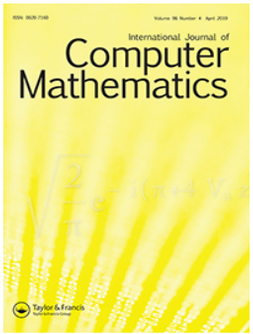

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ARTICLE

On optimal parameter of Laguerre's family of zero-finding methods

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ABSTRACT

A one parameter Laguerre's family of iterative methods for solving nonlinear equations is considered. This family includes the Halley, Ostrowski and Euler methods, most frequently used one-point third-order methods for finding zeros. Investigation of convergence quality of these methods and their ranking is reduced to searching optimal parameter of Laguerre's family, which is the main goal of this paper. Although methods from Laguerre's family have been extensively studied in the literature for more decades, their proper ranking was primarily discussed according to numerical experiments. Regarding that such ranking is not trustworthy even for algebraic polynomials, more reliable comparison study is presented by combining the comparison by numerical examples and the comparison using dynamic study of methods by basins of attraction that enable their graphic visualization. This combined approach has shown that Ostrowski's method possesses the best convergence behaviour for most polynomial equations.

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1. Introduction

Solving nonlinear equations, a part of Numerical analysis, is of great importance in the theory and practice, e.g. in applied mathematics, engineering sciences, physics, the theory of control systems, digital processing, finance and economy, stability of dynamical systems, and other branches. During the last three centuries, many zero-finding methods were developed and published in a vast number of papers and books, see references in the books [13,14]. Although several efficient and robust methods were constructed, even they have certain disadvantages manifested in solving some specific problems. For this reason, no one could assert that the best zero-finding algorithm was finally stated, capable to solve every equation.

A one parameter Laguerre's family of iterative methods [11] for solving a single equation $f(x) = 0$ certainly belongs to the above-mentioned class of powerful methods, especially in the case of algebraic polynomials, see, e.g. [6–8,17,22]. A number of numerical experiments have shown that convergence breakdown is extremely rare solving polynomial equations, making this method very close to being globally convergent. Extensive studies of Laguerre's method can be found in [9,13,16] (see, also, [4,8,12,17,18,20]).

Let f be a function of x with a simple zero α and let

$$u(x) = \frac{f(x)}{f'(x)}, \quad A_2(x) = \frac{f''(x)}{2f'(x)}.$$

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Laguerre's iteration function is defined by the iterative formula

$$L(x;\lambda) := x - \frac{\lambda u(x)}{1 + \operatorname{sgn}(\lambda - 1)\sqrt{(\lambda - 1)^2 - 2\lambda(\lambda - 1)A_2(x)u(x)}} \quad (\lambda \neq 0, 1), \tag{1}$$

where λ is a fixed real parameter.

The choice of the sign in front of the square root in Equation (1) is equivalent to Henrici's criterion [9, p.532] that reads: *the argument of the root is to be chosen to differ by less than* $\pi/2$ *from the argument of* $(\lambda - 1)f'(x)$. This approach provides the largest magnitude for the denominator of (1).

Later we will see that for $\lambda = 0$ Laguerre's method (1) gives in the limit process Halley's method, given by the iteration function (3) (see Section 2). The choice $\lambda = 1$ also leads to the limit process giving quadratically convergent Newton's method

$$N(x) = x - \frac{f(x)}{f'(x)}. \tag{2}$$

The subject of this paper is to bring more light into the choice of optimal parameter λ appearing in Equation (1). Although there is a lot of papers concerned with the family (1), this theme was rarely considered in the literature. We primarily consider algebraic polynomials since the structure of functions which are combinations of algebraic and transcendental functions is too complicated to provide reliable conclusions, as numerical examples from Section 7 have shown.

Our main goal is to compare convergence characteristics of the methods which are obtained from the Laguerre family (1) choosing different values of the parameter λ. More precisely, we search for the optimal parameter λ that could give approximately the 'best' method, at least for some classes of functions. In our investigation, we vary the parameter λ and study results obtained by methodologies described in short in Section 3 and implemented in Sections 4–7.

This paper is organized as follows: In Section 2 we present some facts on Laguerre's family (1) and list iterative methods that follow for different values of λ. The used comparison methodologies are described in Section 3. In Section 4 we study the dependence of the approximation error $|x - \alpha|$ on the parameter λ in the close neighbourhood of the zero α of f. The comparison of the accuracy of approximations produced by methods from Laguerre's family on the set of six tested algebraic polynomials is given in Section 5. The dynamic study of the same methods is the subject of Section 6. Section 7 is devoted to the comparison of the accuracy of approximations in the case of non-polynomial functions. According to the results exposed in Sections 4–7, in Section 8 we give a short conclusion on the quality of the considered methods of Laguerre's family.

2. Laguerre's family of iterative methods

Theoretical and empirical evidences have shown that the choice of a complex value of the parameter λ in the iteration function (1) does not bring any advantage so that we will deal with real value of λ in this paper. Starting from a suitable initial value $x = x_0$, the iterative process is defined in the form $x_{k+1} = L(x_k;\lambda)$ $(k = 0, 1, \ldots)$ given by Equation (1). For simplicity, we omit the iteration index k and denote the new approximation with $\hat{x}$. Note that in most papers Laguerre's formula (1) was derived taking f to be an algebraic polynomial. An interesting derivation of Equation (1) is given in [8].

Starting from Laguerre's formula (1) we can obtain various cubically convergent methods choosing different values of the parameter λ, among them the following ones:

Halley's method [8, 24, p. 91, 1], $\lambda = 0$, limit case:

$$H(x) = x - \frac{u(x)}{1 - A_2(x)u(x)}. \tag{3}$$

Euler's method [8, 24, p. 94], $\lambda = 2$:

$$E(x) = x - \frac{2u(x)}{1 + \sqrt{1 - 4A_2(x)u(x)}}. \tag{4}$$

Ostrowski's method [8,16], $\lambda \to \pm\infty$, limit case:

$$O(x) = x - \frac{u(x)}{\sqrt{1 - 2A_2(x)u(x)}}. \tag{5}$$

In this paper we will also often consider two other special cases of Laguerre's family obtained for $\lambda = -2$

$$L(x; -2) = x - \frac{2u(x)}{3\sqrt{1 - \frac{4}{3}A_2(x)u(x)} - 1} \tag{6}$$

and for $\lambda = 0.9$

$$L(x; 0.9) = x - \frac{9u(x)}{10 - \sqrt{1 + 18A_2(x)u(x)}}. \tag{7}$$

We consider the iterative method (7) to study methods from Laguerre's family with the convergence behaviour similar to quadratically convergent Newton's method (2) ($\lambda = 1$) having in mind that 0.9 is close to 1.

