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Construction of optimal order nonlinear solvers using inverse interpolation

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ABSTRACT

There is a vast literature on finding simple roots of nonlinear equations by iterative methods. These methods can be classified by order, by the information used or by efficiency. There are very few optimal methods, that is methods of order 2^m requiring $m + 1$ function evaluations per iteration. Here we give a general way to construct such methods by using inverse interpolation and any optimal two-point method. The presented optimal multipoint methods are tested on numerical examples and compared to existing methods of the same order of convergence.

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1. Introduction

There is a vast literature on the solution of nonlinear equations and nonlinear systems, see for example Ostrowski [1], Traub [2], Neta [3] and references there. In general, methods for the solution of polynomial equations are treated differently and will not be discussed here. The methods can be classified as bracketting or fixed point methods. The first class include methods that at every step produce an interval containing a root, whereas the other class produces a point which is hopefully closer to the root than the previous one. Here we construct families of fixed point methods. We define the efficiency index, I, of a method (see Traub [2]) as

$$I = p^{1/d}, \tag{1}$$

where p is the order of the method and d is the number of function- and derivative-evaluation per cycle. Here we will show that our methods are of order $p = 2^m$ and require m function- and one derivative-evaluation per cycle. Thus $d = m + 1$ and the efficiency index is $I_m = 2^{m/(m+1)}$, which supports the Kung–Traub conjecture on the upper bound of the efficiency index of multipoint methods without memory, see [4]. Methods with this property will be called *optimal methods*.

In Section 2 we consider optimal two-point methods that serve as the base of our multipoint iterative schemes at the first two steps. Construction of optimal three-point methods of the order eight, relied on the inverse interpolation, is presented in Section 3. The same approach is applied in Section 4 to derive optimal four-point methods of order 16. In this way we can continue to construct a general m-point optimal method of order 2^m. The presented multipoint methods are tested and compared with existing methods of the same order in Section 5.

2. Optimal two-point methods

In this paper we construct some classes of optimal m-point methods ($m \geqslant 3$) with the optimal order 2^m requiring $m + 1$ function evaluations. These classes rely on optimal two-point methods applied at the first two steps and the inverse interpolatory polynomial of degree m used in latter steps.

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The first optimal two-point methods were constructed by Ostrowski [1], Jarratt [5,6], King [7] and Kung and Traub [4]. New optimal two-point methods have been developed at the beginning of this century, see, e.g., [8–18]. Kung and Traub [4] and Sharma and Goyal [19] have developed the fourth order two-step optimal methods that do not require any derivative. All of these methods are optimal of order four and possess the efficiency index $I_2 = 2^{2/3} \approx 1.587$. Since we use the first derivative when applying inverse interpolation, in this paper we will not consider derivative-free multipoint methods.

A general form of optimal two-point methods with a derivative is as follows:

$$\begin{cases} w_n = N(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{(Newton's method)}, \\ x_{n+1} = \psi(x_n, w_n), \end{cases} \quad (n = 0, 1, \ldots), \tag{2}$$

where ψ is suitably chosen real function such that

(1) requires already calculated values $f(x_n)$ and $f'(x_n)$ and the new entry $f(w_n)$; (3)
(2) provides the fourth order convergence of the sequence $\{x_n\}$.

In this paper, we will mainly restrict ourselves to a rather wide family of optimal two-point methods of order four of the form

$$\begin{cases} w_n = N(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \mu(t_n)\frac{f(w_n)}{f'(x_n)}, \quad t_n = \frac{f(w_n)}{f(x_n)}, \end{cases} \quad (n = 0, 1, \ldots), \tag{4}$$

(see [16]). Here μ and its derivatives μ' and μ'' are real functions such that they are continuous in the neighborhood of 0 and satisfy the conditions $\mu(0) = 1$, $\mu'(0) = 2$, $|\mu''(0)| < \infty$. In fact, we substitute the derivative $f'(w_n)$ in the second step by its approximation $f'(x_n)/\mu(t_n)$, where $t_n = f(w_n)/f(x_n)$. We note that Chun [9] has approximated $f'(w_n)$ by $f'(x_n)h(t_n)$, but the approximation applied in (4) is slightly better since it directly (without any expansion of μ) generates several existing and new optimal two-point methods listed in Table 1. The function μ is called a *multiplicative function* or a *multiplier*. The six forms of μ, denoted with $\mu_1, \ldots, \mu_6$, are given in Table 1 defining in this way different two-point methods. The entry w in Table 1 denotes Newton's approximation $w = x - f(x)/f'(x)$.

