	4.043×10^{-174}			
		0	1.87	1.62893	0.130000			
		1	1.99999999393956	8.327×10^{-8}	6.060×10^{-9}	0.07429458432	0.0399096822	7.69542
WBX1	F_2	2	2.00000000000000	9.980×10^{-67}	7.262×10^{-68}	0.03990968332		8.00000
		3	2.00000000000000	8.086×10^{-174}	2.425×10^{-173}			
		0	$\begin{pmatrix} 1.97 \\ -0.36 \end{pmatrix}^*$	0.0608640	0.0789358			
		1	$\begin{pmatrix} 1.99999999638476 \\ -0.433012689128059 \end{pmatrix}$	1.148×10^{-8}	1.326×10^{-8}	8.801529463	4.234524089	7.71185
WDY7	F_3	2	$\begin{pmatrix} 2.00000000000000 \\ -0.433012701892219 \end{pmatrix}$	3.518×10^{-63}	4.062×10^{-63}	4.234524636		8.00000
		3	$\begin{pmatrix} 2.00000000000000 \\ -0.433012701892219 \end{pmatrix}$	0.0×10^{-160}	5.717×10^{-174}			

Note: MT = method, $\begin{pmatrix} 1.97 \\ -0.36 \end{pmatrix}^* = 1.96 - 0.36\mathrm{i}$, $\mathrm{i} = \sqrt{-1}$.

Table 5. Additional test functions $f_i(x)$ with zeros α and initial guesses x_0.

i	$f_i(x)$	α	x_0
1	$x\sin(x^2) - \log[1 + \frac{1}{x^2} - \frac{1}{\pi}]$	$\sqrt{\pi}$	1.7
2	$\cos[(x-3)^2+3] - \log[(x-3)^2+4] - 1$	$3 + \mathrm{i}\sqrt{3}$	2.95 + 1.76i
3	$x^5 + \log[1 + \sin x]$	0	0.05
4	$\sin^{-1}(\frac{1}{x} - 1) - 4x^2 + 3$	0.884690687180673	1.0
5	$x^3 - \pi^3 + \sin x\sqrt{(x+1)}$	π	2.9
6	$4x^2 + e^{-x} + \sin(1 + \frac{1}{x}) - 4$	0.830382156106894	0.75
7	$\log x - \sqrt{x} + x^3$	1	0.9

Note: Here, $\log z$ ($z \in \mathbb{C}$) representsaprincipalanalyticbranchwith $-\pi \le \mathrm{Im}(\log z) < \pi$.

Table 5 lists additional test functions to ensure the convergence behaviour of proposed scheme (2).

In Table 6, we compare numerical errors $|x_n - \alpha|$ of proposed methods **WAX1** through **WFY6** with that of method **SA**. The least errors within the prescribed error bound are highlighted in bold face. Although we are limited to the selected current experiments, within two iterations, a strict comparison shows that methods **WFY6, WDY7, WCY1, WDX2** display slightly better convergence for test functions f_1, f_2, f_5, f_6, respectively, and method **WFY4** for three test functions f_3, f_4, f_7.

In view of a close inspection of the asymptotic error constant $\eta(\theta_i, L_f, K_f) = |x_{n+1} - \alpha|/|x_n - \alpha|^8$, we should be aware that the local convergence is dependent on the function $f(x)$, an initial value x_0, the zero α itself and the weight functions L_f and K_f. Accordingly, for all given set of test functions, the convergence of one method is hardly expected to be always better than the others.

The efficiency index *EI* [40] is found to be $8^{1/4} \approx 1.68179$ for the proposed family of methods (2), which evidently show a better performance than that of classical Newton's method.

Proper initial values generally influence the convergence behaviour of iterative methods. To guarantee the convergence of Newton-like iterative map (54) with a weight function $H_p(z)$, it requires good initial values close to zero α. It is, however, not a simple task to determine how close the initial values are to zero α, since initial values are generally sensitive to computational precision, error bound and the given function $f(x)$ under consideration.

We now introduce the notion of the *basin of attraction* that is the set of initial guesses leading to long-time behaviour approaching the attractors (e.g. periodic, quasi-periodic or chaotic behaviours of different types) under the action of the iterative function. Hence, one effective way of selecting stable initial values would be directly using visual basins of attraction. Since the area of convergence can be seen on the basins of attraction, it would be reasonable to say that a method having a larger area of convergence implies a more robust method. It is no doubt for us to employ a quantitative analysis for measuring the size of area of convergence. Evidently, convergence behaviour of global character can be conveniently observed on the basin of attraction. The basic topological structure of such a basin of attraction as a region can vary greatly from system to system with various forms of weight functions.

To show the performance of the listed methods, we present Tables 7–9 featuring a statistical data giving the average number of iterations per point, CPU time (in seconds) and number of points requiring 40 iterations. In the following examples, we take a 6×6 square centred at the origin and containing all the zeros of the given functions. We then take 601×601 equally spaced points in the square as initial points for the iterative methods. We colour the point based on the root it converged to. This way we can figure out if the method converged within the maximum number of iteration allowed and if it converged to the root closer to the initial point.

Table 6. Comparison of $|x_n - \alpha|$ for selected methods applied to various test functions.

