Intern. J. Computer Math., 1986, Vol. 20, pp. 67–75
0020-7160/86/2001-0067 $18.50/0
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Printed in Great Britain

Families of Backward Differentiation Methods Based on Trigonometric Polynomials

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(*Received July, 1985; in final form September, 1985*)

Backward differentiation methods based on trigonometric polynomials for the initial value problems whose solutions are known to be periodic are constructed. It is assumed that the frequency w can be estimated in advance. The resulting methods depend on a parameter $v = hw$, where h is the step size, and reduce to classical backward methods if $v \to 0$. Neta and Ford [6] constructed Nyström and generalized Milne–Simpson type methods. Those methods require the Jacobian matrix to have purely imaginary eigenvalues. The methods we construct here will not suffer of this deficiency.

KEY WORDS: Periodic initial value problems, linear multistep methods.

C.R. CATEGORIES: G1.7.

1. INTRODUCTION

Gautschi [3] constructed methods of Adams and Störmer type for problems with oscillatory solutions whose frequency is known. However, these methods are sensitive to changes in the frequency w. Neta and Ford [6] developed Nyström and generalized Milne–

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Simpson type methods. These methods showed less sensitivity to perturbation in w but require the eigenvalues of the Jacobian to be purely imaginary. Many authors developed various methods for this problem (see Stiefel and Bettis [7], Bettis [1], Lyche [5], van der Houwen and Sommeijer [4]).

In this paper we consider the initial value problem

$$y'(x)=f(x,y(x)), \quad y(x_0)=y_0, \tag{1}$$

whose solution is known to oscillate with a known frequency w. We construct backward differentiation formulae. These methods will not suffer from the restriction in [6].

Let us recall some definitions and notations, see e.g. [3]. Let $C^s[a,b]$ $(s\geqq 0)$ denote the linear space of functions $y(x)$ having s continuous derivatives in the finite closed interval $[a,b]$. We assume that the space is normed by

$$\|y\|=\sum_{i=0}^{s}\max_{x\in[a,b]}|y^{(i)}(x)|. \tag{2}$$

DEFINITION A linear functional $\mathscr{L}$ in $C^s[a,b]$ is said to be of *algebraic order p*, if

$$\mathscr{L}x^r\equiv 0, \quad r=0,1,\ldots,p, \quad \mathscr{L}x^{p+1}\not\equiv 0. \tag{3}$$

DEFINITION The method

$$\sum_{j=0}^{k}\alpha_j y_{n+j}=h\beta_k(v)f_{n+k}, \quad v=wh, \quad \alpha_k=+1, \tag{4}$$

is said to be of *trigonometric order q relative to the frequency w* if the associated linear difference operator

$$\mathscr{L}y(x)=\sum_{j=0}^{k}\alpha_j y(x+jh)-h\beta_k(v)y'(x+kh), \tag{5}$$

satisfies

$$\mathscr{L}1=0 \quad \text{and} \quad \mathscr{L}\cos rwx\equiv\mathscr{L}\sin rwx\equiv 0, \quad r=1,2,\ldots,q. \tag{6}$$

$\mathscr{L}\cos((q+1)wx)$ and $\mathscr{L}\sin((q+1)wx)$ not both identically zero.

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2. CONSTRUCTION OF METHODS

The methods constructed are of the form

$$y_{n+k}+\sum_{j=0}^{k-1}\alpha_j y_{n+j}=h\beta_k(v)y'_{n+k}. \qquad (7)$$

The operator $\mathscr{L}$ is then defined by

$$\mathscr{L}y(x)=y(x+kh)+\sum_{j=0}^{k-1}\alpha_j y(x+jh)-h\beta_k(v)y'(x+kh). \qquad (8)$$

2.1 $k=2$. The parameters α_0, α_1 and β_2 can be calculated from

$$\alpha_0+\alpha_1+1=0$$

$$\alpha_0+\alpha_1\cos v+\cos 2v+\beta_2 v\sin 2v=0$$

$$\alpha_1\sin v+\sin 2v-\beta_2 v\cos 2v=0. \qquad (9)$$

The solution obtained via MACSYMA (Project MAC's SYmbolic MAnipulation system written in LISP and used for performing symbolic as well as numerical mathematical manipulation [2]).

The solution is given in terms of $x=\cos v$,

$$\alpha_0=\frac{1}{1+2x},$$

$$\alpha_1=-2\frac{1+x}{1+2x},$$

$$\beta_2=\frac{2\sin v}{v(2x+1)}, \qquad (10)$$

and the method

$$\frac{1}{1+2\cos v}y_n-2\frac{1+\cos v}{1+2\cos v}y_{n+1}+y_{n+2}=h\frac{2\sin v}{v(2x+1)}f_{n+2}.$$

This is a method of trigonometric order 1.