Remark 2.1: It is interesting to note that Laguerre's family can be derived from Halley's method (3) by *irrationalizing* the rational denominator using a simple approximation. Applying Halley's method (3) to the function λf, where λ is a parameter ($\lambda \neq 0, 1$), we obtain

$$\hat{x} = x - \frac{\lambda f(x)}{\lambda f'(x) - \lambda \frac{f(x)f''(x)}{2f'(x)}} = x - \frac{\lambda f(x)}{f'(x) + (\lambda - 1)f'(x) - \lambda \frac{f(x)f''(x)}{2f'(x)}}. \tag{8}$$

Assume that x is a reasonably good approximation to the zero α so that the quantity

$$\left| \frac{\lambda}{\lambda - 1} \frac{f(x)f''(x)}{f'(x)^2} \right|$$

is sufficiently small. Then, by using the approximation $\sqrt{1 - t} \approx 1 - t/2$, for small $|t|$, and applying it to the appropriate part of the denominator of (1), we obtain

$$\begin{aligned} \hat{x} &= x - \frac{\lambda f(x)}{f'(x) + (\lambda - 1)f'(x)\left[1 - \frac{1}{2}\frac{\lambda}{\lambda - 1}\frac{f(x)f''(x)}{f'(x)^2}\right]} \\ &= x - \frac{\lambda u(x)}{1 + \operatorname{sgn}(\lambda - 1)\sqrt{(\lambda - 1)^2 - 2\lambda(\lambda - 1)A_2(x)u(x)}} = L(x; \lambda). \end{aligned}$$

Remark 2.2: Using an original approach, Hansen and Patrick have derived in [8] the following iterative formula

$$\hat{x} = x - \frac{(\nu + 1)f(x)}{\nu f'(x) + \operatorname{sgn}(\nu)\sqrt{f'(x)^2 - (\nu + 1)f(x)f''(x)}}. \tag{9}$$

The above formula follows from Equation (1) by letting $\lambda = 1/\nu + 1$, which means that Hansen–Patrick method (9) can be obtained from Laguerre's family (1). In fact, the families (1) and (9) are equivalent. Although many authors treat (9) as the original method, it is fair and reasonable to give the priority to Laguerre who derived family (1) a century before Hansen and Patrick, see the references [8,11]. The family (9) can be regarded as the rediscovered Laguerre family.

3. Comparison methodologies

What is a good way to compare zero-finding methods from Laguerre's family using methodologies available at present time? Obviously, a satisfactory answer is to determine the optimal parameter λ in Laguerre's iteration function $L(x;\lambda)$, at least for some classes of functions.

Prior to the beginning of the twenty-first century, when computer graphics have not been sophisticated enough and computer algebra systems (shorter CAS) (such as Maple, Mathematica, Saga, Maxima, etc.) have not been sufficiently developed, the quality of zero-finding methods has been estimated by using only numerical experiments on a given set of tested functions and by analysing computational efficiency based on convergence order and computational costs. Such analysis has had a serious drawback since the mentioned convergence characteristics strongly depend on the choice of initial approximation x_0, the shape of considered functions, and location of zeros. For this reason, a proper ranking of compared methods performed only by the described approach is not reliable and it can differ for different starting points. In many papers authors neglect this fact assuming *a priori* that the initial approximation x_0 is close enough to the sought zero of a given function so that tested methods are always convergent.

Recall that *computational efficiency* of an iterative method *IM* (introduced by Ostrowski [16, p. 20]) is defined by the so-called *efficiency index*

$$\mathcal{E}(IM) = r^{1/d}, \tag{10}$$

where r is the order of convergence of *IM* and d is the number of function evaluations. According to Traub [24, Theorem 5–3], any one-point method of order r must depend explicitly on f and the first $r-1$ derivatives of f. Thus, the best case of Equation (10) appears if $\mathcal{E}(IM) = r^{1/r}$. It is easy to find that the function $\psi(r) = r^{1/r}$ $(= \mathcal{E}(IM))$ attains its maximum for $r = e$ (the base of natural logarithm). Since $e \approx 2.718$, it is clear that one-point methods of third-order possess the maximal computational efficiency. Hence, there follows the interest for that class of one-point zero-finding methods. On the other hand, methods of Laguerre's type, presented above, have the same efficiency index $\mathcal{E} = 3^{1/3} \approx 1.442$ in the class of one-point third-order methods, which means that this measure is irrelevant for comparison procedure. Therefore, the comparison of accuracy of approximations produced by tested methods on the set of the same numerical examples is not sufficient for their proper ranking although this methodology is certainly useful for crude estimation of the quality of their convergence characteristics. For this reason, in finding the optimal parameter λ in Equation (1) we apply the following methodologies:

(i) the dependence of the approximation error $|x - \alpha|$ on the parameter λ in a close neighbourhood of α,
(ii) the comparison by numerical examples and
(iii) the comparison by dynamic study that enables graphic visualization.

We emphasize that the application of the dynamic study (methodology (iii)) provides not only a visualization of convergencebehaviour of compared iterative methods but also gives additional information on iterations such as the total CPU time for the whole basin of attraction and average number of iterations for each tested method regarding all starting points. It also detects whether the method converges to the closest zero. Obviously, the determination of optimal or almost optimal parameter λ is helpful for the user.

4. Approximation error as a function of the parameter λ

In this section we will consider the dependence of error $\varepsilon(\lambda) = |L(x;\lambda) - \alpha|$ of the new approximation $L(x;\lambda)$ as a function of the parameter λ in Equation (1) in a close neighbourhood of the zero α of a polynomial p. We define the close neighbourhood choosing the current approximation

Table 1. Tested functions.

$p(x)$	α
$p_1(x) = (x^4 - 16)(x^4 + 16)(x^{10} + x^9 + 1)$	2
$p_2(x) = (x^3 - 1)(x^3 - 2)(x^3 - 3)(x^3 - 4)(x^3 - 5)(x^3 - 6)$	1
$p_3(x) = (x - 8)(x - 0.9)(x - 1)(x - 1.1)(x - 1.2)(x - 1.3)(x - 1.4)(x - 1.5)$	1
$p_4(x) = x^{17} - 1$	1

$x = \alpha \pm 10^{-m}$, where $m \geq 2$. A number of numerical experiments shown that the choice of the sign and the choice $m > 2$ in $x = \alpha \pm 10^{-m}$ slightly affect the error $\varepsilon(\lambda) = |L(x;\lambda) - \alpha|$ considered as a function of λ in most examples. For this reason, in our study we take the sign '+' and $m = 2$, that is, we deal with the fixed approximation $x = \alpha + 10^{-2}$. For demonstration, we present results obtained for algebraic polynomials displayed in Table 1.