		$f(x)$; x_0						
Method	$\|x_n-\alpha\|$	f_1; 1.7	f_2; 2.95 + 1.76i	f_3; 0.05	f_4; 1.0	f_5; 2.9	f_6; 0.75	f_7; 0.9
WAX1	$\|x_1-\alpha\|$	1.35e−8*	4.21e−9	8.10e−14	1.05e−10	4.45e−10	4.94e−11	3.56e−10
	$\|x_2-\alpha\|$	7.13e−63	5.71e−67	1.90e−109	1.42e−82	4.09e−80	9.43e−85	8.88e−78
WAX2	$\|x_1-\alpha\|$	2.57e−8	7.63e−9	9.62e−14	1.03e−10	7.54e−10	3.24e−11	4.18e−10
	$\|x_2-\alpha\|$	2.88e−60	1.50e−64	2.13e−108	1.10e−82	5.09e−80	2.70e−86	3.88e−77
WAX6	$\|x_1-\alpha\|$	1.87e−8	3.09e−9	6.74e−14	1.06e−10	8.99e−10	7.95e−11	2.92e−10
	$\|x_2-\alpha\|$	2.37e−61	7.13e−68	1.85e−109	1.66e−82	2.36e−77	5.20e−83	1.54e−78
WAY4	$\|x_1-\alpha\|$	2.79e−8	6.17e−9	7.02e−14	1.05e−10	4.62e−10	4.11e−11	3.17e−10
	$\|x_2-\alpha\|$	5.19e−60	2.12e−65	2.29e−109	1.38e−82	5.68e−80	2.01e−85	2.96e−78
WAY5	$\|x_1-\alpha\|$	3.08e−8	4.73e−9	4.47e−14	1.07e−10	1.61e−9	4.96e−11	2.21e−10
	$\|x_2-\alpha\|$	8.46e−60	1.75e−66	1.70e−110	1.73e−82	4.66e−75	1.00e−84	9.39e−80
WAY6	$\|x_1-\alpha\|$	3.20e−8	3.29e−9	1.96e−14	1.10e−10	2.69e−9	5.81e−11	1.30e−10
	$\|x_2-\alpha\|$	1.27e−59	5.49e−68	3.86e−113	2.16e−82	4.98e−73	3.85e−84	3.11e−82
WBX1	$\|x_1-\alpha\|$	1.57e−8	7.86e−9	1.65e−13	1.04e−10	4.15e−9	2.851e−11	7.11e−10
	$\|x_2-\alpha\|$	3.88e−62	1.92e−64	1.13e−105	1.12e−82	2.18e−71	9.32e−87	4.79e−75
WBX6	$\|x_1-\alpha\|$	2.60e−8	5.22e−9	6.27e−14	1.09e−10	8.36e−10	9.17e−11	2.80e−10
	$\|x_2-\alpha\|$	4.31e−60	7.23e−66	1.38e−109	2.28e−82	9.54e−78	1.78e−82	1.01e−78
WBY6	$\|x_1-\alpha\|$	5.44e−8	9.17e−9	5.06e−14	1.01e−10	1.59e−9	1.03e−10	2.20e−10
	$\|x_2-\alpha\|$	2.93e−57	1.07e−63	3.76e−110	1.21e−82	3.89e−75	4.65e−82	1.24e−79
WBZ5	$\|x_1-\alpha\|$	7.75e−9	1.28e−9	7.54e−14	1.17e−10	5.78e−11	7.97e−11	3.50e−10
	$\|x_2-\alpha\|$	2.36e−65	3.40e−71	3.02e−109	3.92e−82	1.71e−87	5.68e−83	6.90e−78
WCX4	$\|x_1-\alpha\|$	1.15e−8	3.57e−9	5.13e−14	1.02e−10	5.58e−10	5.18e−11	2.36e−10
	$\|x_2-\alpha\|$	1.27e−63	1.15e−67	4.19e−110	1.13e−82	3.64e−79	1.36e−84	2.03e−79
WCY1	$\|x_1-\alpha\|$	9.61e−9	1.67e−9	4.34e−14	9.73e−11	3.02e−11	6.01e−11	1.97e−10
	$\|x_2-\alpha\|$	5.87e−64	7.31e−71	1.34e−110	7.26e−83	**1.94e−90**	4.57e−84	4.21e−80
WCY2	$\|x_1-\alpha\|$	3.52e−8	6.10e−9	5.95e−14	1.08e−10	1.21e−9	4.32e−11	2.80e−10
	$\|x_2-\alpha\|$	3.88e−59	1.89e−65	1.06e−109	1.71e−82	3.53e−76	3.15e−85	8.74e−79
WCY6	$\|x_1-\alpha\|$	1.93e−8	1.76e−9	3.95e−14	9.45e−11	3.02e−10	6.42e−11	1.79e−10
	$\|x_2-\alpha\|$	2.74e−61	3.68e−70	6.95e−111	5.69e−83	1.05e−81	7.78e−84	1.80e−80
WDX2	$\|x_1-\alpha\|$	5.49e−8	1.41e−8	1.55e−13	1.03e−10	2.58e−9	3.00e−14	6.92e−10
	$\|x_2-\alpha\|$	2.73e−57	4.35e−62	6.01e−106	9.37e−83	2.98e−73	**1.06e−110**	3.53e−75
WDY7	$\|x_1-\alpha\|$	2.05e−8	5.45e−10	7.97e−14	1.19e−10	3.92e−10	7.56e−11	3.76e−10
	$\|x_2-\alpha\|$	1.35e−61	**1.50e−74**	3.16e−109	4.63e−82	1.78e−80	3.69e−83	1.27e−77
WDY9	$\|x_1-\alpha\|$	2.01e−7	2.37e−8	7.48e−14	1.17e−10	6.94e−9	3.15e−11	3.79e−10
	$\|x_2-\alpha\|$	2.26e−52	4.77e−60	2.85e−109	2.77e−82	1.49e−69	1.46e−86	1.13e−77
WEX2	$\|x_1-\alpha\|$	8.60e−9	4.96e−9	1.22e−13	1.05e−10	2.34e−9	4.48e−11	5.26-10
	$\|x_2-\alpha\|$	1.66e−64	2.70e−66	4.99e−107	1.40e−82	1.26e−73	4.11e−85	3.19e−76
WEX6	$\|x_1-\alpha\|$	4.29e−8	1.17e−8	1.43e−13	1.04e−10	2.32e−9	1.20e−11	6.31e−10
	$\|x_2-\alpha\|$	3.01e−58	8.11e−63	2.60e−106	1.05e−82	1.16e−73	8.23e−90	1.57e−75
WEY6	$\|x_1-\alpha\|$	2.20e−8	9.82e−9	1.36e−13	1.00e−10	2.72e−9	1.90e−11	5.80e−10
	$\|x_2-\alpha\|$	9.47e−61	1.52e−63	1.52e−106	7.79e−83	4.89e−73	3.28e−88	7.76e−76
WEZ1	$\|x_1-\alpha\|$	2.61e−8	5.84e−9	1.88e−13	9.48e−11	6.53e−9	3.32e−11	7.68e−10
	$\|x_2-\alpha\|$	1.25e−60	1.14e−65	4.42e−105	4.89e−83	1.33e−69	2.90e−86	1.06e−74
WEZ7	$\|x_1-\alpha\|$	1.79e−8	5.28e−9	9.66e−14	1.17e−10	1.60e−9	9.55e−11	4.23e−10
	$\|x_2-\alpha\|$	1.88e−61	8.08e−66	2.21e−108	4.07e−82	4.28e−75	2.62e−82	4.27e−77
WFY4	$\|x_1-\alpha\|$	3.15e−8	1.38e−8	1.19e−14	7.57e−11	1.77e−9	7.61e−12	7.64e−11
	$\|x_2-\alpha\|$	2.49e−59	3.66e−62	**7.95e−115**	**5.58e−84**	1.15e−74	1.00e−91	**3.22e−84**
WFY5	$\|x_1-\alpha\|$	1.45e−8	7.85e−9	2.84e−14	8.75e−11	9.80e−10	2.49e−11	1.39e−10
	$\|x_2-\alpha\|$	2.01e−62	1.93e−64	6.25e−112	2.43e−83	5.37e−77	2.68e−87	1.52e−81
WFY6	$\|x_1-\alpha\|$	4.56e−9	1.98e−9	4.53e−14	9.87e−11	1.11e−10	5.80e−11	2.07e−10
	$\|x_2-\alpha\|$	**9.29e−67**	4.73e−71	1.82e−110	8.16e−83	2.72e−85	3.44e−84	6.32e−80
SA	$\|x_1-\alpha\|$	2.66e−8	5.43e−9	6.13e−14	1.08e−10	5.68e−10	9.29e−11	2.65e−10
	$\|x_2-\alpha\|$	5.10e−60	9.89e−66	1.16e−109	2.16e−82	4.34e−79	1.97e−82	6.58e−79

* $1.35e-8 \equiv 1.35 \times 10^{-8}$.