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2.2 $k=3$. In this case we have a one-parameter family of methods of trigonometric order 1. The system

$$\alpha_0+\alpha_1+\alpha_2+1=0,$$

$$\alpha_0+\alpha_1\cos v+\alpha_2\cos 2v+\cos 3v+\beta_3 v\sin 3v=0,$$

$$\alpha_1\sin v+\alpha_2\sin 2v+\sin 3v-\beta_3 v\cos 3v=0, \tag{11}$$

is solved by MACYSMA, using α_0 as a parameter,

$$\alpha_1=\frac{1}{1+2x}-4\alpha_0\frac{x(1+x)}{1+2x},$$

$$\alpha_2=-2\frac{1+x}{1+2x}+\alpha_0\left(2x-\frac{1}{1+2x}\right),$$

$$\beta_3=\frac{2\sin v}{v(1+2x)}(\alpha_0+1). \tag{12}$$

The parameter α_0 can be chosen so as to increase the algebraic order. With the help of MACSYMA, one can show that the truncation error is

$$-\left(\frac{11}{9}\alpha_0+\frac{2}{9}\right)w^2h^3y'_n-\left(\frac{22}{18}\alpha_0+\frac{4}{8}\right)h^3y''''_n-(1+4\alpha_0)w^2\frac{h^4}{2}y''_n+0(h^5).$$

Choosing $\alpha_0=\frac{-2}{11}$ will yield a method of algebraic order 3. The truncation error is then,

$$-\frac{3}{22}w^2h^4y''_n+0(h^5), \tag{13}$$

and the method.

$$-2y_n+\frac{11+8\cos v(1+\cos v)}{1+2\cos v}y_{n+1}-2\frac{12+9\cos v-4\cos^2 v}{1+2\cos v}y_{n+2}$$

$$+11y_{n+3}=h\frac{18\sin v}{v(1+2\cos v)}f_{n+3}. \tag{14}$$

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2.3 $k=4$. The five parameters are computed from

$$\alpha_0+\alpha_1+\alpha_2+\alpha_3+1=0,$$

$$\sum_{j=0}^{3}\alpha_j\cos(vjr)+\cos(4rv)+\beta_4 rv\sin(4rv)=0,\quad r=1,2,$$

$$\sum_{j=1}^{3}\alpha_j\sin(vjr)+\sin(4rv)-\beta_4 rv\cos(4rv)=0,\quad r=1,2, \tag{15}$$

by MACSYMA. Let

$$D=16\cos^3 v+12\cos^2 v-2\cos v-1, \tag{16}$$

then

$$\alpha_0=(2\cos v+1)/D, \tag{17}$$

$$\alpha_1=8\cos^2 v(-\cos v-1)/D, \tag{18}$$

$$\alpha_2=4\cos^2 v(2\cos v+1)^2/D, \tag{19}$$

$$\alpha_3=-8\cos^2 v(\cos v+1)(2\cos v+1)/D, \tag{20}$$

$$\beta_4=\frac{-\cos v(\cos^2 v-1)(2\cos 2v+1)}{D\,v\sin v}. \tag{21}$$

In the next section we describe some of the numerical experiments performed and compare our results with those obtained by previously published methods.

3. NUMERICAL EXPERIMENTS

In our numerical experiments we solved the following differential equation (see [6]):

$$y'''+\lambda y''+y'+\lambda y=0,\quad 0\leqq t\leqq 12\pi, \tag{22}$$

whose exact solution is

$$y(x)=C_1\cos x+C_2\sin x+C_3 e^{-\lambda x}. \tag{23}$$

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In order to get a small perturbation to the periodic solution we choose the initial values

$$y(0)=y'(0)=1+10^{-10},\ y''(0)=-1+10^{-10}. \tag{24}$$

Therefore the constants in (23) are

$$c_1=1+10^{-10}-2\cdot 10^{-10}/(1+\lambda^2),$$

$$c_2=1+10^{-10}+2\cdot 10^{-10}\lambda/(1+\lambda^2),$$

$$c_3=2\cdot 10^{-10}/(1+\lambda^2). \tag{25}$$

In Table I we compare results obtained by Adams method, Nyström method, and generalized Milne–Simpson with a backward differentiation method all of trigonometric order 1 for various values of λ and $h=\pi/60$, $w=1$.