The dependence of the function $\varepsilon(\lambda) = |L(x;\lambda) - \alpha|$ of the parameter/argument λ is presented in Figure 1 for the listed polynomials $p_1 - p_4$ (Table 1). First of all, we observe that the error $\varepsilon(\lambda)$ is considerably large for $\lambda = 1$. This is clear since for $\lambda = 1$ Laguerre's method (1) reduces to only quadratically convergent Newton's method (2). Furthermore, $\varepsilon(\lambda)$ is monotonically decreasing for $\lambda < 1$ for all tested polynomials, which means that the choice of λ in the region $(-\infty, \gamma)$ $(\gamma < 1)$ is recommendable. This is in accordance with the fact that letting $\lambda \to -\infty$ we obtain Ostrowski's method (5), which shows good convergence behaviour for polynomials, see Sections 5 and 6.

Considering the graphs in Figure 1 for $\lambda > 1$ we observe that the behaviour of the error $\varepsilon(\lambda)$ is not monotone, but this error is acceptably small, especially for the polynomials p_1, p_2 and p_4. The error $\varepsilon(\lambda)$ increases very slowly in the case of p_3 not exceeding 1.4×10^{-4} for any $\lambda > 2$. For $\lambda = 2$, thus for Euler's method (4), $\varepsilon(2)$ attains the minimum $\approx 1.5 \times 10^{-7}$ (which is hard to observe from the figure). Since p_3 has the set of close zeros (0.9, 1, 1.1, 1.2, 1.3, 1.4, 1.5), Hence, the hypothesis that Euler's method shows better convergence behaviour related to other methods from Laguerre's family (1) in the case of cluster of zeros might be stated. This question is discussed in Section 6 by analysing the dynamics of considered methods from the family (1) (Figures 2–6).

In overall, although the described approach gives only a crude estimation, it could be concluded that Laguerre's method shows a good convergence characteristics for a very wide range of values of the parameter λ. However, this approach does not lead to the optimal value of λ. The methodologies (ii) and (iii), considered in the next sections, give more precise answer in the search of optimal parameter λ. It is certain that the choice of λ in the interval $[1 - \delta, 1 + \delta]$ for $0 < \delta \ll 1$ should be avoided since

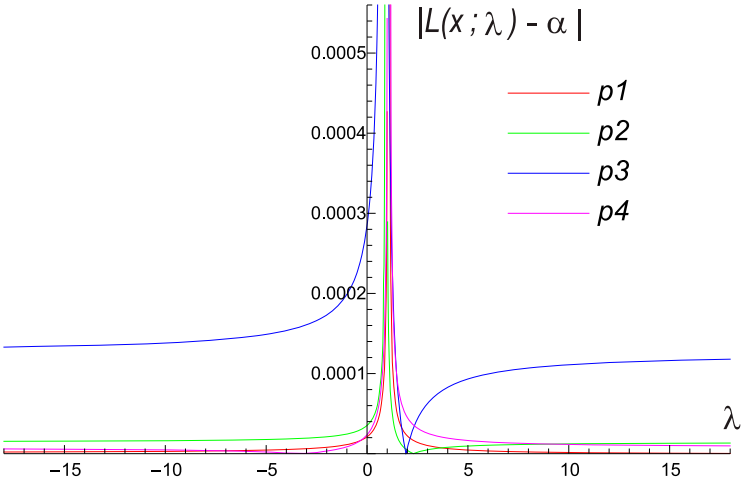


Figure 1. Errors of approximations as a function of λ.

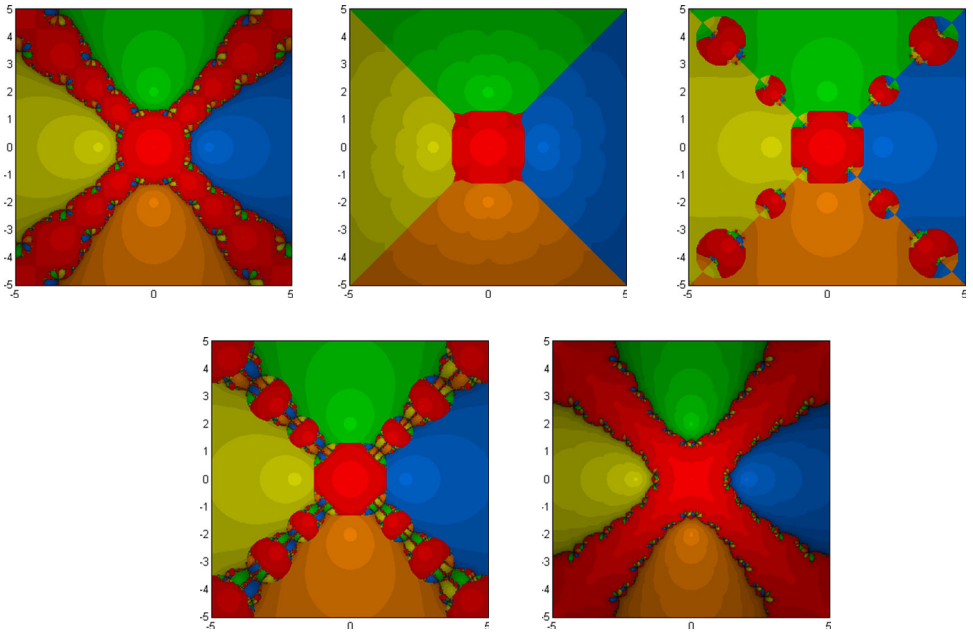


Figure 2. Basins for the polynomial $q_1(x) = x(x^4 - 16)$.

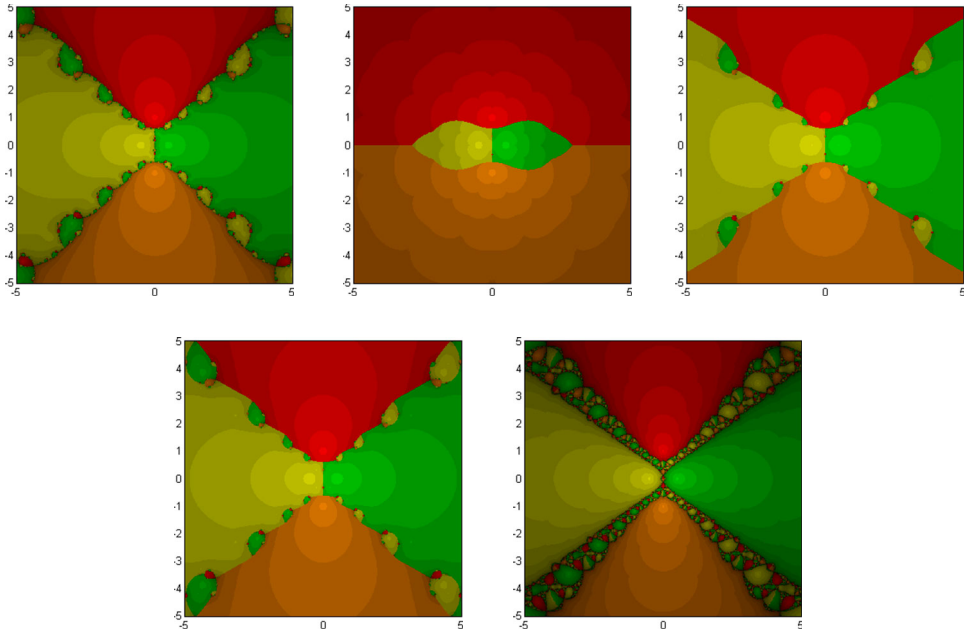


Figure 3. Basins for polynomial $q_2(x) = (x^2 - 0.25)(x^2 + 1)$.