Although we have experimented with all methods listed in Table 2, because of space limitations, we are only able to present the dynamics of 20 selected iterative maps **WAX1, WAX6, WAY4, WAY5, WAY6, WBX1, WBX6, WBY6, WBZ5, WCX4, WCY2, WCY6, WDY9, WEX6, WEY6, WEZ1, WEZ7, WFY4, WFY6** and **SA**, when applied to various polynomials $p_k(z)$, $(1 \le k \le 6)$ and one non-polynomial equation.

Example 5.1: As a first example, we have taken a quadratic polynomial with all real roots:

$$p_1(z) = z^2 - 1. \tag{55}$$

Clearly, the roots are ± 1. Basins of attraction for **WAX1–WFY6** and **SA** are given in Figure 1. Consulting Tables 7–9, we find that the method **SA** uses the least number of iterations per point on average (ANIP), it also has the least number of black points. The methods **WAX6, WCY6** and **WEZ7** have almost the same ANIP as SA. The fastest method is **SA** with 153.130 s.

Table 7. Average number of iterations per point for each example (1–7).

	Example							
Map	1	2	3	4	5	6	7	Average
WAX1	2.25	2.52	2.68	3.09	3.11	2.96	2.24	2.69
WAX6	2.18	2.46	2.64	3.11	3.30	3.13	2.29	2.73
WAY4	2.31	2.62	2.81	3.10	3.06	3.02	2.29	2.75
WAY5	2.35	2.74	2.95	3.20	3.21	3.15	2.43	2.86
WAY6	2.42	2.90	3.11	3.59	3.65	3.46	2.45	3.08
WBX1	2.22	2.54	2.71	3.45	4.55	4.02	2.47	3.14
WBX6	2.22	2.58	2.79	3.40	3.52	3.33	2.44	2.90
WBY6	2.28	2.73	2.72	3.77	4.07	3.94	2.69	3.17
WBZ5	2.28	2.63	2.81	3.40	3.67	3.16	2.46	2.92
WCX4	2.28	2.60	2.78	3.05	3.07	2.96	2.34	2.73
WCY2	2.39	2.77	2.99	3.16	3.15	3.08	2.37	2.84
WCY6	2.18	2.47	2.70	3.51	3.72	3.60	2.59	2.97
WDY9	2.28	2.57	2.76	3.72	5.08	3.96	2.63	3.29
WEX6	2.28	2.51	2.70	3.11	3.12	3.28	2.47	2.78
WEY6	2.33	2.59	2.75	5.83	6.24	5.12	2.49	3.91
WEZ1	2.25	2.51	2.65	3.74	4.14	3.42	2.40	3.01
WEZ7	2.18	2.51	2.67	3.38	3.71	3.11	2.25	2.83
WFY4	2.40	2.66	2.91	3.31	3.52	3.49	2.80	3.01
WFY6	2.24	2.57	2.76	3.27	3.33	3.22	2.41	2.83
SA	2.16	2.46	2.62	2.98	3.03	2.89	2.28	2.63

Table 8. CPU time (in seconds) required for each example (1–7) using a Dell Multiplex-990.

	Example							
Map	1	2	3	4	5	6	7	Average
WAX1	201.116	313.562	292.050	340.425	399.175	983.539	359.536	412.772
WAX6	192.630	313.999	274.749	348.117	418.722	1018.156	354.404	417.253
WAY4	179.417	314.139	277.307	331.190	362.250	956.364	336.041	393.815
WAY5	197.762	336.588	310.629	358.272	391.766	1019.825	363.654	425.499
WAY6	195.048	339.692	304.733	377.148	431.000	1091.351	358.240	442.459
WBX1	182.147	304.560	260.678	343.592	520.326	1258.648	367.569	462.503
WBX6	187.731	310.598	274.998	352.905	426.398	1062.882	354.934	424.349
WBY6	200.758	347.227	280.240	415.446	499.999	1270.940	409.627	489.177
WBZ5	184.299	323.187	271.972	351.891	446.911	1018.858	368.599	423.674
WCX4	169.963	289.834	256.092	297.104	350.222	936.537	326.572	375.189
WCY2	192.864	327.103	283.719	334.545	382.624	992.900	356.353	410.015
WCY6	188.402	310.535	274.889	370.580	453.776	1168.276	375.027	448.784
WDY9	190.259	314.139	280.006	392.217	592.211	1278.116	391.578	491.218
WEX6	170.961	279.725	246.919	307.884	344.216	1024.989	354.075	389.824
WEY6	199.572	317.742	277.192	637.857	749.054	1651.052	375.915	601.198
WEZ1	186.093	306.776	265.544	396.368	486.208	1099.385	351.549	441.703
WEZ7	191.476	317.025	284.655	368.271	449.158	1003.742	337.024	421.622
WFY4	205.844	336.323	300.271	363.014	437.208	1135.063	419.066	456.684
WFY6	194.471	315.262	292.767	359.067	405.041	1037.720	348.943	421.896
SA	153.130	290.364	241.786	294.529	338.740	1252.002	333.967	414.931

Table 9. Number of points requiring 40 iterations for each example (1–7).

	Example							
Map	1	2	3	4	5	6	7	Average
WAX1	747	50	0	1201	1	0	330	333
WAX6	769	57	0	1345	1451	1086	694	772
WAY4	795	117	0	1201	5	0	914	433
WAY5	949	534	476	1201	6	158	842	595
WAY6	1175	1217	1024	2001	959	1027	1314	1245
WBX1	753	46	0	1201	1	0	533	362
WBX6	749	15	0	1201	1	5	1714	526
WBY6	761	19	0	1201	19	23	1884	558
WBZ5	769	109	0	1337	1619	5	1580	774
WCX4	765	70	0	1201	1	0	556	370
WCY2	1433	1162	1260	1201	1	1	972	861
WCY6	769	11	0	1201	6	10	1564	509
WDY9	761	53	0	1261	843	0	1598	645
WEX6	765	23	0	1201	1	0	1141	447
WEY6	1281	549	16	28,661	32,016	16,657	1191	11,482
WEZ1	741	84	0	1433	1879	2	950	727
WEZ7	773	83	0	1257	1450	2	705	610
WFY4	1021	14	0	1201	200	5	2395	691
WFY6	745	10	0	1201	1	2	1212	453
SA	601	54	0	1201	1	0	514	339

Example 5.2: In our second example, we have taken a cubic polynomial:

$$p_2(z) = z^3 + 4z^2 - 10. \tag{56}$$

Basins of attraction are given in Figure 2. We now consult the tables to find that the method with the fewest ANIP are **SA** and **WAX6** with 2.46 iteration. All the others require between 2.51 and 2.90. In terms of CPU time in seconds, the fastest is **WEX6** (279.725 s) and the slowest is **WBY6** (347.227 s). The method **WAY6** has the most black points (1216) and **WFY6** has the least (10 points).

Example 5.3: As a third example, we have taken another cubic polynomial:

$$p_3(z) = z^3 - z. \tag{57}$$

Now all the roots are real. The basins for this example are plotted in Figure 3. Based on Table 7, we see that again **SA** has the lowest ANIP followed closely by **WAX6**. The fastest method is again **SA** (241.786 s) followed by **WEX6** (246.919 s) and the slowest are **WAY5** (310.629 s) and **WAY6** (304.733 s). Most of the methods have no black points except **WCY2** with 1260, **WAY6** with 1024, **WAY5** with 476 and **WEY6** with 16 black points.

