CONGRESSUS NUMERANTIUM

VOLUME 42

MAY, 1984

WINNIPEG, CANADA

HIGHER ORDER HYBRID STÖRMER-COWELL METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS

Beny Neta
Texas Tech University
Department of Mathematics
Lubbock, TX 79409

Abstract

Hybrid Störmer-Cowell methods are constructed for the numerical solution of second order ordinary differential equations not containing y'. The order p of such stable k-step method is not limited to k+1 (k+2). These methods are combined with predictor k-step methods of the same order as the corrector or higher. The coefficients, the order and the error constant for the corrector and predictor(s) will be given.

1. Introduction

In this paper we construct hybrid Störmer-Cowell k-step methods for the numerical solution of a special class of second-order ordinary differential equations,

$$y''(x)=f(x,y(x)),y(x_0)=y_0,y'(x_0)=y_0'. \tag{1}$$

In [1], Neta and Lee have shown how to construct hybrid formulae for the solution of (1). These formulae used the value of f at an offstep point. To be more precise, let

$$x_n = x_0+nh \tag{2}$$

$$y_n = y(x_n) \tag{3}$$

$$f_n = f(x_n,y_n) \tag{4}$$

then one can find $\alpha_i,\beta_i,r,\beta_r$ such that the method

$$y_{n+k} + \sum_{i=0}^{k-1}\alpha_i y_{n+i} = h^2\sum_{i=0}^{k'}\beta_i f_{n+i} + h^2\beta_r f_{n+r} \tag{5}$$

CONGRESSUS NUMERANTIUM, Vol. 42 (1984), pp. 251-264

is of order $p \geq k'+3$, where $k'=k$ (implicit) or $k'=k-1$ (explicit). It was shown in [1] that if the characteristic polynomials

$$\rho(\zeta) = \sum_{i=0}^{k-1} \alpha_i \zeta^i + \zeta^k = \sum_{i=2}^{k} a_i (\zeta-1)^i \tag{6}$$

and

$$\sigma(\zeta) = \sum_{i=0}^{k'} \beta_i \zeta^i = \sum_{i=0}^{k'} b_i (\zeta-1)^i \tag{7}$$

then if the coefficients a_i are given one can find r, β_r and b_i as follows:

$$r = k'+1 + \frac{d_{k'+2}}{d_{k'+1}} (k'+2) \tag{8}$$

$$\beta_r = \frac{d_{k'+1}}{\binom{r}{k'+1}} \tag{9}$$

$$b_i = d_i - \beta_r \binom{r}{i} \quad i=0,1,\ldots,k' \tag{10}$$

where

$$d_i = \sum_{j=0}^{\min(k-2,i)} a_{j+1} \delta_{i-j} \tag{11}$$

and a table for δ_ℓ is given. We choose here a_i so that the methods obtained are of Störmer-Cowell type, i.e.

$$a_i = \binom{k-2}{i-2}, \; i=2,\ldots,k. \tag{12}$$

Method (5) should be complemented by a predictor method to obtain y_{n+r} and if $k'=k$ also by a predictor to obtain y_{n+k}. These predictors should be of at least the same order. Therefore one might need to use points to the left of x_n, i.e.

$$y_{n+r} + \sum_{i=m_1}^{\ell} \alpha_i^p y_{n+i} = h^2 \sum_{i=m_2}^{\ell} \beta_i^p f_{n+i}, \tag{13}$$

where m_1 or m_2 or both may take the value of -1 and ℓ is not larger than k-1. For implicit methods one requires another predictor

$$y_{n+k} + \sum_{i=m_3}^{k-1} \alpha_i^e y_{n+i} = h^2 \sum_{i=m_4}^{k-1} \beta_i^e f_{n+i} + h^2 \beta_r^e f_{n+r}, \tag{14}$$

where m_3 or m_4 may take value of -1. Note that this predictor is using the value f_{n+r}. This form of predictor is different from the one suggested in [1]. The addition of last term makes it possible to take $m_4=0$ for all k and $m_3=0$ for $k>4$.