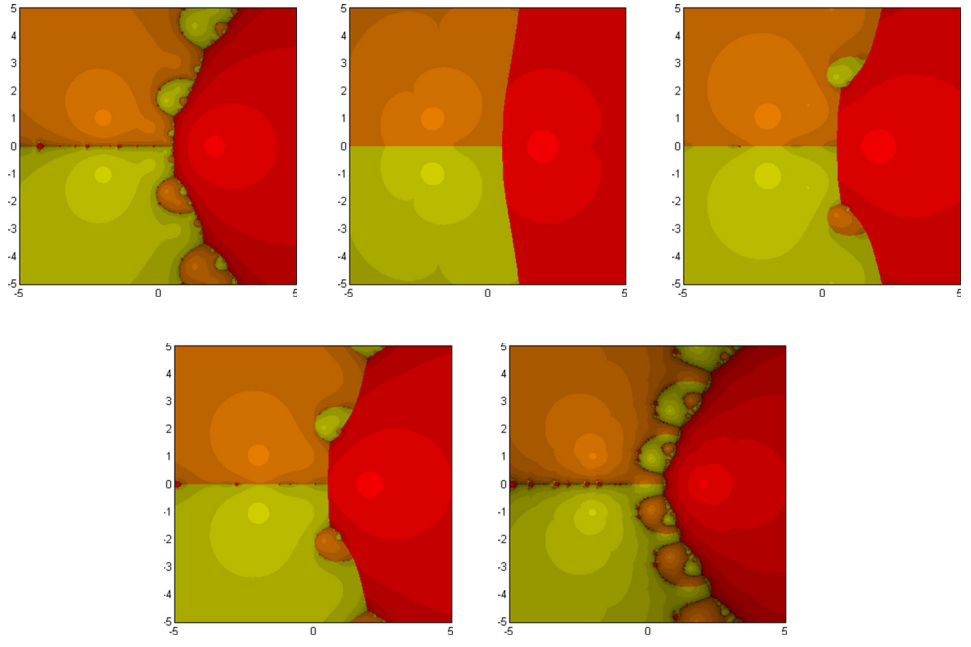


Figure 4. Basins for the polynomial $q_3(x) = (x-2)(x^2+4x+5)$.

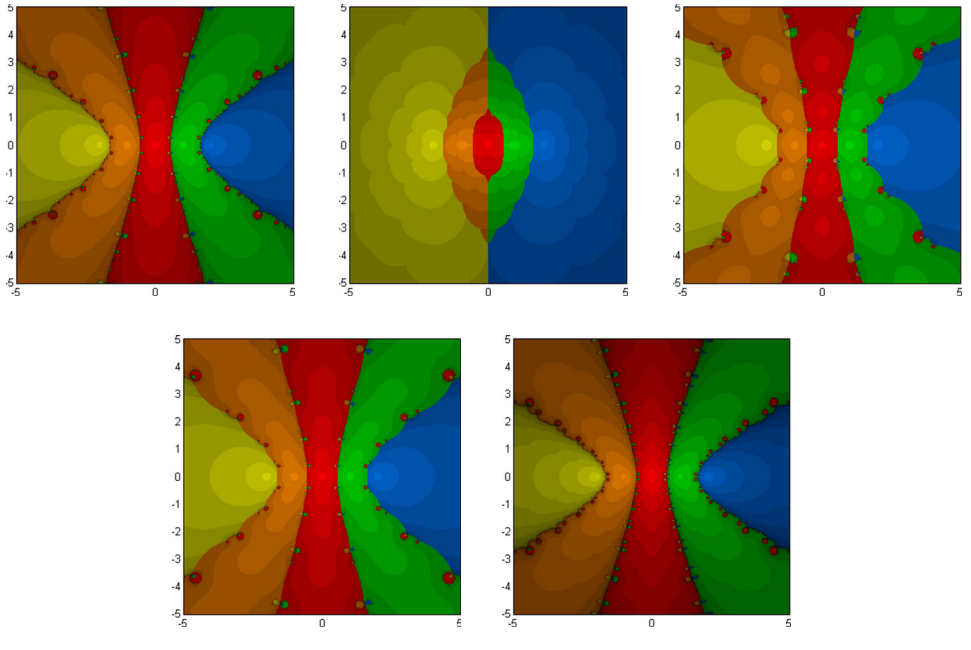


Figure 5. Basins for the polynomial $q_4(x) = x(x^2-1)(x^2-4)$.

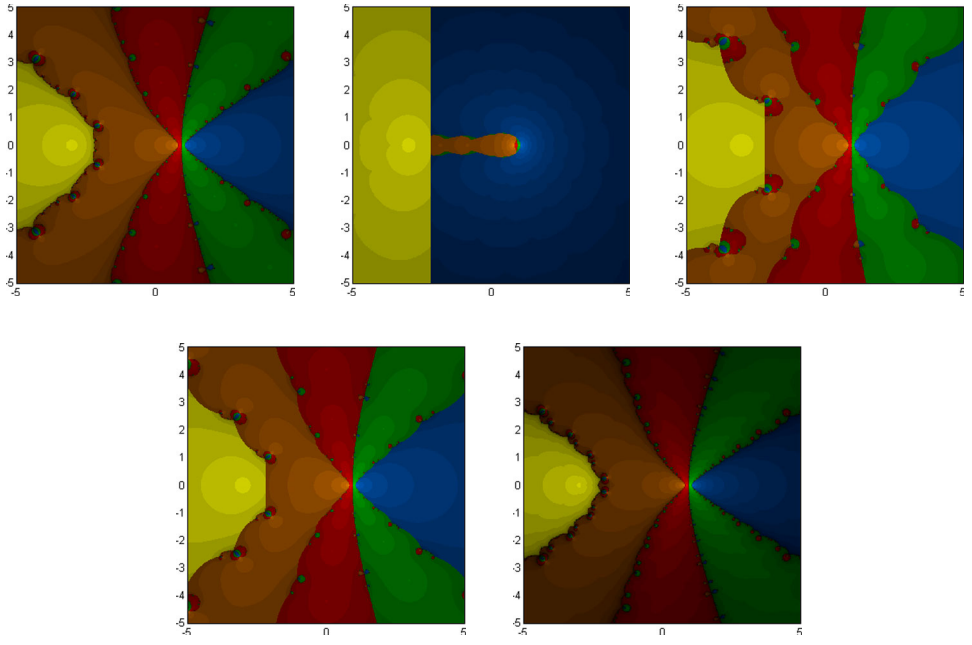


Figure 6. Basins for the polynomial $q_5(x) = (x+3)(x-0.8)(x-0.9)(x-1)(x-1.1)$.

this range of λ defines methods whose convergence behaviour is similar to the second-order Newton method ($\lambda = 1$). In Sections 5 and 6 the value $\lambda = 0.9$ is used to demonstrate this fact.