Example 5.4: As a fourth example, we have taken a quartic polynomial:

$$p_4(z) = z^4 - 1. \tag{58}$$

The basins are given in Figure 4. We now see that **WBZ5**, **WDY9**, **WEY6**, **WEZ1** and **WEZ7** are the worst. The best are those with smaller lobes along the diagonals. In terms of ANIP, **SA** is the best (2.98) and **WEY6** is the worst (5.83). The fastest is again **SA** (294.529 s) followed by **WCX4** (297.104 s) and the slowest is **WEY6** (637.857 s). Most of the methods have 1201 black point with the worst being **WEY6** with 28,661 points.

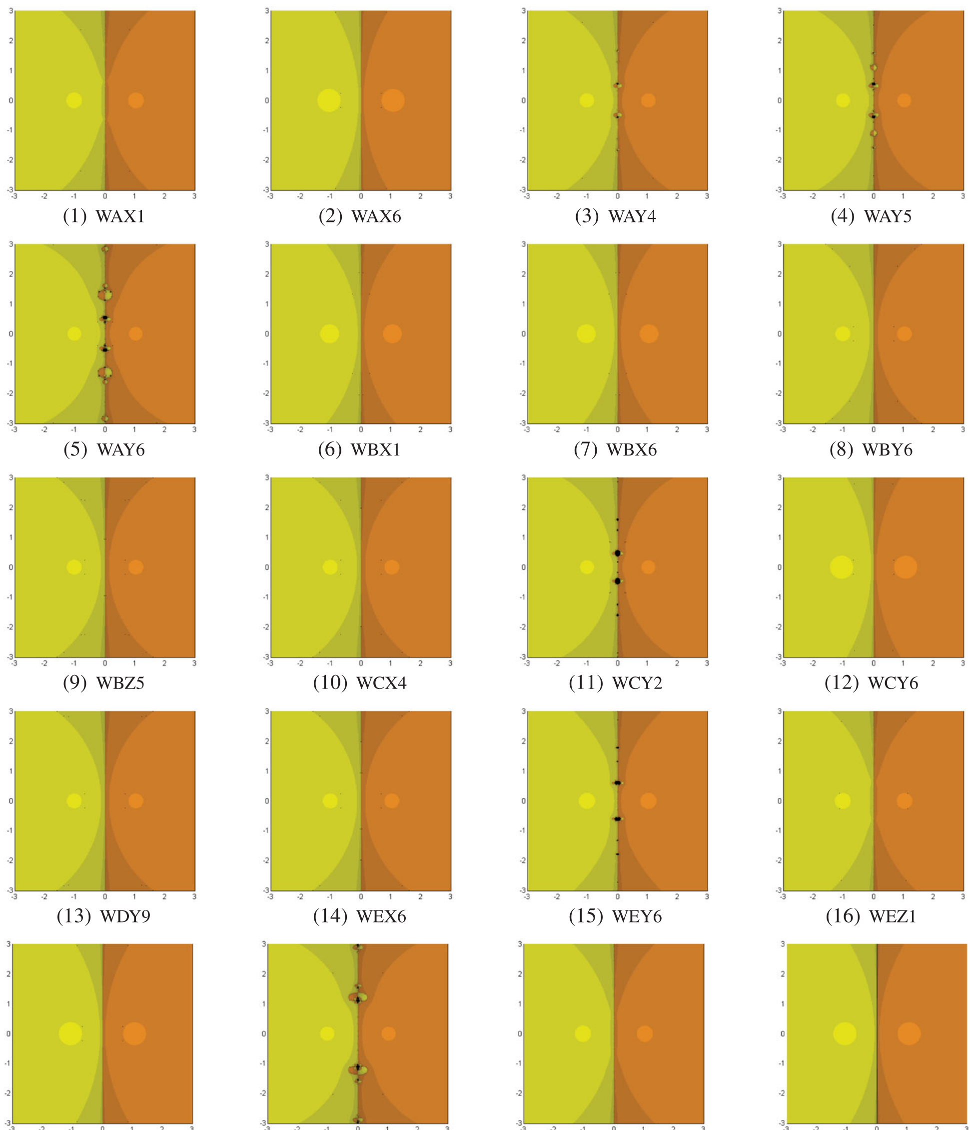

Figure 1. The top row for **WAX1** (left), **WAX6** (centre left), **WAY4** (centre right) and **WAY5** (right). The second row for **WAY6** (left), **WBX1** (centre left), **WBX6** (centre right) and **WBY6** (right). The third row for **WBZ5** (left), **WCX4** (centre left), **WCY2** (centre right) and **WCY6** (right). The fourth row for **WDY9** (left), **WEX6** (centre left), **WEY6** (centre right) and **WEZ1** (right). The bottom row for **WEZ7** (left), **WFY4** (centre left), **WFY6** (centre right) and **SA** (right), for the roots of the polynomial $(z^2 - 1)$.

Example 5.5: As a fifth example, we have taken a quintic polynomial:

$$p_5(z) = z^5 - 1. \tag{59}$$

The basins for the best methods left are plotted in Figure 5. The worst are **WDY9**, **WEY6**, **WEZ1**, **WEZ7**, **WBZ5** and **WBX1**. In terms of ANIP, the best is **SA** (3.03) followed closely by **WAY4** (3.06) and **WCX4** (3.07) and the worst are **WEY6** (6.24) and **WDY9** (5.08). The fastest is **SA** using 338.74 s followed by **WEX6** using 344.216 s and the slowest is **WEY6** (749.054 s). There are 16 methods with less than 10 black points. The highest number is for **WEY6** (32,016) preceded by **WEZ1** with 1879 black points.