In the next section we give corrector methods of step $k=3,\ldots,10$. Section 3 will be dvoted to predictor methods. In the last section we quote some of the numerical experiments with some of the methods developed here.

2. Corrector Methods

In this section we list all methods (5) with step $k=3,4,\ldots,10$. The coefficients α_i are given by

$$\alpha_k = \alpha_{k-2} = 1, \alpha_{k-1} = -2, \alpha_i = 0 \text{ for } i<k-2. \tag{15}$$

The parameter r specifying the location of the offstep point, the order p of the method, the error constant C_{p+2} and the coefficients β_i, $i=0,\ldots,k'$, β_r are given in table 1 for explicit methods and in table 2 for implicit ones. Note that the implicit 2-step and 3-step methods are not admissible (see [1]).

k	3	4	5	6
r	2.8	3.73684210	4.70767196	5.69177288
p	5	6	7	8
C_{p+2}	-.10000000-2	-.36462684-3	-.16204766-3	-.80804574-4
β_r	-.12400794	.14516580	.15758052	.16540094
β_0	-.59523809-2	.14084507-2	-.49054041-3	.20950070-3
β_1	.11111111	-.13782051-1	.45665358-2	-.20312433-2
β_2	.77083333	.13030303	-.22638821-1	.95388181-2
β_3		.73690476	.14505551	-.31976901-1
β_4			.71592679	.15683471
β_5				.70202417

Table 1a

Explicit methods k=3,4,5,6

k	7	8	9	10
r	6.68229495	7.67634671	8.67251060	9.67001689
p	9	10	11	12
C_{p+2}	-.43287127-4	-.24221408-4	-.13845229-4	-.79061388-5
β_r	.17054348	.17400933	.17636925	.17797044
β_0	-.10119455-3	.52850086-4	-.28970778-4	.16282699-4
β_1	.10489682-2	-.59021225-3	.34895104-3	-.21126402-3
β_2	-.51209658-2	.30781356-2	-.19603740-2	.12806198-2
β_3	.16196153-1	-.10102852-1	.68694141-2	-.48348316-2
β_4	-.41377820-1	.24254098-1	-.17086510-1	.12835695-1
β_5	.16637432	-.50495023-1	.33295804-1	-.25880675-1
β_6	.69243705	.17411045	-.59028738-1	.42787438-1
β_7		.68568323	.18033057	-.66712287-1
β_8			.68089060	.18523791
β_9				.67751067

Table 1b

Explicit methods k=7,8,9,10

Note that the offstep point is close to the last one but always to the left. This is not always the case for implicit method (see table 2_a,k=5).

Note also that the 3-step method has already appeared in [1] but without the error constant.

k	3	4	5	6
r	not admissible	4.26190476	5.01809955	5.89754386
p		7	8	9
C_{p+2}		.12726982-3	.47341230-4	.21493471-4
β_r		-.48111699-1	-.11625695+1	.27817386
β_0		.72160149-3	-.20243820-3	.74695943-4
β_1		-.88807786-2	.22823046-2	-.84967203-3
β_2		.11359649	-.14310404-1	.48215643-2
β_3		.78396226	.12530652	-.20373494-1
β_4		.15871212	.76315917	.13569099
β_5			.12863343+1	.74575181
β_6				-.14328975