5. Methodology (ii): comparison by numerical examples

The iterative methods (3)–(7) have been tested on 6 algebraic polynomials starting with a suitably chosen initial approximation x_0. The tested polynomials are displayed in Table 2, together with the initial approximation x_0 and the sought zero α of f_k. For all examples we have applied multi-precision arithmetic in CAS *Mathematica*.

In Table 3 we have presented the errors of approximations $\varepsilon_k = |x_k - \alpha|$ $(k = 1, 2, 3, 4)$ produced by the third-order methods (3)–(7). The minimal errors are marked in bold. Since the convergence behaviour of iterative methods in practice depends on many factors, some of them are commented below, to check the theoretical order of convergence we have displayed in Table 3 the so-called *computational order of convergence* r_c (COC, for brevity) using the approximate formula [10]

$$r_c = \frac{\log|f(x_{k+1})/f(x_k)|}{\log|f(x_k)/f(x_{k-1})|}. \tag{11}$$

Table 2. Tested functions.

$f(x)$	x_0	α
$f_1(x) = (x^8 - 256)(x^7 + x^5 + x^3 + 1)$	2.2+0.2i	2
$f_2(x) = (x^3 - 1)(x^3 + 1)(x^{10} + x^5 + 1)$	1.2	1
$f_3(x) = (x^{10} + 1)(x^6 - i)$	$-1.2i$	$-i$
$f_4(x) = (x-4)(x+1)(x^4 - 16)(x^2 + 9)(x^2 + 2x + 5)$ $\times(x^2 + 2x + 2)(x^2 - 2x + 2)(x^2 - 4x + 5)$	0.2+3.2i	3i
$f_5(x) = \prod_{k=1}^{20}(x - k)$	13.5	13
$f_6(x) = x^{17} - 1$	1.2	1

Table 3. Errors of approximations produced by the methods (3)–(7).

f	Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	r_c by (11)
	Halley's IM (3)	9.68(−2)	1.12(−2)	1.84(−5)	7.88(−14)	3.019
	Euler's IM (4)	0.115	2.37(−2)	1.68(−4)	5.66(−11)	2.966
f_1	Ostrowski's IM (5)	1.03(−2)	4.31(−7)	3.21(−20)	**1.33(−59)**	3.000
	Laguerre's IM (6), $\lambda=-2$	6.20(−2)	1.12(−3)	6.26(−9)	1.10(−24)	2.999
	Laguerre's IM (7), $\lambda=0.9$	0.146	5.79(−2)	1.48(−2)	2.63(−4)	3.599*
	Halley's IM (3)	7.13(−2)	6.20(−3)	5.17(−6)	3.05(−15)	2.978
	Euler's IM (4)	0.138	5.86(−2)	2.40(−2)	4.17(−4)	2.781
f_2	Ostrowski's IM (5)	1.03(−2)	6.75(−6)	1.86(−15)	**3.87(−44)**	2.999
	Laguerre's IM (6), $\lambda=-2$	4.44(−2)	4.40(−4)	2.85(−10)	7.69(−29)	2.999
	Laguerre's IM (7), $\lambda=0.9$	0.108	3.77(−2)	4.29(−3)	1.68(−5)	2.293
	Halley's IM (3)	7.86(−2)	1.08(−2)	3.72(−5)	1.44(−12)	3.002
	Euler's IM (4)	0.150	6.68(−2)	1.63(−2)	1.90(−4)	2.556
f_3	Ostrowski's IM (5)	3.02(−2)	3.17(−4)	2.90(−10)	**2.33(−28)**	3.000
	Laguerre's IM (6), $\lambda=-2$	5.48(−2)	1.99(−3)	7.24(−8)	3.47(−21)	3.001
	Laguerre's IM (7), $\lambda=0.9$	0.112	4.50(−2)	7.85(−3)	1.29(−3)	2.024
	Halley's IM (3)	8.22(−2)	5.66(−3)	1.98(−6)	8.24(−17)	3.007
	Euler's IM (4)	0.102	1.64(−2)	4.06(−5)	6.13(−13)	2.997
f_4	Ostrowski's IM (5)	1.36(−2)	1.95(−6)	5.74(−18)	**1.46(−52)**	2.999
	Laguerre's IM (6), $\lambda=-2$	5.19(−2)	5.80(−4)	7.96(−10)	2.05(−27)	2.999
	Laguerre's IM (7), $\lambda=0.9$	0.131	4.11(−2)	5.61(−3)	1.46(−5)	3.241*
	Halley's IM (3)	0.377	0.124	2.90(−3)	4.06(−8)	2.945
	Euler's IM (4)	9.74(−2)	1.42(−3)	4.02(−9)	**9.15(−26)**	3.000
f_5	Ostrowski's IM (5)	0.178	8.62(−3)	9.87(−7)	1.48(−18)	2.998
	Laguerre's IM (6), $\lambda=-2$	0.217	1.59(−2)	6.36(−6)	4.06(−16)	2.996
	Laguerre's IM (7), $\lambda=0.9$	0.677	0.271	2.87(−2)	9.49(−5)	2.422
	Halley's IM (3)	7.76(−2)	8.87(−3)	1.54(−5)	8.74(−14)	2.965
	Euler's IM (4)	0.142	6.68(−2)	3.21(−2)	1.20(−3)	2.535
f_6	Ostrowski's IM (5)	1.18(−2)	1.38(−5)	2.12(−14)	**7.61(−41)**	2.999
	Laguerre's IM (6), $\lambda=-2$	5.01(−2)	6.81(−4)	8.57(−10)	1.68(−27)	2.999
	Laguerre's IM (7), $\lambda=0.9$	0.113	4.41(−2)	6.66(−3)	6.27(−5)	2.141

From the last column of Table 3 we can conclude that the computational order of convergence r_c, given by Equation (11), mainly matches well the theoretical order, except in the case of equations $f_1(x)=0$ and $f_4(x)=0$ for which r_c of the method $L(x;0.9)$ (marked by *) exceeds 3 in spite of bad approximations. This is accidental outcome appearing sometimes due to the fact that (11) is an approximate formula which is perfect only for good approximations x_k. The line above the entries of r_c for f_1, f_3, f_5 points that r_c is very close to 3 but not exactly equal to 3. The same is valid for entries r_c in Table 8.

From Table 3 we observe that Ostrowski's method (5) produces the most accurate approximations to the zeros for 5 of 6 tested polynomials. Similar results have been obtained for another 20 tested polynomials (not listed here to save the space) of different degree, including polynomials with complex coefficients. This means that Ostrowski's method (5) is the most serious candidate for the best method from Laguerre's family (1) in the case of algebraic polynomials.