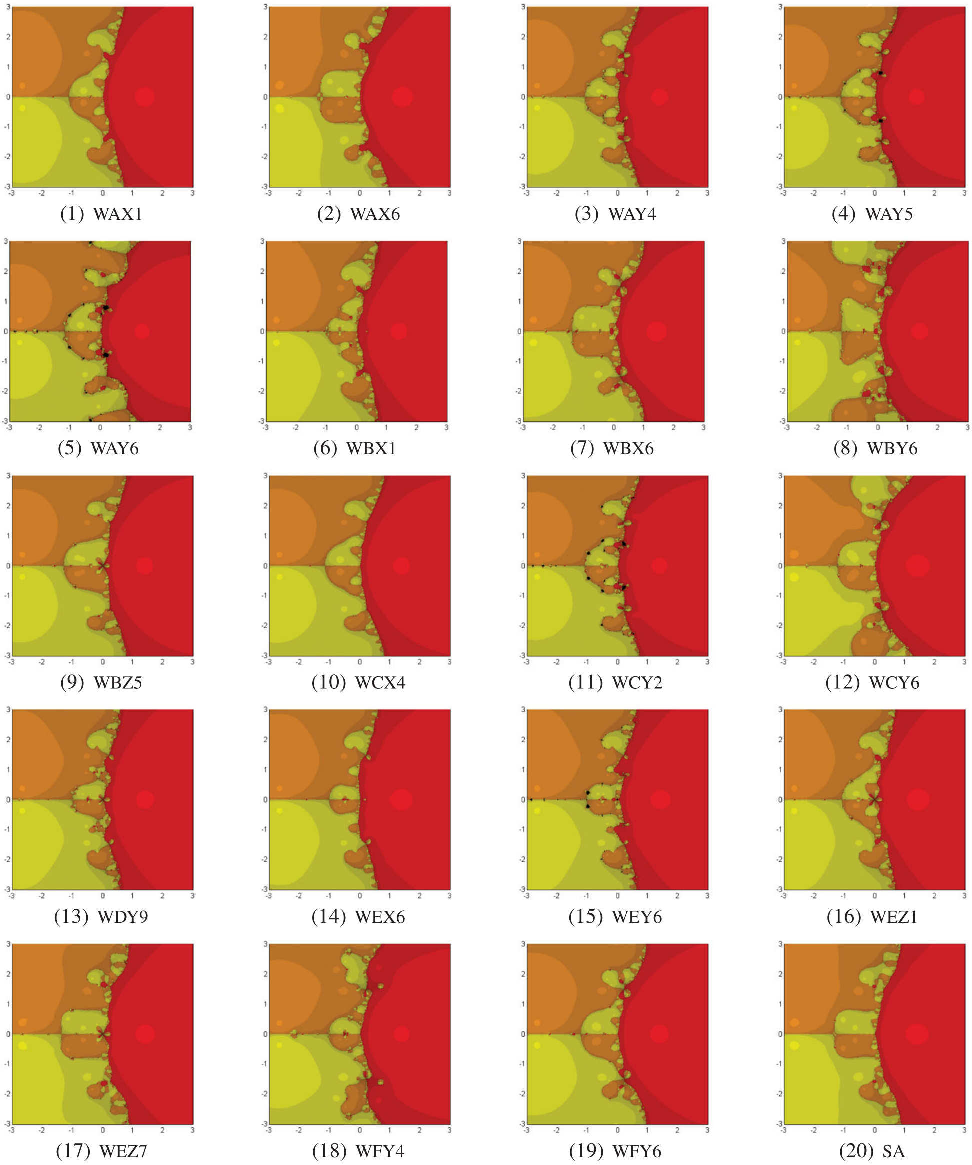

Figure 2. The top row for **WAX1** (left), **WAX6** (centre left), **WAY4** (centre right) and **WAY5** (right). The second row for **WAY6** (left), **WBX1** (centre left), **WBX6** (centre right) and **WBY6** (right). The third row for **WBZ5** (left), **WCX4** (centre left), **WCY2** (centre right) and **WCY6** (right). The fourth row for **WDY9** (left), **WEX6** (centre left), **WEY6** (centre right) and **WEZ1** (right). The bottom row for **WEZ7** (left), **WFY4** (centre left), **WFY6** (centre right) and **SA** (right), for the roots of the polynomial $(z^3 + 4z^2 - 10)$.

Example 5.6: As a sixth example, we have taken a sextic polynomial with complex coefficients:

$$p_6(z) = z^6 - \frac{1}{2}z^5 + \frac{11(\mathrm{i}+1)}{4}z^4 - \frac{3\mathrm{i}+19}{4}z^3 + \frac{5\mathrm{i}+11}{4}z^2 - \frac{\mathrm{i}+11}{4}z + \frac{3}{2} - 3\mathrm{i}. \tag{60}$$

The basins for the best methods left are plotted in Figure 6. It seems that the best methods are **WAX6, WAY6, WBX6, WBY6, WCY6** and **WFY6**. The worst are **WBX1** and **WDY9**. Based on Table 7, we find that **SA** has the lowest ANIP (2.89) followed by **WAX1** and **WCX4** (2.96). The fastest method is **WCX4** (936.537s) followed by **WAY4** (956.364s) and **WAX1** (983.539s). There are 10 methods