All of methods listed in Table 1 are optimal of order four and possess the efficiency index $2^{2/3}$. From this table we observe that Ostrowski's, Kou's and Chun's methods are special cases of King's method obtained for $\beta = 0$, $\beta = 1$, and $\beta = 2$, respectively. Kung–Traub's method, which can be derived using a quadratic inverse interpolation, is a special case of the two-point method defined by the multiplicative function μ_2. Finally, Maheshwari's method is obtained from the two-point method defined by the multiplier μ_5.

3. Construction of optimal eighth order scheme

There are several ways of constructing multipoint methods. Bi et al. [18] used an approximation of derivatives to lower the number of function evaluations. Another possibility is the use of undetermined coefficients (see [20,15]). Petković [14] has developed a class of optimal order multipoint root-finders by using Hermite interpolation to replace the first derivative in a Newton substep following a two-step fourth order method. Interpolation by a nonlinear fraction is used in [17]. Here we suggest to start with any two step optimal method of order four using 2 function- and 1 derivative-evaluation and add

Table 1
List of optimal two-point methods.

Number j	Multiplier $\mu_j(t)$	Two-point method	Author(s)
1	$\frac{1+\beta t}{1+(\beta-2)t}$	$w - \frac{f(w)}{f'(x)} \cdot \frac{f(x)+\beta f(w)}{f(x)+(\beta-2)f(w)}$	King [7]
	$\beta = 0$	$w - \frac{u(x)f(w)}{f(x)-2f(w)}$	Ostrowski [1]
	$\beta = 1$	$x - \frac{f(x)^2+f(w)^2}{f'(x)[f(x)-f(w)]}$	Kou et al.[11]
	$\beta = 2$	$x - u(x)\left[1 + \frac{f(w)}{f(x)} + \frac{2f(w)^2}{f(x)^2}\right]$	Chun [9]
2	$\left(1 + \frac{2}{\lambda}t\right)^{\lambda}$	$w - \frac{f(w)}{f'(x)}\left(1 + \frac{2}{\lambda} \cdot \frac{f(w)}{f(x)}\right)^{\lambda}$	
	$\lambda = -2$	$x - \frac{f(w)}{f'(x)} \cdot \frac{1}{(1-f(w)/f(x))^2}$	Kung and Traub [4]
3	$\frac{1+\gamma t^2}{1-2\gamma}$	$x - \frac{f(w)}{f'(x)f(x)} \cdot \frac{f(x)^2+\gamma f(w)^2}{f(x)-2f(w)}$	
4	$\frac{1}{1-2t+at^2}$	$w - \frac{f(w)}{f'(x)} \cdot \frac{1}{1-2\frac{f(w)}{f(x)}+a\left(\frac{f(w)}{f(x)}\right)^2}$	
5	$\frac{t^2+(c-2)t-1}{ct-1}$	$w - \frac{f(w)}{f'(x)}\left[1 + \frac{f(w)(f(w)-2f(x))}{f(x)(cf(w)-f(x))}\right]$	
	$c = 1$	$x - u(x)\left[\frac{f(w)^2}{f(x)^2} - \frac{f(x)}{f(w)-f(x)}\right]$	Maheshwari [13]
6	$\frac{1}{t}\left(\frac{2}{1+\sqrt{1-4t}} - 1\right)$	$x - \frac{2u(x)}{1+\sqrt{1-4f(w)/f(x)}}$	Petkovićs [12]

substeps resulting from inverse interpolation to arrive at optimal methods of any order. This idea was used by the author in 1981 [21] to obtain an optimal method of order 16.