Remark 5.1: Experimenting with the parameter λ of large magnitude we found that there are some values of λ that give better results than Ostrowski's method. For example, determining approximations to the zero $\alpha=2$ of the function f_1 Ostrowski's method produced $|x_4-\alpha|\approx 1.33\times 10^{-59}$ (see Table 3), while the method $L(x;-60)$ (for $\lambda=-60$) gave smaller error $|x_4-\alpha|\approx 3.08\times 10^{-66}$. But in the cases similar to the previous one, we meet not only more complicated iterative formula but also the difficult problem *how to choose a proper λ which guarantees better results than Ostrowski's method.*

We end this section with an important comment. Numerical examples rarely can confirm that a concrete method (from the considered class of methods) is always the best, they rather offer empirical evidence, often very useful but not absolutely reliable. Convergence behaviour differs for various

methods and strictly depends on the choice of starting point but also on the structure of a tested function and the location of its zeros. Ostrowski's method produced the most accurate approximations in the considered numerical experiments for most tested polynomials, but one cannot expect that it is the best for all polynomials. Most likely, one or more exceptions can appear, which is the case for the polynomial f_5 where Euler's method (4) has produced the best approximation. For this reason, additional comparison methods are necessary, among which one of the most powerful is presented in Section 6.

6. Methodology (iii): dynamic study

As mentioned above, the rapid growth of computer graphics about the beginning of the third millennium provided a new approach for visual study of convergence behaviour of zero-finding methods – dynamic study. This new methodology, relies on basins of attractions, presents not only visualization of iterative processes but also gives very useful information on convergence characteristics of these processes such as the total CPU time and average number of iterations for all starting points.

Definition 6.1: Let f be a given sufficiently many times differentiable function in some complex domain $Q \subseteq \mathbb{C}$ with simple or multiple zeros $\alpha_1, \alpha_2, \ldots, \alpha_m \in Q$, and let a (convergent) zero-finding iteration be defined by

$$x_{k+1} = \varphi(f; x_k).$$

The basin of attraction for the zero α_i of f is defined as follows:

$$\mathcal{B}_{f,\varphi}(\alpha_i) = \{\xi \in Q \,|\, \text{the iteration } x_{k+1} = \varphi(f; x_k) \text{ with } x_0 = \xi \text{ converges to } \alpha_i\}.$$

Note that the notion of basin of attraction is known for a century, but only recently has been applied for visual study of zero-finding methods, see, e.g. [2,3,5,15,19,21,23,25,26]. We have performed the dynamic study of the methods (3)–(7) on PC with Intel processor i7-2600 working on 3.4 GHz. These methods have been tested for the 640,000 equally spaced points of the square $Q = \{-5, 5\} \times \{-5, 5\}$ centred at the origin. Basins of attraction have been plotted for five algebraic polynomials q_1–q_5 listed in Table 4 together with their zeros.

Observe that the region Q can be always chosen in such a way that contains all zeros of a tested polynomial using the following Henrici's result [9, p. 457]:

All (simple or multiple) zeros $\alpha_1, \ldots, \alpha_m$ of a polynomial

$$P_n(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \quad (c_n \neq 0)$$

of degree n ($\geq m$) are contained in the disk $D = \{x \,|\, |x| \leq R\}$ centred in the origin and having the radius R, where

$$R = 2 \max_{1 \leq j \leq n} |c_{n-j}/c_n|^{1/j}.$$

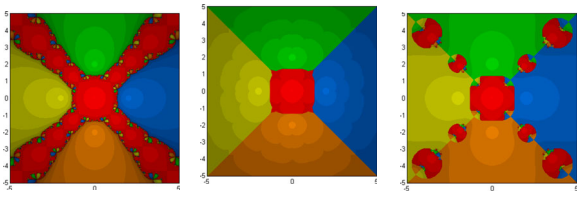
According to Henrici's results, we take for the region Q the tangent square to the disk D with sides parallel to the coordinate axes. In the case of the listed polynomials $q_1 - q_5$ the value of R is given

Table 4. Tested polynomials and their zeros.

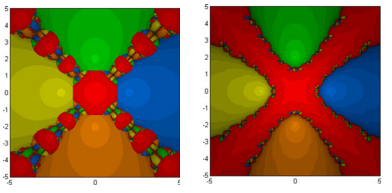
$q(x)$	Zeros	R
$q_1(x) = x(x^4 - 16)$	$0, \pm 2, \pm 2i$	4
$q_2(x) = (x^2 - 0.25)(x^2 + 1)$	$\pm 0.5, \pm i$	2
$q_3(x) = (x - 2)(x^2 + 4x + 5)$	$2, -2 \pm i$	4.31
$q_4(x) = x(x^2 - 1)(x^2 - 4)$	$0, \pm 1, \pm 2$	4.46
$q_5(x) = (x + 3)(x - 0.8)(x - 0.9)(x - 1)(x - 1.1)$	$-3, 0.8, 0.9, 1, 1.1$	4.92

in the last column of Table 4, wherefrom we see that the square $Q = \{-5, 5\} \times \{-5, 5\}$ contains all zeros of the polynomials $q_1 - q_5$.

Basins of attraction are given in Figures 2–6 in the following scheme



Halley's IM (3) Euler's IM (4) Ostrowski's IM (5)



Laguerre's IM (6) Laguerre's IM (7)

For comparison, dynamic study of the considered methods from Laguerre's family has been performed using algebra computer systems Maple and Mathematica. As expected, the obtained basins of attraction are the same, except a negligible number of black points for some methods, see Table 6. This is a consequence of different computational techniques and numerical algorithms for the execution of arithmetic operations of the applied programming languages, the subjects that are out of the scope of this paper. The same causes lead to different CPU times in plotting basins; in general, different CAS consume different CPU times, which is a characteristic of the applied CAS rather than an indicator of the quality of tested methods. In addition, the CPU time is heavily influenced by the characteristics of the micro-processor embedded in digital computer. For the mentioned reasons we have omitted absolute vales of CPU times (expressed in seconds) in Table 5. However, representation of normalized CPU times compared to a fixed method has a sense in comparing tested methods. In our case, we have chosen Ostrowski's method as a reference method, see Table 6. The normalization of CPU times is carried out in the following way:

Let L_1, L_2, L_3, L_4, L_5 denote the methods (3), (4), (5), (6), (7), respectively, and let $t_{L_m}^{(k)}$ be the measured CPU time expressed in seconds, where $k \in \{1, 2, 3, 4, 5\}$ is the index of the example, see Table 5. Taking Ostrowski's method L_3 for the reference method, the normalized CPU time for the method $m \in \{1, 2, 3, 4, 5\}$ is calculated as

$$T_m = \frac{\sum_{k=1}^{5} t_{L_m}^{(k)}}{\sum_{k=1}^{5} t_{L_3}^{(k)}} \quad (m = 1, 2, 3, 4, 5).$$

It is clear that $T_3 = 1$ (normalized value).