Table 2a

Implicit methods k=4,5,6

k	7	8	9	10
r	6.82612677	7.77918355	8.74616319	9.72180140
p	10	11	12	13
C_{p+2}	.11056649-4	.62071745-5	.37194074-5	.23447000-5
β_r	.20060996	.18250507	.17692934	.17560563
β_0	-.32867724-4	.16345007-4	-.88923170-5	.51816403-5
β_1	.38996793-3	-.20489775-3	.11832348-3	-.73253484-4
β_2	-.22246023-2	.12174863-2	-.74236859-3	.48768778-3
β_3	.84536504-2	-.46406284-2	.29446022-2	-.20415997-2
β_4	-.26987808-1	.13270984-1	-.84334359-2	.60804591-2
β_5	.14506952	-.34082690-1	.19347750-1	-.13954183-1
β_6	.73114235	.15364528	-.41597575-1	.26742561-1
β_7	-.56420173-1	.71877521	.16155723	-.49479752-1
β_8		-.30502150-1	.70820568	.16890562
β_9			-.18320654-1	.69908985
β_{10}				-.11368200-1

Table 2b

Implicit methods k=7,8,9,10

In the next section we give the predictor(s) for each method.

3. Predictor Methods

In this section we first obtain predictors for y_{n+r},

$$y_{n+r} + \sum_{i=m_1}^{\ell} \alpha_i^{P} y_{n+i} = h^2 \sum_{i=m_2}^{\ell} \beta_i^{P} f_{n+i}. \tag{16}$$

Since the error constants for the predictors were much larger than those for the correctors we compensate for that by using methods of order higher than the corrector. We, of course, try to take m_1 and m_2 equal to zero. This was not possible for k=3,4,5. The value of m_2 is zero except for k=3. The value of m_1 is zero except for k=3,4,5. The value of ℓ is k-1 except for k=7,9. In table 3 we list all predictors for explicit methods. In table 4 we list all predictors for implicit ones. The numerical computation were done on IBM 3033 using quadruple precision.

k	3	4	5	6
p	6	7	9	10
m_1	-1	-1	-1	0
m_2	-1	0	0	0
ℓ	2	3	4	5
error constant	-.120644	-.781580	-.126110+1	-.511035+1
α^p_{-1}	.342221	-.258793	.208853	-
β^p_{-1}	-.194560-2	-	-	-
α^p_0	.611842+1	-.396424+1	.120792+2	.367629
β^p_0	.969882	-.695354	.101038+1	.877969-2
α^p_1	-.124635+2	.130984+2	.239818+2	.115265+2
β^p_1	.560138+1	-.318615+1	.154738+2	.117248+1
α^p_2	.500287+1	-.125320+2	-.724675+2	.191310+2
β^p_2	.129577+1	.316797+1	.412074+2	.146835+2
α^p_3		.265667+1	.243363+2	-.631349+2
β^p_3		.100631+1	.156096+2	.367044+2
α^p_4			.108613+2	.216244+2
β^p_4			.134448+1	.138740+2
α^p_5				.948540+1
β^p_5				.123554+1

Table 3a
Predictors for y_{n+r}, k=3,4,5,6
Explicit methods

k	7	8	9	10
p	10	14	14	18
ℓ	5	7	7	9
error constant	.225082+2	-.196303+2	.433475+3	-.114931+3
α_0^p	-.153395+4	.188920	.709131+4	.205771
β_0^p	.338469-2	.227664-2	0	.302836-2
α_1^p	-.292726+4	.176000+2	.111532+6	.282802+2
β_1^p	-.960976+2	.111279+1	.264220+3	.147642+1
α_2^p	.600557+4	.149721+3	.878572+5	.485862+3
β_2^p	-.188826+4	.304916+2	.157969+5	.623928+2
α_3^p	.264761+4	.780667+2	-.490597+6	.185815+4
β_3^p	-.537193+4	.196918+3	.138769+6	.710567+3
α_4^p	-.393145+4	-.486276+3	.219783+6	-.598117+2
β_4^p	-.425259+4	.381792+3	.280850+6	.293923+4
α_5^p	-.261519+3	.694778+2	.799926+5	-.462766+4
β_5^p	-.627614+3	.200924+3	.741289+5	.477573+4
α_6^p		.152752+3	-.150648+5	-.537155+2
β_6^p		.320643+2	-.116481+5	.293761+4
α_7^p		.174690+2	-.596110+3	.185325+4
β_7^p		.149393+1	-.192434+4	.711539+3
α_8^p				.486939+3
β_8^p				.630044+2
α_9^p				.274888+2
β_9^p				.178819+1