To get a three-step optimal eighth order method using 4 function evaluations, we use the following cubic polynomial

$$x = R(f(x)) = a + b(f(x) - f(x_n)) + c(f(x) - f(x_n))^2 + d(f(x) - f(x_n))^3. \tag{5}$$

Clearly when substituting $f(x) = f(x_n)$ we have

$$x_n = R(f(x_n)) = a. \tag{6}$$

If we differentiate (5) we get

$$1 = R'(f(x))f'(x) = \left(b + 2c(f(x) - f(x_n)) + 3d(f(x) - f(x_n))^2\right)f'(x). \tag{7}$$

Therefore

$$b = R'(f(x_n)) = \frac{1}{f'(x_n)}. \tag{8}$$

To find the parameters c and d, we substitute $f(x) = f(w_n)$ and $f(x) = f(z_n)$ in (5), where z_n is the result of any optimal fourth order method (2). Upon using the values of a and b above, we get

$$\begin{aligned} w_n &= x_n + \frac{\Delta_f^n(w,x)}{f'(x_n)} + c\left(\Delta_f^n(w,x)\right)^2 + d\left(\Delta_f^n(w,x)\right)^3,\\ z_n &= x_n + \frac{\Delta_f^n(z,x)}{f'(x_n)} + c\left(\Delta_f^n(z,x)\right)^2 + d\left(\Delta_f^n(z,x)\right)^3, \end{aligned} \tag{9}$$

where we denote $\Delta_f^n(w,z) = f(w_n) - f(z_n)$. We can rewrite this system as

$$\begin{aligned} c + d\Delta_f^n(w,x) &= \frac{1}{\Delta_f^n(w,x)f[w_n,x_n]} - \frac{1}{f'(x_n)\Delta_f^n(w,x)},\\ c + d\Delta_f^n(z,x) &= \frac{1}{\Delta_f^n(z,x)f[z_n,x_n]} - \frac{1}{f'(x_n)\Delta_f^n(z,x)} \end{aligned} \tag{10}$$

where we use the divided difference

$$f[x,y] = \frac{f(x) - f(y)}{x - y}. \tag{11}$$

Subtracting the second equation of (10) from the first gives

$$d\Delta_f^n(w,z) = \frac{1}{\Delta_f^n(w,x)f[w_n,x_n]} - \frac{1}{\Delta_f^n(z,x)f[z_n,x_n]} + \frac{1}{f'(x_n)\Delta_f^n(z,x)} - \frac{1}{f'(x_n)\Delta_f^n(w,x)}. \tag{12}$$

Therefore

$$d = \frac{1}{\Delta_f^n(w,x)\Delta_f^n(w,z)f[w_n,x_n]} - \frac{1}{\Delta_f^n(z,x)\Delta_f^n(w,z)f[z_n,x_n]} + \frac{1}{f'(x_n)\Delta_f^n(z,x)\Delta_f^n(w,z)} - \frac{1}{f'(x_n)\Delta_f^n(w,x)\Delta_f^n(w,z)}, \tag{13}$$

$$c = \frac{1}{\Delta_f^n(w,x)f[w_n,x_n]} - \frac{1}{f'(x_n)\Delta_f^n(w,x)} - d\Delta_f^n(w,x). \tag{14}$$

Once we have c and d we get the optimal three-point method

$$x_{n+1} = R(0) = x_n - \frac{f(x_n)}{f'(x_n)} + c[f(x_n)]^2 - d[f(x_n)]^3. \tag{15}$$

We would like to show that this method is of order 8. To this end, we quote a theorem due to Traub [2].

Theorem 1 [2]. *Let $x_{n-m}, x_{n-m+1}, \ldots, x_n$ be $m+1$ approximations to a zero α of f. Let $Q_{m,\gamma}$ be interpolatory polynomial at $y_{n-m}, y_{n-m+1}, \ldots, y_n$ in the sense of*

$$Q_{m,\gamma}^{(k_j)}(y_{n-j}) = \mathscr{F}^{(k_j)}(y_{n-j}) \quad \text{for } j = 0,1,2,\ldots,m, \quad k_j = 0,1,\ldots,\gamma_j,\ \gamma_j \geqslant 1, \tag{16}$$

where $\mathscr{F}$ is the inverse of f. Define a new approximation to α by

$$x_{n+1} = Q_{m,\gamma}(0), \tag{17}$$

and let

$$e_n = x_n - \alpha, \tag{18}$$

then

$$e_{n+1} = M_n \prod_{j=0}^{m} e_{n-j}^{\gamma_j} \tag{19}$$

for suitable constants M_n.

