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FINITE ELEMENTS VERSUS FINITE DIFFERENCES FOR FLUID FLOW PROBLEMS

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ABSTRACT

In this article, we discuss finite element methods based on bilinear basis functions on rectangles and linear functions on various triangular elements. We compare the results with second and fourth order finite differences and the so-called A, B and C schemes. The model used for this comparative study is the quasi-geostrophic approximation in the shallow water equations on a β-plane.

It is shown that the finite element methods (on isosceles triangles or rectangle) produce better estimates to the frequency than the finite differences.

1. INTRODUCTION

Advective processes are dominant in atmospheric and oceanic circulation systems, while diffusive effects are important only in boundary layer regions. Any numerical model for these systems should treat advective effects accurately. In [8] we analyzed various finite element formulations of the linearized advection equation in two dimensions, which can be written in the form

$$\frac{\partial F}{\partial t} + V \cos \theta \frac{\partial F}{\partial x} + V \sin \theta \frac{\partial F}{\partial y} = 0 , \tag{1}$$

where V is the mean flow speed and θ is the wind direction relative to the x-axis. The quantity F(x,y,t) should be interpreted as vorticity or temperature, for example. The schemes considered in [8] employ leapfrog time differencing and finite element spatial approximations. Linear elements on the following triangles are treated: isosceles (Figure 1), biased (Figure 2), criss-crossed (Figure 3), and unbiased (Figure 4).

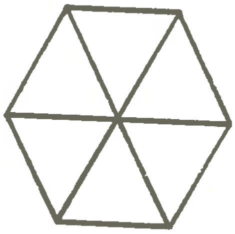

Fig. 1

Isosceles Triangles

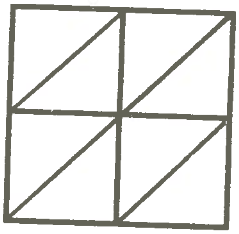

Fig. 2

Biased Grid

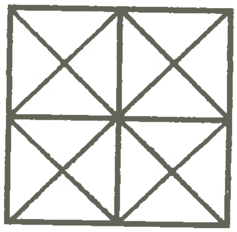

Fig. 4

Unbiased Grid

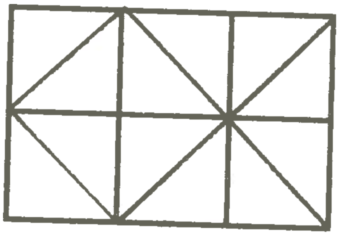

Fig. 3

Criss-Cross Grid

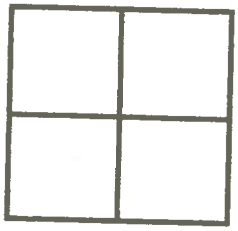

Fig. 5

Rectangular Grid

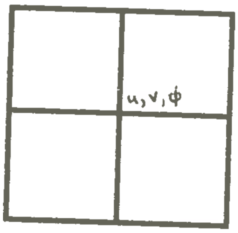

Fig. 6

Grid A

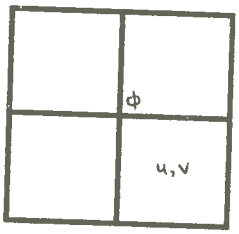

Fig. 7

Grid B

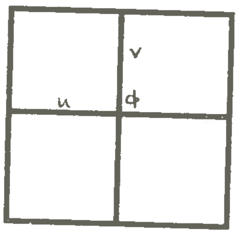

Fig. 8

Grid C

Bilinear basis functions on rectangles (Figure 5) are also examined. The computational stability conditions are derived for each method and the computational phase speed is compared with second- and fourth-order finite differences and the exact value.

The various finite element and finite difference schemes were compared for the case where the mean flow was directed along the x axis ($\theta = 0$). The finite element formulation which is based on isosceles triangles was clearly superior to the formulations based on right triangles. The phase speed for the rectangular finite element scheme was found to be independent of the y-wavenumber. The isosceles triangle scheme gave better phase speeds than the rectangular scheme for low y-wavenumbers, but the situation is reversed for the higher ones. The finite difference schemes gave poorer results. One can conclude that there is little difference between the finite element formulation with isosceles triangles and rectangles. However, in the case of variable grids, the triangles will be changed so that they are no longer isosceles. Also using rectangles, the resulting equations are easier to solve (Staniforth [10]).

2. LINEARIZED SHALLOW WATER MODEL

The linear equations of motion of an inviscid stratified flow can be separated into vertical modes, with the horizontal flow governed by the shallow water equations. See Gill and Clarke [3].