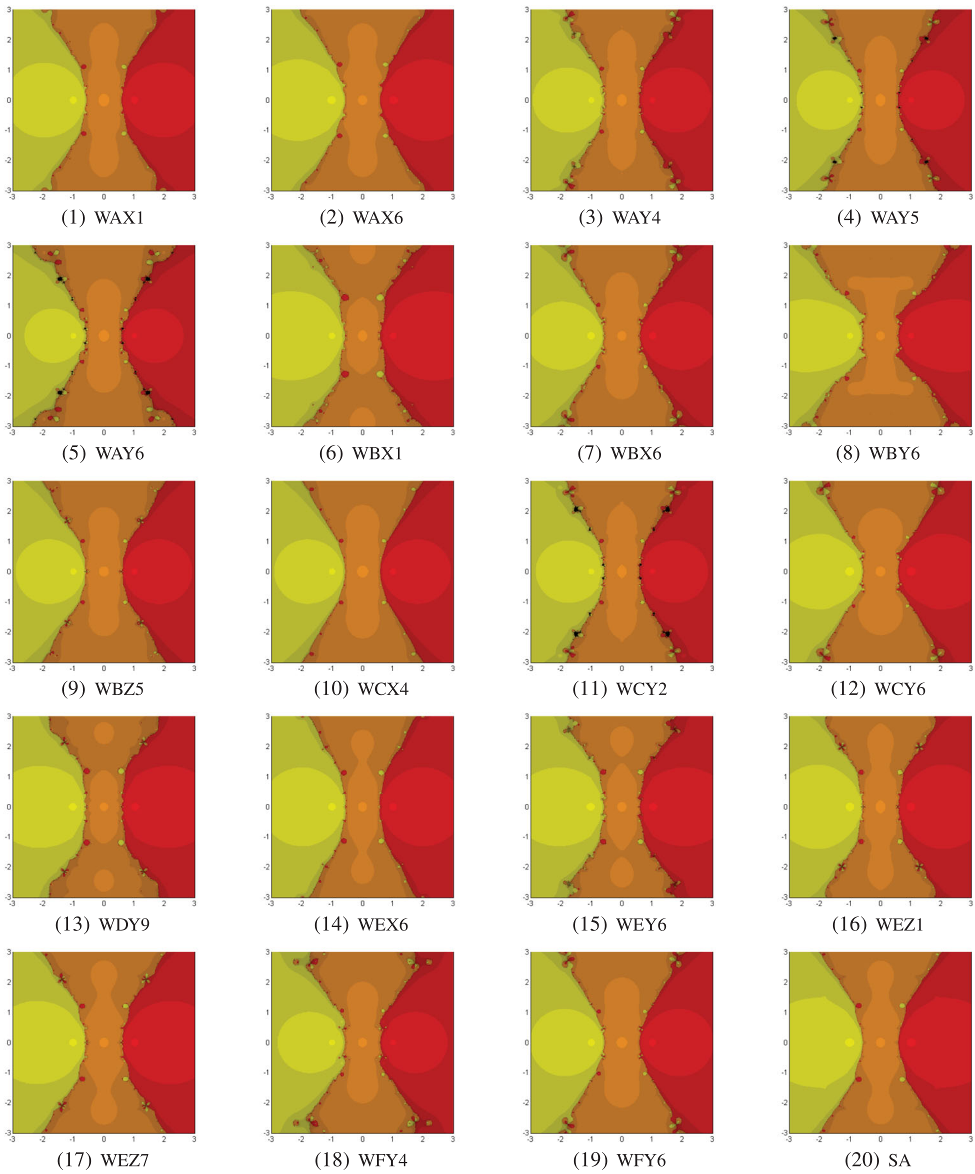

Figure 3. The top row for **WAX1** (left), **WAX6** (centre left), **WAY4** (centre right) and **WAY5** (right). The second row for **WAY6** (left), **WBX1** (centre left), **WBX6** (centre right) and **WBY6** (right). The third row for **WBZ5** (left), **WCX4** (centre left), **WCY2** (centre right) and **WCY6** (right). The fourth row for **WDY9** (left), **WEX6** (centre left), **WEY6** (centre right) and **WEZ1** (right). The bottom row for **WEZ7** (left), **WFY4** (centre left), **WFY6** (centre right) and **SA** (right), for the roots of the polynomial $(z^3 - z)$.

without black points and 10 methods with 10 or less. The highest number is for **WEY6** with 16,657 black points.

Example 5.7: As a last example, we have taken a non-polynomial equation:

$$p_7(z) = (e^{z+1} - 1)(z + 1). \tag{61}$$

The basins for this example are plotted in Figure 7. the roots are at ± 1 and it is expected that the boundary will be close to the imaginary axis as in Example 1. All methods show a larger basin for the

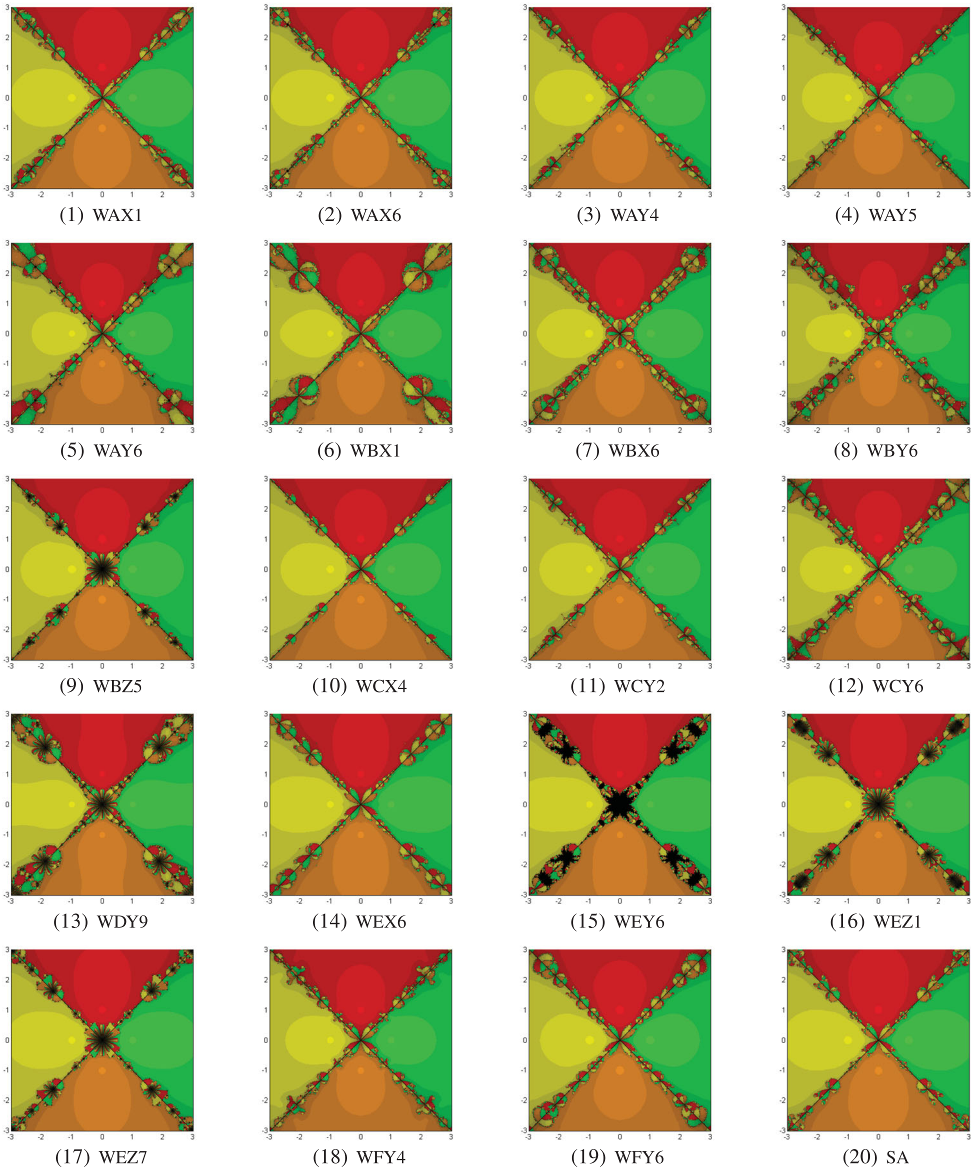

Figure 4. The top row for **WAX1** (left), **WAX6** (centre left), **WAY4** (centre right) and **WAY5** (right). The second row for **WAY6** (left), **WBX1** (centre left), **WBX6** (centre right) and **WBY6** (right). The third row for **WBZ5** (left), **WCX4** (centre left), **WCY2** (centre right) and **WCY6** (right). The fourth row for **WDY9** (left), **WEX6** (centre left), **WEY6** (centre right) and **WEZ1** (right). The bottom row for **WEZ7** (left), **WFY4** (centre left), **WFY6** (centre right) and **SA** (right), for the roots of the polynomial for the roots of the polynomial $(z^4 - 1)$.

root at -1. The methods with the largest basin for $+1$ are **WAX1, WAY4, WCY2, WEY6** and **WEZ1**. In terms of ANIP, **WAX1** is best (2.24) followed closely by **WEZ7** (2.25), **SA** (2.28) and **WAX6,** and **WAY4** with 2.29. The worst is **WFY4** with 2.80. The fastest method is **WCX4** (326.572s) and the slowest is **WFY4** (419.066s). **WAX1** has the least number of black points and **WFY4** has the highest (2395) such number. Based on these seven examples we see that **SA** has six examples with the lowest ANIP, **WAX1** and **WAX6** each with one example. **WCX4** is the fastest in two examples, **WEX6** in one example and **SA** is the fastest in the other four examples.