In our case $n = 2$ and $\gamma_0 = \gamma_1 = 1$ and $\gamma_2 = 2$. According to Theorem 1 *we have*

$$e_{n+1} = M_n e_n e_{n-1} e_{n-2}^2. \tag{20}$$

Note that $e_{n-2} = x_n - \alpha$, $e_{n-1} = w_n - \alpha$ and $e_n = z_n - \alpha$. Furthermore, $e_{n-1} = e_{n-2}^2$ since the first (Newton's) step is of second order. Also $e_n = e_{n-2}^4$ and therefore

$$e_{n+1} = M_n e_{n-2}^8, \tag{21}$$

which means that the method (15) *is of order 8*.

Remark 1. The three-point methods (15) are optimal of order eight and possess the efficiency index $I_3 = 2^{3/4} \approx 1.682 > I_2 = 2^{2/3} \approx 1.587$.

4. Construction of optimal 16th order scheme

Such an optimal method was developed by the first author in 1981 [21]. The method has 4 substeps. The first two are an optimal fourth order method (2) and the third is (15). If we let t_n be the approximate at this third step, we can use inverse interpolation at the fourth step as follows:

$$x = R(f(x)) = a + b(f(x) - f(x_n)) + c(f(x) - f(x_n))^2 + d(f(x) - f(x_n))^3 + g(f(x) - f(x_n))^4. \tag{22}$$

Clearly when substituting $f(x) = f(x_n)$ we have, as before,

$$x_n = R(f(x_n)) = a. \tag{23}$$

If we differentiate (22) we get

$$1 = R'(f(x))f'(x) = \Big(b + 2c(f(x) - f(x_n)) + 3d(f(x) - f(x_n))^2\Big)f'(x). \tag{24}$$

Therefore

$$b = R'(f(x_n)) = \frac{1}{f'(x_n)}. \tag{25}$$

To find the parameters c, d, and g we substitute $f(x) = f(w_n)$, $f(x) = f(z_n)$ and $f(x) = f(t_n)$ in (22) and we get a system of three equations in the three unknowns

$$\begin{aligned} c + d\Delta_f^n(w,x) + g\Big(\Delta_f^n(w,x)\Big)^2 &= \frac{1}{\Delta_f^n(w,x)f[w_n,x_n]} - \frac{1}{f'(x_n)\Delta_f^n(w,x)}, \\ c + d\Delta_f^n(z,x) + g\Big(\Delta_f^n(z,x)\Big)^2 &= \frac{1}{\Delta_f^n(z,x)f[z_n,x_n]} - \frac{1}{f'(x_n)\Delta_f^n(z,x)}, \\ c + d\Delta_f^n(t,x) + g\Big(\Delta_f^n(t,x)\Big)^2 &= \frac{1}{\Delta_f^n(t,x)f[t_n,x_n]} - \frac{1}{f'(x_n)\Delta_f^n(t,x)}. \end{aligned} \tag{26}$$

The solution is

$$\begin{aligned} g &= \frac{\frac{\phi_t - \phi_z}{\Delta_f^n(t,z)} - \frac{\phi_w - \phi_z}{\Delta_f^n(w,z)}}{\Delta_f^n(t,w)}, \\ d &= \frac{\phi_t - \phi_z}{\Delta_f^n(t,z)} - g\Big(\Delta_f^n(t,x) + \Delta_f^n(z,x)\Big), \\ c &= \phi_t - d\Delta_f^n(t,x) - g(\Delta_f^n(t,x))^2, \end{aligned} \tag{27}$$

where

$$\phi_t = \frac{1}{f[t,x]\Delta_f^n(t,x)} - \frac{1}{f'(x_n)\Delta_f^n(t,x)}. \tag{28}$$

Once the coefficients are computed, the optimal four-point iterative method is defined by

$$x_{n+1} = R(0) = x_n - \frac{f(x_n)}{f'(x_n)} + cf^2(x_n) - df^3(x_n) + gf^4(x_n). \tag{29}$$

Using Theorem 1, we have

$$e_{n+1} = M_n e_n e_{n-1} e_{n-2} e_{n-3}^2, \tag{30}$$

where $e_{n-2} = e_{n-3}^2,\ e_{n-1} = e_{n-3}^4$, and $e_n = e_{n-3}^8$. Therefore

$$e_{n+1} = M_n e_{n-3}^{8+4+2+2} = M_n e_{n-3}^{16}, \tag{31}$$

which proves that the order of convergence of the four-point method (29) is 16.