Each basin of attraction of a zero is coloured in the following manner: the basin is painted by one of five different colours and shaded by lighter (darker) colour as the number of iterations N_{it} is smaller (larger). To satisfy the required accuracy of approximation to the zero, the program executes not more than $N_{it} = 30$ iterations for every starting point after which proclaims that the point is *convergent*; otherwise, if $N_{it} > 30$, such point is treated as *divergent* one and painted in black. Clearly, it is desirable that the number of black points is as small as possible (the best case is 0 divergent points) for any method. For each basin we record data concerning the total CPU time (in seconds) for all 640,000 points, average number of iterations (for all starting points of the square Q) necessary to satisfy the termination criterion $|x_k - \alpha| < 10^{-7}$ and the number of divergent ('black') points for each of the methods (3)–(7) (see Table 5).

Table 5. Iteration data for the polynomials $q_1 - q_5$ and the methods (3)–(7).

	Maple 18, Mathematica 10		Maple 18
Examples	Methods	B	C (%)
	Halley's IM (3)	5.36	0
Example 1	Euler's IM (4)	5.11	0
$q_1(x)$	Ostrowski's IM (5)	3.98	0
	Laguerre's IM (6), $\lambda = -2$	4.81	0
	Laguerre's IM (7), $\lambda = 0.9$	6.41	16 (0.0025%)
	Halley's IM (3)	6.39	0
Example 2	Euler's IM (4)	6.43	0
$q_2(x)$	Ostrowski's IM (5)	4.74	0
	Laguerre's IM (6), $\lambda = -2$	5.48	0
	Laguerre's IM (7), $\lambda = 0.9$	8.99	141 (0.022%)
	Halley's IM (3)	4.29	45 (0.007%)
Example 3	Euler's IM (4)	3.68	0
$q_3(x)$	Ostrowski's IM (5)	3.57	0
	Laguerre's IM (6), $\lambda = -2$	3.86	0
	Laguerre's IM (7), $\lambda = 0.9$	5.60	12 (0.018%)
	Halley's IM (3)	6.13	0
Example 4	Euler's IM (4)	6.18	0
$q_4(x)$	Ostrowski's IM (5)	5.01	0
	Laguerre's IM (6), $\lambda = -2$	5.48	0
	Laguerre's IM (7), $\lambda = 0.9$	7.72	0
	Halley's IM (3)	9.56	0
Example 5	Euler's IM (4)	9.91	0
$q_5(x)$	Ostrowski's IM (5)	7.33	0
	Laguerre's IM (6), $\lambda = -2$	8.39	0
	Laguerre's IM (7), $\lambda = 0.9$	12.54	18 (0.0028)

Notes: B – Average number of iterations for all starting points; C – Number of 'black' points, expressed also as a percentage (in parenthesis).

Table 6. Iteration data for the polynomials $q_1 - q_5$ and the methods (3)–(7).

	Maple 18			
	Methods	A	B	C (%)
	Halley's IM (3)	**0.982**	6.35	9 (0.0014%)
Average data	Euler's IM (4)	1.258	6.26	**0**
over all	Ostrowski's IM (5)	1	**5.07**	**0**
examples	Laguerre's IM (6), $\lambda = -2$	1.151	5.60	**0**
	Laguerre's IM (7), $\lambda = 0.9$	1.651	8.25	37.4 (0.006%)
	Mathematica 10			
	Halley's IM (3)	1.276	6.35	**0**
Average data	Euler's IM (4)	1.304	6.26	**0**
over all	Ostrowski's IM (5)	**1**	**5.07**	**0**
examples	Laguerre's IM (6), $\lambda = -2$	1.144	5.60	0.768 (0.00012%)
	Laguerre's IM (7), $\lambda = 0.9$	1.856	8.25	14.336 (0.00224%)

Notes: A – Normalized CPU time compared to Ostrowski's method (= 1); B – Average number of iterations for all starting points; C – Number of 'black' points, expressed also as a percentage (in parenthesis).

Obviously, it is desirable that the basin of attraction for each zero has an area which is (a) *unvaried*, (b) *as large as possible*, and (c) *without (or with a few) divergent points*. Besides, it is preferable that (d) *the number of blobs and fractals on the boundary of basin is as small as possible*. Although very complicated dependence of convergence behaviour of applied zero-finding methods on starting points causes that basins of attraction often have a complex and intricate structure, according to the size, hue and shape of basins of attraction, it is possible in most cases to draw a proper conclusion about the quality of considered methods and their reliable ranking.

According to Figures 2–6 we can draw the following conclusions. It is evident that the basins of attraction of Euler's method (4) (the middle picture in the first row of Figures 2–6) have varied area to a minimal extent and less blobs and fractals (the characteristics (a) and (d)) compared to the basins of attraction of the remaining four methods for the polynomials $q_1 - q_4$. There are no divergent points for this method for all tested polynomials, which also holds for Halley's method (3) and Ostrowski's method (5). At first sight, this fact supports the hypothesis mentioned in Section 4 about the advantage of Euler's method in the case of clusters of zeros. However, its basin of attraction for the polynomial q_5, that has the cluster of zeros $(0.8, 0.9, 1, 1, 1)$, has shortcomings. The basin for Euler's method has very small regions of convergence for the zeros 0.9 and 1, which is not the case for the remaining four methods, see Figure 6. Similar exceptions have appeared for other sets of polynomials. Therefore, the cluster-hypothesis for Euler's method is not sufficiently supported. Ostrowski's method (5), with slightly more complicated structure and more blobs of its basins of attractions than Euler's method, is a competitive follower of Euler's method, sometimes with better characteristics (a)–(d) than Euler's method. Therefore, it is impossible to determine which method from Laguerre's family (1) is absolutely the best considering only basins of attraction.

On the other hand, the values of normalized CPU times and the average number of iterations, given in Table 6, provide better and more reliable insight into the quality of the methods (3)–(7). Ostrowski's method (5) requires:

(1) the smallest average number of iterations considering all starting points for all tested polynomials $q_1 - q_5$ and the shortest CPU time when CAS Mathematica is employed and shares

(2) the shortest CPU time together with Halley's method (3) running Maple.

At the same time, this method, together with Euler's method,

(3) converges for all points from the square Q.

The mentioned minimal values are marked bold in Table 6. Having in mind the advantages (1)–(3), convergence behaviour of Ostrowski's method (5) is significantly better compared to the methods (3), (4), (6) and (7). Taking into account the results of Sections 4–6, we conclude that, among the considered methods from Laguerre's family (1), Ostrowski's method (5) shows the best convergence characteristics for most algebraic polynomials.

7. Non-algebraic functions

In previous sections we have seen that iterative methods from Laguerre's family behave very well in solving polynomial equations for almost every parameter λ, except in the neighbourhood of $\lambda = 1$, the value that defines quadratically convergent Newton's method. For this reason, the methods given by (1) with λ close to 1 usually cannot reach cubical convergence. Also, by the help of the methodologies (i)–(iii) presented in Sections 4–6, we have concluded that Ostrowski method (5) gives the best results for most algebraic polynomials.