A similar comparative study for the shallow water fluid model with topography is given in [9]. The system of equations consists of three equations with the three forecast variables ϕ, u and v. The equations are

$$\begin{aligned} &\frac{\partial\phi}{\partial t} + \frac{\partial}{\partial x}[u(\phi-\phi_B)] + \frac{\partial}{\partial y}[v(\phi-\phi_B)] = 0 \ , \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + \frac{\partial\phi}{\partial x} = 0 \ , \qquad (2) \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + \frac{\partial\phi}{\partial y} = 0 \ , \end{aligned}$$

where $\phi = gh$ is the geopotential height (h = height of free surface), ϕ_B is the bottom topography (assumed to be independent of time), u and v are the components of the wind speed and f is the Coriolis parameter.

Cullen and Hall [1] showed that the accuracy of the Galerkin finite element solution was better for the vorticity-divergence formulation of the shallow-water equations than for an increase in resolution with the primitive equations (2).

Williams and Schoenstadt [13] noted that staggered variable formulations of the primitive equations and the unstaggered vorticity-divergence formulation gave the best treatment of geostrophic adjustment for small-scale features. The analysis in [9] uses the linearized vorticity divergence formulation

$$\frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x} + V\frac{\partial\phi}{\partial y} + D(\Phi-\phi_B) = u\frac{\partial}{\partial x}(\Phi-\phi_B) + v\frac{\partial}{\partial y}(\Phi-\phi_B) ,$$

$$\frac{\partial\zeta}{\partial t} + U\frac{\partial\zeta}{\partial x} + V\frac{\partial\zeta}{\partial y} + fD = 0 , \qquad (3)$$

$$\nabla^2\phi = f\zeta$$

with the geostrophic relations

$$fu = -\frac{\partial\phi}{\partial y}$$

$$fv = \frac{\partial\phi}{\partial x} \qquad (4)$$

where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the relative vorticity, $D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ is the divergence and f is constant.

The finite elements again perform better than the finite differences with the rectangles than with the isosceles triangles.

3. FREE PLANETARY WAVES

The atmosphere is almost always in a state of near-equilibrium. When this state is disturbed the admosphere adjusts back to near-equilibrium through the action of inertia-gravity waves and planetary waves. The properties of inertia-gravity waves were discussed by Lord Kelvin (Thomson [11]). The slower planetary waves, which are caused by the dependence of the Coriolis parameter f on latitude, are discussed by Longuet-Higgins [4-6].

Mesinger and Arakawa [7] have discussed various finite difference schemes used in oceanic and atmospheric models. These schemes differ in the location of the variables on the grid. Here we discuss A-, B- and C-grids (Figures 6-8). The B-grid is generally used in coarse-resolution models whereas the C-grid is used in fine-resolution models. In the A-grid the variables are unstaggered. The staggering of variables on the B- and C-grid is shown in Figures 7 and 8, respectively.

The basic equations of motion describing the horizontal structure of a normal mode are (see Wajsowicz [12] and Gill [2])

$$\frac{\partial u}{\partial t} - fv + \frac{\partial \phi}{\partial x} = 0 ,$$

$$\frac{\partial v}{\partial t} + fu + \frac{\partial \phi}{\partial y} = 0 , \qquad (5)$$

$$\frac{\partial \phi}{\partial t} + \Phi(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0 ,$$

where

$$f = f_o + \beta y. \qquad (6)$$

To obtain the frequency σ we assume each of the forecast varibles to be of the form

$$e^{i(\mu x + \nu y - \sigma t)} \qquad (7)$$

It was shown by Williams and Neta [14] that the frequency of the Rossby wave is given by

$$\sigma = - \frac{\beta \Phi \theta}{\Phi \delta + f_o^2 \alpha} \qquad (8)$$

where α, θ, δ depend on the numerical scheme used [14].

In the next 8 figures, we contoured the frequency σ as a function of $\mu d/\pi$ and $\nu d/\pi$ for various finite elements and finite difference schemes. It can be seen that the isosceles triangles give results closer to the analytic one than any other method. The rectangular finite elements compete with C grid for second place. For other grid resultions one has similar results (see Williams and Neta [14]).

A comparative study of the numerical zonal and meridional group velocities was also performed in [14]. These group velocities are defined by

$$G_M = \frac{\partial \sigma}{\partial \nu} ,$$

$$G_z = \frac{\partial \sigma}{\partial \mu} . \qquad (9)$$

The authors conclude that for coarser grids the finite elements are better than the finite differences. For finer resolutions the group velocities are well approximated by all but the A and B schemes.