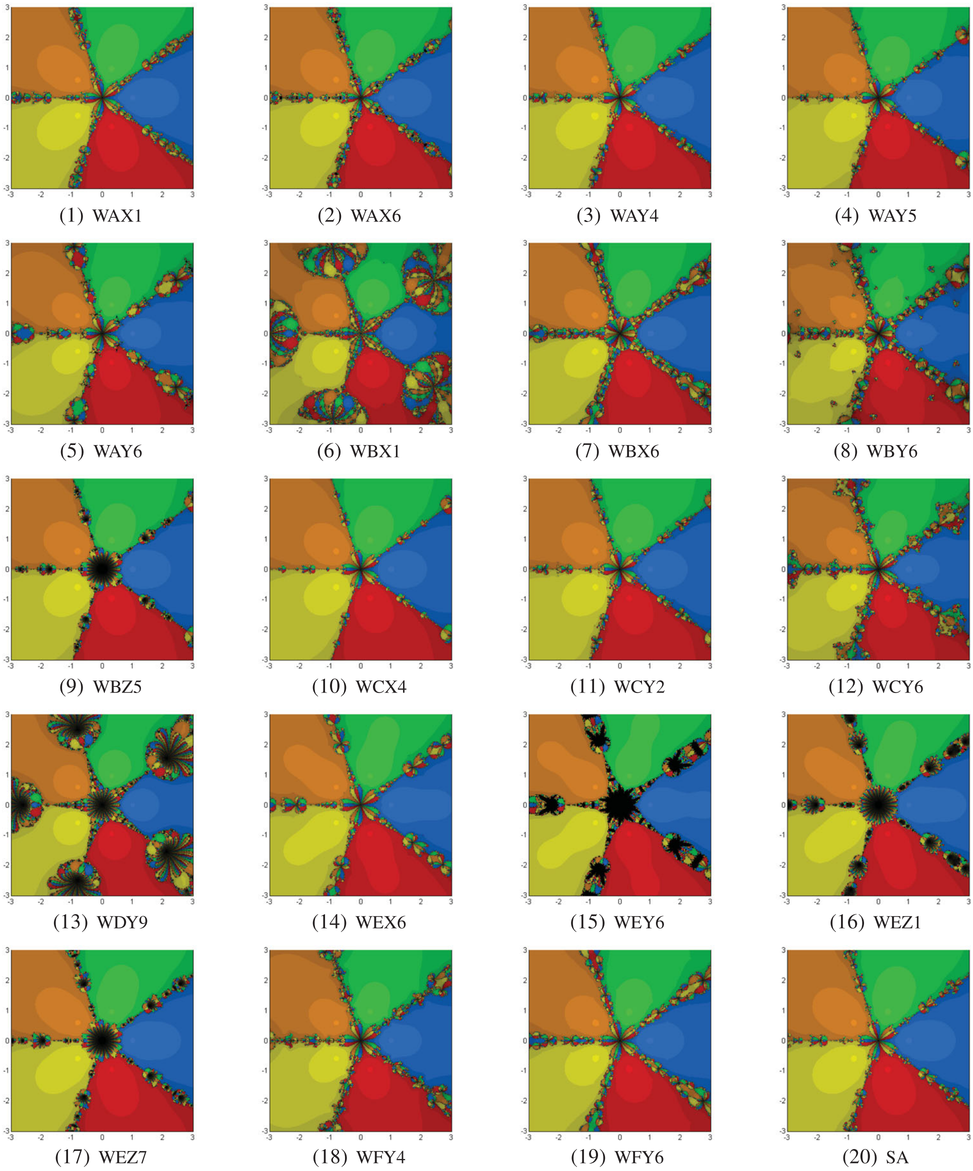

Figure 5. The top row for **WAX1** (left), **WAX6** (centre left), **WAY4** (centre right) and **WAY5** (right). The second row for **WAY6** (left), **WBX1** (centre left), **WBX6** (centre right) and **WBY6** (right). The third row for **WBZ5** (left), **WCX4** (centre left), **WCY2** (centre right) and **WCY6** (right). The fourth row for **WDY9** (left), **WEX6** (centre left), **WEY6** (centre right) and **WEZ1** (right). The bottom row for **WEZ7** (left), **WFY4** (centre left), **WFY6** (centre right) and **SA** (right), for the roots of the polynomial $(z^5 - 1)$.

We now average all these results across the seven examples to try and pick the best method. **SA** has the lowest ANIP (2.63) followed by **WAX1** with 2.69, **WAX6** and **WCX4** with 2.73. The fastest method is **WCX4** followed by **WEX6** (389.824s). **WAX1** has the lowest number of black points on average (333) followed by **SA, WBX1** and **WCX4.**

Based on this, we recommend **WCX4** since it is the only method mentioned as close to the top at all three categories. **SA** and **WAX1** are close to the top at 2 out of the three categories.

As concluding remarks of our study, we state the following results. Theorem 2.1 verifies that convergence order of proposed family of methods (2) has been increased to 8 by means of weight