TABLE I

	Adams				
λ	Explicit	Implicit	Nyström	Generalized Milne–Simpson	Backward differentiation
0	0.17–9	0.17–9	0.17–9	0.17–9	−0.15–14
0.1	0.18–9		0.19–9	0.18–9	0.44–11
0.2	0.19–9		0.20–9	0.20–9	0.10–12
0.5	0.20–9		0.36–3	0.14–8	−0.15–14
1.0	0.20–9	0.20–9	unstable	unstable	−0.15–14
5.0	0.18–9				−0.15–14
10.0	0.17–9				−0.15–14
17.5	0.17–9				−0.15–14
20.0	unstable	0.18–9 unconditionally stable			−0.15–14

The results in Table I are taken from [6] except the last column which was obtained with quadruple precision on IBM 3033. This shows that the method is unconditionally stable.

In Table II we compare our backward differentiation method with Adams implicit and generalized Milne–Simpson all of trigonometric

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order 1. The solution of the following system is approximated

$$Y'(t)=F(t,Y),\ Y(0)=(0,1,1,0)^T \tag{26}$$

where

$$Y=(y_1,y_2,y_3,y_4)^T,$$

$$F=(y_2,-y_1/r^3,y_4,-y_3/r^3), \tag{27}$$

and

$$r^2=y_1^2+y_3^2.$$

The exact solution is

$$Y_e=(\sin t,\cos t,\cos t,-\sin t)^T. \tag{28}$$

Clearly, $w=1$. We have computed the solution at $t=12\pi$ using various values of w. In Table II we have compared the L_2 norm of the error at $t=12\pi$ using $h=\pi/60$.

TABLE II

w	Adams	G.M-S	B. D.
0.90	0.231–2	0.230–5	0.323–1
0.95	0.119–2	0.124–5	0.166–1
1.00	0.189–12	0.262–10	0.202–7
1.05	0.125–2	0.144–5	0.174–1
1.10	0.256–2	0.310–5	0.356–1

Note that the method performs equally well if w is over or underestimated. Note also that the results are not as good as those of the Generalized Milne–Simpson method. This phenomenon was pointed out by van der Houwen and Sommeijer [4].

In our last experiment we solve the "almost periodic" problem studied by Stiefel and Bettis [7].

$$z''+z=0.001\,e^{it},\qquad i=\sqrt{-1},\qquad 0\leqq t\leqq 40\pi, \tag{29}$$

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$$z(0)=1, \tag{30}$$

$$z'(0)=0.9995i, \tag{31}$$

whose theoretical solution is

$$z(t)=\cos t+0.0005t\sin t+i(\sin t-0.0005t\cos t). \tag{32}$$

The solution represents motion on a perturbation of a circular orbit in the complex plane: the point $z(t)$ spirals slowly outwards. We write the equations in the form

$$\left.\begin{aligned} y_1' &= y_2, \\ y_2' &= -y_1+0.001\cos t, \\ y_3' &= y_4, \\ y_4' &= y_3+0.001\sin t, \end{aligned}\right\} \tag{33}$$

$$y_1'(0)=1, \qquad y_2(0)=y_3(0)=0, \qquad y_4(0)=0.9995. \tag{34}$$

The exact solution of this system is

$$\left.\begin{aligned} y_1(t) &= \cos t+0.0005t\sin t, \\ y_2(t) &= -0.9995\sin t+0.0005t\cos t, \\ y_3(t) &= \sin t-0.0005t\cos t, \\ y_4(t) &= 0.9995\cos t+0.0005t\sin t. \end{aligned}\right\} \tag{35}$$

The system was solved numerically using backward differentiation method of trigonometric order 2. The results for $h=\pi/60$ are presented in Table III for various values of w.

Note that in this almost periodic problem the results are about the same for all w used. The accuracy in y_1 and y_4 is better than that in the other components.

It is suggested that backward differentiation formulae be used only

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TABLE III

w	G.M–S	Adam Imp.	B. D.
0.90	0.134–3	0.842–3	0.463–2
0.95	0.129–3	0.841–3	0.464–2
1.00	0.115–3	0.839–3	0.464–2
1.05	0.133–3	0.837–3	0.466–2
1.10	0.120–3	0.835–3	0.468–2

in case the Jacobian has purely imaginary eigenvalues. Otherwise, Nyström or generalized Milne–Simpson method may be used.

Acknowledgement

This work was completed while the author was a National Research Council, Research Associate Fellow at the Naval Postgraduate School.

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