The above conclusion is no longer valid for Ostrowski's method (5) in finding zeros of transcendental functions or combinations of transcendental and algebraic functions. Unfortunately, the methodology (iii) – the dynamic study, most frequently cannot be applied in searching for optimal parameter λ since it is very hard in most cases to find *all zeros* of non-algebraic functions in the selected region of starting points. Besides, basins of attraction often have strange shape, as illustrated by Figure 7 for Euler's method (4) applied to the function $f_1(x) = (e^{x^2+6x-16} - 1)\sin(x-3)$ (see Table 7) considering the argument x as a complex variable. In addition to the real zeros $2, 3, 3-\pi$ in the square $Q = \{-5, 5\} \times \{-5, 5\}$, we can observe small basins for a pair of complex conjugate zeros $2.038723989 \pm 0.6234897288\, i$ and many very small basins of the zeros of the function $g(x) = \sin(x-3)$ inside Q. Therefore, we can apply only the methodology (ii) – the determination of approximations to the zeros.

Many numerical examples have shown that some methods from Laguerre's family (1) are better for certain classes of functions, other methods are better for other classes. For demonstration, we have tested four functions given in Table 7. The produced results presented in Table 8 confirm the above

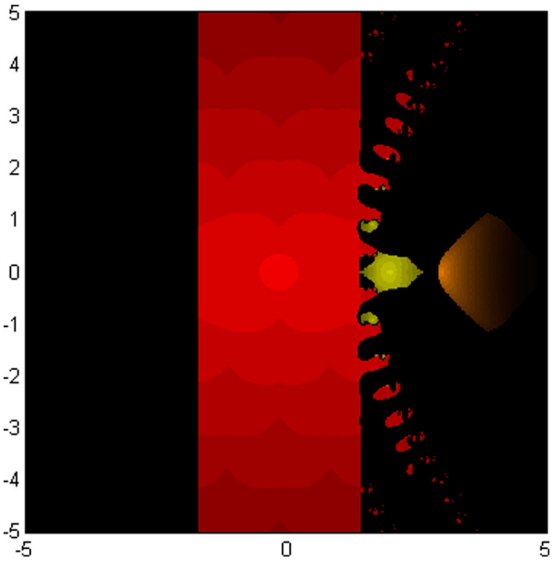


Figure 7. Basins for the polynomial $f_1(x) = (e^{x^2+6x-16} - 1)\sin(x-3)$.

Table 7. Tested functions.

$f(x)$	x_0	α
$f_1(x) = (e^{x^2+6x-16} - 1)\sin(t-3)$	2.2	2
$f_2(x) = x\,e^{x^2} - \sin^2 t + 3\cos t + 5$	−0.9	−1.20764782713 . . .
$f_3(x) = 2\,e^{\sin(x^2-1)} - (x^2+1)\cos^2(1-t)$	1.2	1
$f_4(x) = (x^2+16)\sin[x+2-i]$	−1.7	$-2+i$

Table 8. Errors of approximations produced by the methods (3)–(7).

f	Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	r_c (11)
	Halley's IM	4.18(−2)	4.51(−4)	5.49(−10)	9.91(−28)	2.999
	Ostrowski's IM	3.42(−2)	1.57(−4)	1.52(−11)	1.38(−32)	2.999
f_1	Euler's IM	0.129	2.91(−2)	4.38(−4)	1.17(−9)	2.965
	Laguerre's IM ($\lambda = -2$)	1.35(−2)	1.21(−6)	1.14(−18)	**9.55(−55)**	3.000
	Laguerre's IM ($\lambda = 0.9$)	8.39(−2)	1.43(−2)	1.81(−4)	5.62(−10)	2.862
	Halley's IM	1.85(−3)	1.59(−9)	1.02(−27)	**2.66(−82)**	3.000
	Ostrowski's IM	1.67(−2)	4.19(−6)	6.43(−17)	2.33(−49)	3.000
f_2	Euler's IM	2.82(−2)	4.97(−5)	2.46(−13)	2.96(−38)	3.000
	Laguerre's IM ($\lambda = -2$)	1.17(−2)	7.98(−7)	2.53(−19)	8.04(−57)	3.000
	Laguerre's IM ($\lambda = 0.9$)	0.131	1.13(−2)	1.67(−5)	4.81(−14)	3.022
	Halley's IM	2.68(−2)	1.49(−4)	3.07(−11)	2.67(−31)	2.999
	Ostrowski's IM	1.97(−2)	2.29(−5)	3.77(−14)	1.68(−40)	2.999
f_3	Euler's IM	1.16(−3)	4.71(−9)	3.14(−25)	**9.30(−74)**	3.000
	Laguerre's IM ($\lambda = -2$)	2.26(−2)	5.45(−5)	8.35(−13)	3.01(−36)	2.999
	Laguerre's IM ($\lambda = 0.9$)	4.31(−2)	1.94(−3)	4.41(−7)	5.51(−18)	2.989
	Halley's IM	0.127	3.58(−4)	7.65(−12)	**7.45(−35)**	2.999
	Ostrowski's IM	0.138	3.86(−4)	8.16(−12)	**7.68(−35)**	3.000
f_4	Euler's IM	0.161	5.54(−4)	1.98(−11)	9.09(−34)	2.999
	Laguerre's IM ($\lambda = -2$)	0.129	3.26(−4)	5.19(−12)	**2.09(−35)**	3.000
	Laguerre's IM ($\lambda = 0.9$)	0.221	1.07(−2)	4.88(−7)	4.67(−20)	2.999

assumption: it is impossible to ploclaim approximately the best method in the case of non-algebraic functions. It is interesting to note that in solving the equation $f_4(x) = 0$ the first approximation x_1 is not accurate enough for all tested methods. The explanation is simple: we have taken a real number

for initial approximation although the sought zero is complex. However, all methods show good convergence behaviour in the subsequent iterations.

8. Conclusion

In this paper we have compared convergence characteristics of Laguerre's family (1) of iterative methods $L(x;\lambda)$ for various values of the real parameter λ using three methodologies: the dependence of the approximation error $|x-\alpha|$ on the parameter λ in the close neighbourhood of the zero α, the comparison by numerical examples and the comparison by the dynamic study that provides graphic visualization. We have found that methods $L(x;\lambda^*)$ with λ^* large in magnitude show the best convergence characteristics for most algebraic polynomials. Among these methods it is preferable to implement Ostrowski's method (5) since it is defined by the simplest formula and the fact discussed in Remark 5.1. In the case of non-algebraic functions we have not found an optimal parameter λ which generates the iterative method $L(x;\lambda)$ providing the most accurate approximations to the zeros for sufficiently wide classes of functions.

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Disclosure statement

No potential conflict of interest was reported by the authors.

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