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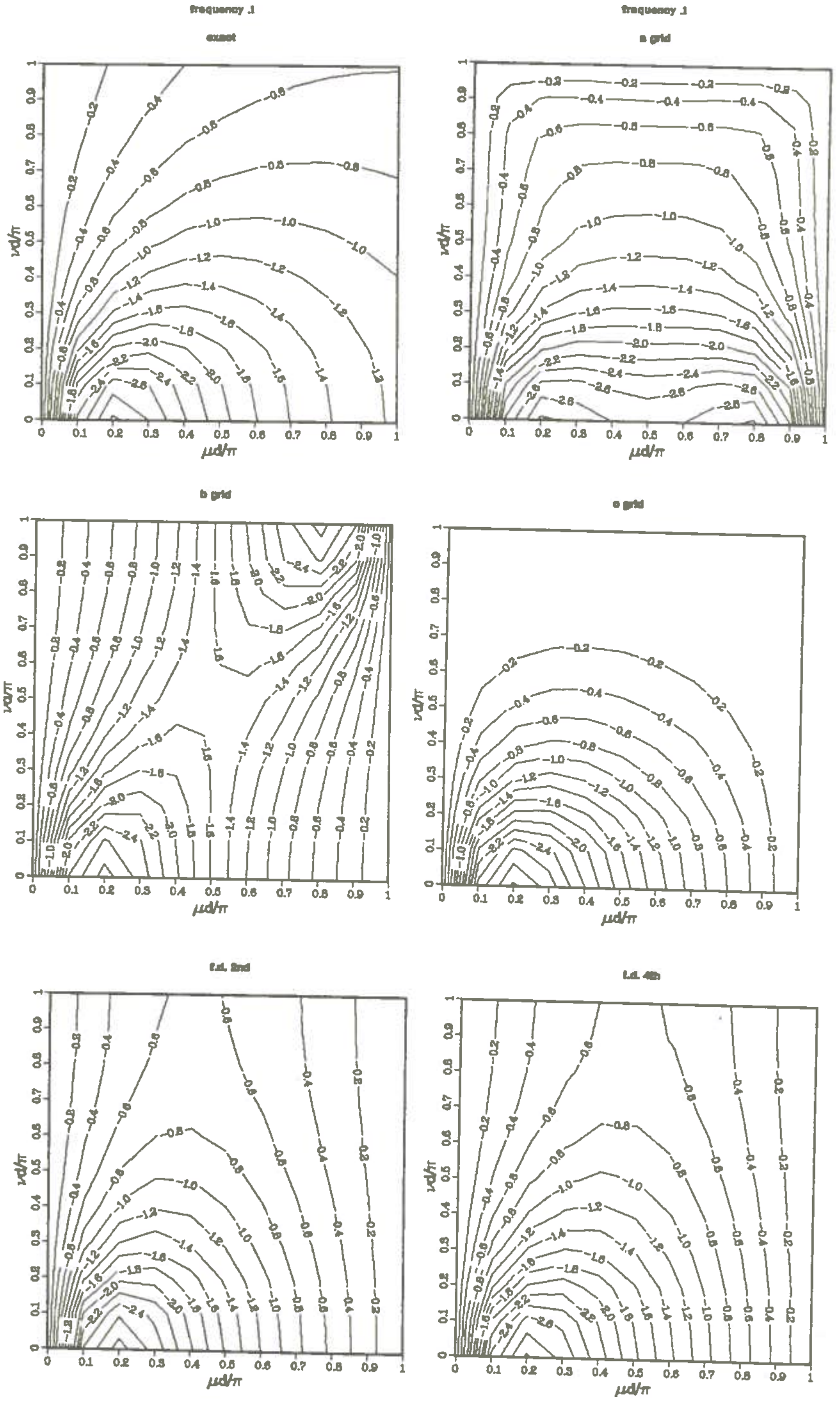
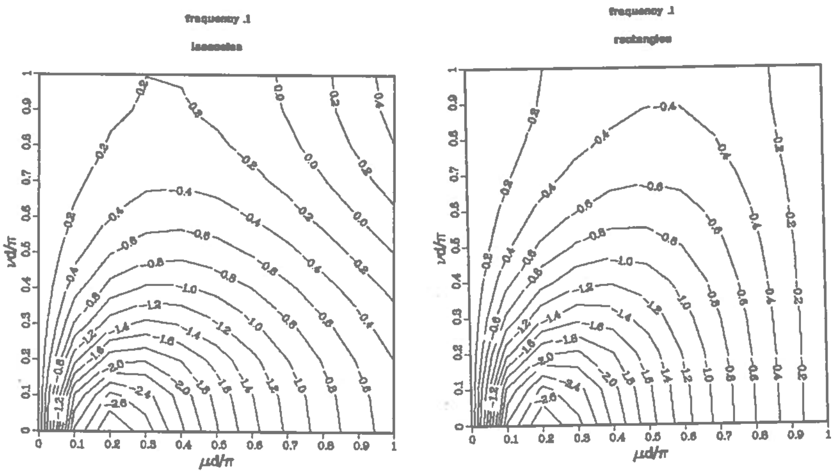

frequency .1
exact
frequency .1
a grid
b grid
c grid
f.d. 2nd
f.d. 4th
νd/π
μd/π



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