Table 3b

Predictors for y_{n+r}, k=7,8,9,10

$m_1=m_2=0$

Explicit methods

k	4	5	6	7
p	7	8	10	10
m_1	-1	-1	0	0
ℓ	3	4	5	5
error constant	-.253727+1	-.254569+1	-.963735+1	.457761+2
α^p_{-1}	-.319885+1	.109108+1	-	-
α^p_0	-.494399+2	.645624+2	.744155	-.328702+4
β^p_0	-.860104+1	.534738+1	.722796-2	-.126693-3
α^p_1	.153404+3	.132571+3	.356354+2	-.106799+5
β^p_1	-.405863+2	.831073+2	.312982+1	-.176602+3
α^p_2	-.144430+3	-.396663+3	.697001+2	.229024+5
β^p_2	.305067+2	.223712+3	.457636+2	-.453655+4
α^p_3	.426655+2	.132924+3	-.212504+3	-.100184+4
β^p_3	.599872+1	.829270+2	.121101+3	-.148777+5
α^p_4		.645137+2	.705428+2	-.754699+4
β^p_4		.527039+1	.455133+2	-.861967+4
α^p_5			.348815+2	-.387649+3
β^p_5			.318632+1	-.103135+4

Table 4a

Predictors for y_{n+r},k=4,5,6,8.

Implicit methods

Note that the implicit 3-step method is not admissible. For the 7-step method we were not able to obtain a predictor of order higher than the corrector (the coefficients obtained were very large!). We thus quote a method of order equal to that of the corrector. Note also that m_2=0 for all k.

k	8	9	10
p	14	14	18
ℓ	7	7	9
error constant	−.321489+2	.534738+3	−.159123+3
α_0^p	.371291	.919053+4	.278898
β_0^p	.401737−2	0	.370340−2
α_1^p	.359664+2	.144642+6	.409100+2
β_1^p	.225246+1	.342409+3	.210232+1
α_2^p	.307613+3	.114874+6	.713224+3
β_2^p	.624200+2	.204774+5	.907521+2
α_3^p	.158651+3	−.635196+6	.274674+4
β_3^p	.404025+3	.180023+6	.104341+4
α_4^p	−.100107+4	.281196+6	−.761357+2
β_4^p	.781732+3	.365276+6	.433338+4
α_5^p	.151508+3	.104616+6	−.685005+4
β_5^p	.407434+3	.986260+5	.705205+4
α_6^p	.309914+3	−.185909+5	−.734132+2
β_6^p	.638506+2	−.139615+5	.433369+4
α_7^p	.360487+2	−.732881+3	.274205+4
β_7^p	.259549+1	−.238246+4	.104557+4
α_8^p			.714945+3
β_8^p			.917146+2
α_9^p			.404536+2
β_9^p			.242710+1

Table 4b
Predictors for y_{n+r},k=8,9,10.
Implicit methods

We now turn to the predictors

$$y_{n+k} + \sum_{i=m_3}^{k-1} \alpha_i^e y_{n+i} = h^2 \sum_{i=m_4}^{k-1} \beta_i^e f_{n+i} + h^2 \beta_r^e f_{n+r} \qquad (17)$$

required when implicit methods are used. We were able to let m_4=0 for all k and m_3=0 for all k>4. The coefficients are given in table 5.