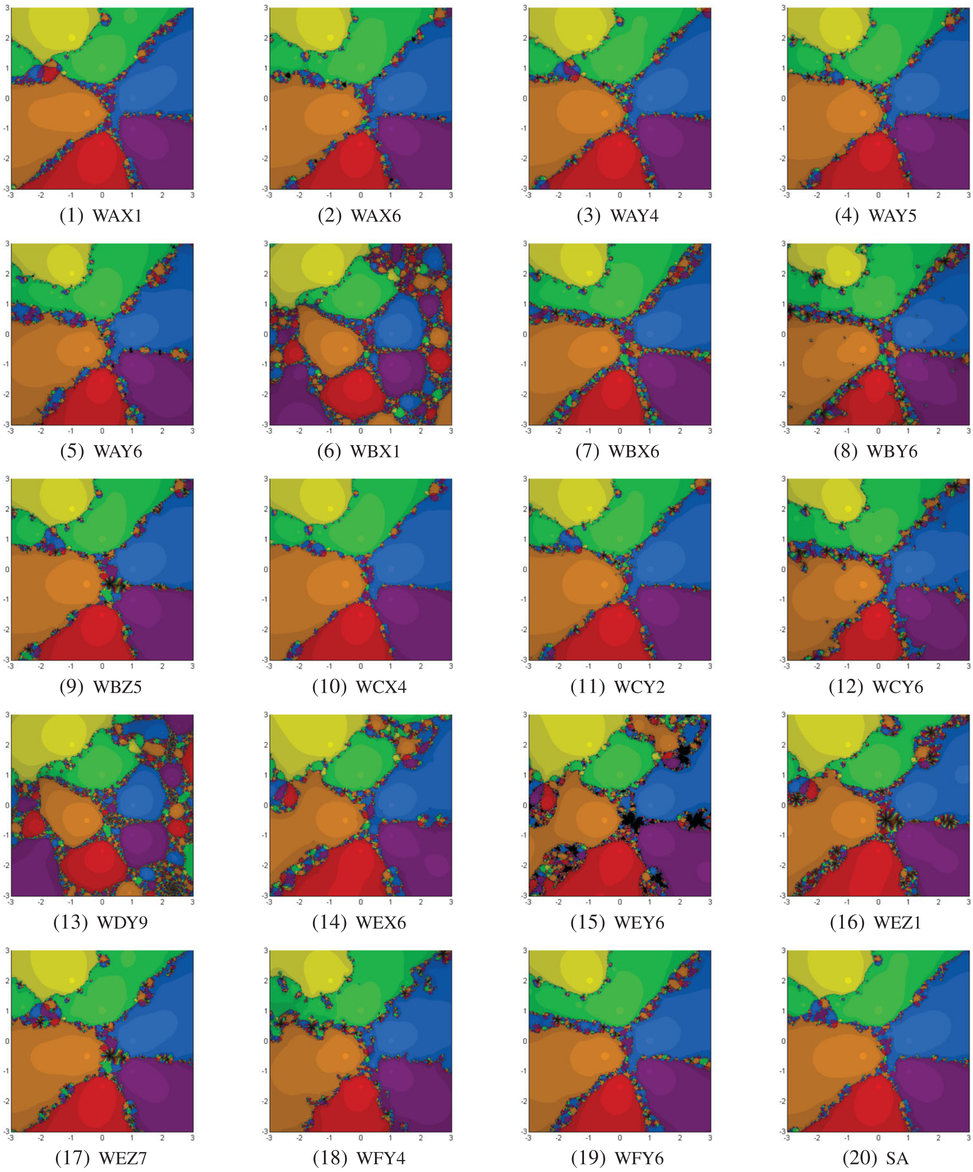

Figure 6. The top row for **WAX1** (left), **WAX6** (centre left), **WAY4** (centre right) and **WAY5** (right). The second row for **WAY6** (left), **WBX1** (centre left), **WBX6** (centre right) and **WBY6** (right). The third row for **WBZ5** (left), **WCX4** (centre left), **WCY2** (centre right) and **WCY6** (right). The fourth row for **WDY9** (left), **WEX6** (centre left), **WEY6** (centre right) and **WEZ1** (right). The bottom row for **WEZ7** (left), **WFY4** (centre left), **WFY6** (centre right) and **SA** (right), for the roots of the polynomial $z^6 - \frac{1}{2}z^5 + 11(i+1)/4z^4 - ((3i+19)/4)z^3 + ((5i+11)/4)z^2 - ((i+11)/4)z + \frac{3}{2} - 3i$.

functions dependent upon function-to-function ratios in their second and third sub-steps. Computational aspects through a variety of test equations for selected cases well agree with the developed theory, verifying the convergence order as well as asymptotic error constants. Dynamical aspects among listed methods have been also illustrated through their basins of attraction not only with a qualitative stability analysis on purely imaginary extraneous fixed points for a prototype quadratic polynomial $f(z) = z^2 - 1$ motivated by the earlier work of Vrscay and Gilbert [42], but also with a quantitative statistical analysis for various polynomials $p_k(z)$ as well as a non-polynomial example.

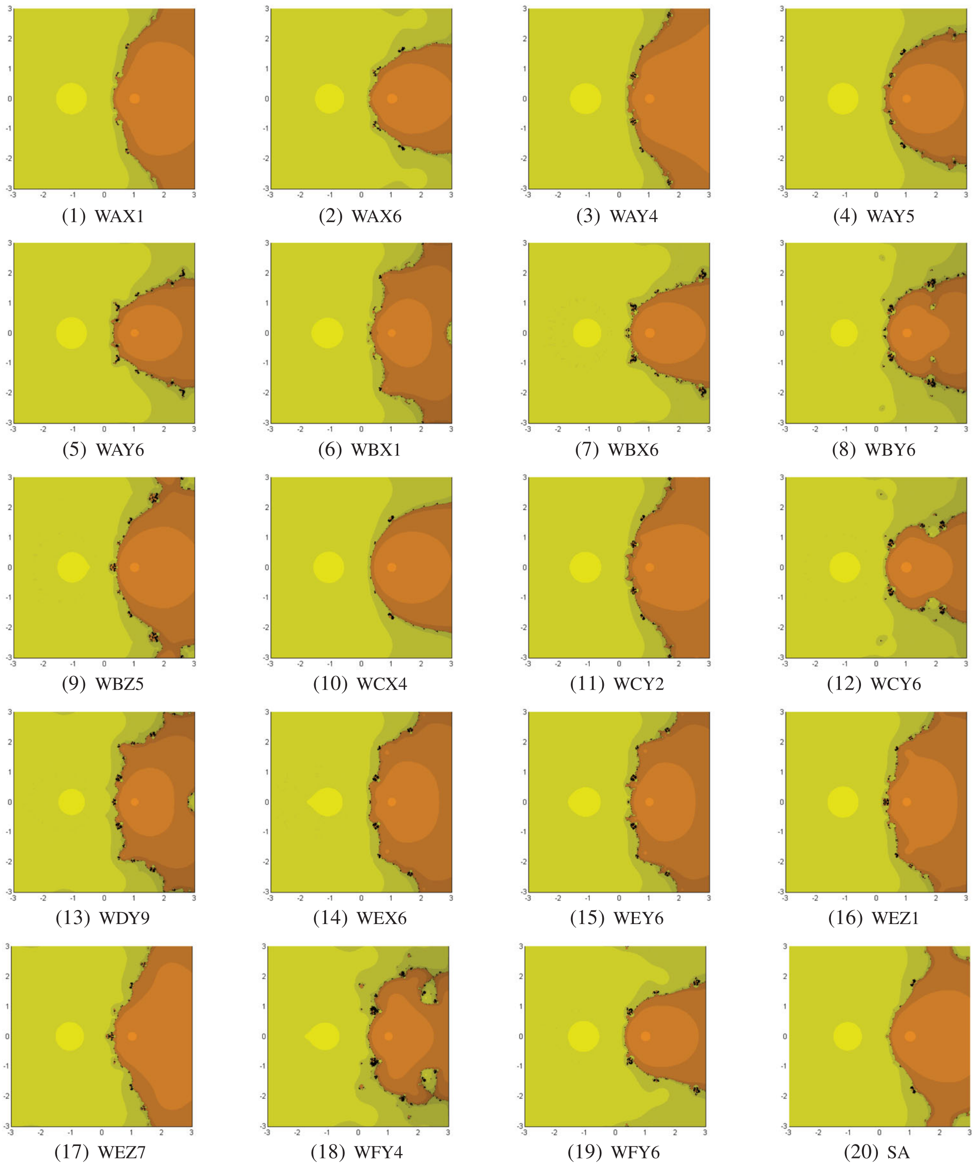

Figure 7. The top row for **WAX1** (left), **WAX6** (centre left), **WAY4** (centre right) and **WAY5** (right). The second row for **WAY6** (left), **WBX1** (centre left), **WBX6** (centre right) and **WBY6** (right). The third row for **WBZ5** (left), **WCX4** (centre left), **WCY2** (centre right) and **WCY6** (right). The fourth row for **WDY9** (left), **WEX6** (centre left), **WEY6** (centre right) and **WEZ1** (right). The bottom row for **WEZ7** (left), **WFY4** (centre left), **WFY6** (centre right) and **SA** (right), for the roots of the non-polynomial equation $(e^{z+1}-1)(z+1)$.

We can determine which members of the proposed family of methods (2) give better convergence from the illustrative basins of attraction.

In our future study, we will extend the current approach with other types of weight functions by means of a different selection of parameters to a high-order family of simple- or multiple-root finders in order to enhance the desired dynamical characteristics behind their purely imaginary extraneous fixed points.

Disclosure statement

No potential conflict of interest was reported by the authors.

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