k	4	5	6	7
p	8	9	11	13
m_3	-1	0	0	0
error constant	-.269468	-.136821+1	-.689958	-.258038+1
β_r^e	-.215780-1	-.464910-1	-.584639-1	-.811941-1
α_{-1}^e	-.938103-2	-	-	-
α_0^e	.129840+1	-.105403+1	.756327	-.396132
β_0^e	.458424-1	-.521267-1	.284739-1	-.149795-1
α_1^e	.641121+1	-.331213+1	.118920+2	-.558051+1
β_1^e	.202068+1	-.150816+1	.167908+1	-.853059
α_2^e	-.156801+2	.151065+2	.117751+2	.283090+1
β_2^e	.735934+1	-.380564+1	.149692+2	-.648265+1
α_3^e	.697987+1	-.150604+2	-.486416+2	.333678+2
β_3^e	.180268+1	.370923+1	.328134+2	-.582212+1
α_4^e		.332010+1	.116081+2	-.412494+2
β_4^e		.145608+1	.151649+2	.162641+2
α_5^e			.116100+2	.235109+1
β_5^e			.188556+1	.103471+2
α_6^e				.767624+1
β_6^e				.157279+1

Table 5a
Predictors for y_{n+k},k=4,5,6,7
Implicit methods

k	8	9	10
p	15	17	19
error constant	.138404-1	-.615773+1	-.680595+1
β_r^e	-.963740-1	-.115590	-.131241
α_0^e	.298987	-.156860	.807340-1
β_0^e	.933504-2	-.490357-2	.222649-2
α_1^e	.100326+1	-.505137+1	.442666+1
β_1^e	.921096	-.476312	.317509
α_2^e	.557976+2	-.226128+2	.494427+2
β_2^e	.154948+2	-.749994+1	.825033+1
α_3^e	.144537+2	.285606+2	.142413+3
β_3^e	.792413+2	-.313340+2	.716069+2
α_4^e	-.154045+3	.902992+2	-.254763+2
β_4^e	.142057+3	-.228257+2	.254241+3
α_5^e	.297029+1	-.100322+3	-.325099+3
β_5^e	.852224+2	.481135+2	.394652+3
α_6^e	.581029+2	-.308724+2	-.441341+2
β_6^e	.185942+2	.472018+2	.269221+3
α_7^e	.113889+2	.320033+2	.133291+3
β_7^e	.166572+1	.122309+2	.854651+2
α_8^e		.715276+1	.578833+2
β_8^e		.137857+1	.127366+2
α_9^e			.617251+1
β_9^e			.124776+1

Table 5b

Predictors for y_{n+k}, k=8,9,10

Implicit methods

$m_3=m_4=0$

4. Numerical Experiments

In this section we solve the following problem

$$y'' = y, \quad 0 < x < 1$$
$$y(0) = 1, \tag{18}$$
$$y'(0) = 1,$$

whose exact solution is

$$y(x) = e^x, \tag{19}$$

using the explicit 6-step method of order 8 and the implicit 6-step method of order 9. These methods are combined with the corresponding predictors. All computations were done on IBM 3033 using quadruple precision. The results for various values of h for each method are given in the following two tables. The order of the method was computed numerically and shown to agree with the theoretical result.

h	max. error	rate
1/6	.233486-11	
1/7	.142545-11	3.2
1/8	.696706-12	5.4
1/9	.318410-12	6.6
1/10	.129530-12	8.5
1/11	.319933-13	14.7
1/12	.212229-13	4.7

Table 6

6-step explicit method of order 8

Note that the roundoff errors are dominating in the last case where h=1/12. Note also that we were able to get excellent results already with h=1/6.

h	max. error	rate
1/10	.868438-10	
1/11	.554339-10	4.7
1/12	.361554-10	4.9
1/13	.240315-10	5.1
1/14	.162167-10	5.3
1/15	.110557-10	5.5
1/16	.756912-11	5.9
1/17	.516353-11	6.3
1/18	.347082-11	6.9
1/19	.225755-11	7.9
1/20	.137270-11	9.7
1/21	.716639-12	13.3

Table 7

6-step implicit method of order 9

Note that the error is not as small as in the explicit one.

References

1. B. Neta, S.C. Lee, Hybrid methods for a special class of second-order differential equations, Congressus Numeratium, 38 (1983), 203-225.