Remark 2. The four-point methods (29) are optimal of order 16 and possess the efficiency index $I_4 = 2^{4/5} \approx 1.741 > I_3 \approx 1.682 > I_2 \approx 1.587$.

5. Numerical experiments

We have experimented with our method (15) taking variants of the two-point optimal method (4) and compared it to the following optimal three-point methods of the eight order.

5.1. Multipoint methods of Kung and Traub

In 1974 Kung and Traub [4] constructed two m-point families of iterative methods of the order 2^{m-1}. We present these families, called here K–T family for the sake of brevity, in the form similar to that given in [4], see, also, [15].

K-T (32): For any m, define iteration function $p_j(f)$ $(j = 0,\ldots,m)$ as follows: $p_0(f)(x) = x$ and for $m > 0$,

$$\begin{cases} p_1(f)(x) = x + \gamma f(x), \gamma \text{ is a nonzero constant}, \\ \quad\vdots \\ p_{j+1}(f)(x) = R_j(0), \end{cases} \tag{32}$$

for $j = 1,\ldots,m-1$, where $R_j(y)$ is the inverse interpolatory polynomial of degree at most j such that $R_j(f(p_\lambda(f)(x)) = p_\lambda(f)(x)$ $(\lambda = 0,\ldots,j)$. The iterative method is defined by $x_{n+1}^{(m)} = p_m(f)(x_n)$ starting with an initial guess x_0. Let us note that the family K–T (32) requires no evaluation of derivatives of f. The order of convergence of the sequence $\left\{x_n^{(m)}\right\}$ is 2^{m-1}.

K–T (33): For any m, define iteration function $q_j(f)$ $(j = 0,\ldots,m)$ as follows: $q_1(f)(x) = x$ for $m > 1$,

$$\begin{cases} q_2(f)(x) = x - f(x)/f'(x), \\ \quad\vdots \\ q_{j+1}(f)(x) = S_j(0), \end{cases} \tag{33}$$

for $j = 2,\ldots,m-1$, where $S_j(y)$ is the inverse interpolatory polynomial of degree at most j such that

$$S_j(f(x)) = x, \quad S_j'(f(x)) = 1/f'(x), \quad S_j(f(q_\lambda(f)(x))) = q_\lambda(x) \quad (\lambda = 2,\ldots,j).$$

The iterative method is defined by $x_{n+1}^{(m)} = q_m(f)(x_n)$ starting with an initial guess x_0. The order of convergence of the sequence $\left\{x_n^{(m)}\right\}$ is 2^{m-1}. In a special case for $m = 3$ one obtains the optimal two-point method displayed in Table 1 for μ_2 and $\lambda = -2$.

For a fixed m, the methods K–T (32) and K–T (33) is constructed using a recurrence procedure on a computer, see [4]. In our tests we have taken $m = 4$ to obtain the three-point methods of the eighth order.

5.2. Three-point methods of Bi, Wu and Ren

We have also tested the family of optimal iterative methods of the order eight proposed by Bi et al. [18], given by divided differences in the form:

$$\begin{cases} w_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = w_n - h(t_n)\frac{f(w_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(x_n)+\beta f(z_n)}{f(x_n)+(\beta-2)f(z_n)} \cdot \frac{f(z_n)}{f[z_n,w_n]+f[z_n,x_n,x_n](z_n-w_n)} \quad (\beta \in R), \end{cases} \tag{34}$$

where $t_n = f(w_n)/f(x_n)$ and $h(t)$ is a suitably chosen real-valued function. We tested two methods belonging to the family (35), obtained by choosing two different forms of the function h in the same way as in [18].

5.3. Three-point methods of Thukral and Petković

Based on King's two-point method, the following family of optimal three-point methods was developed in [15],

$$\begin{cases} w_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = w_n - \frac{f(w_n)}{f'(x_n)} \cdot \frac{f(x_n)+\beta f(w_n)}{f(x_n)+(\beta-2)f(w_n)}, & (n = 0, 1, \ldots), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \left(\varphi\left(\frac{f(w_n)}{f(x_n)}\right) + \frac{f(z_n)}{f(w_n)-af(z_n)} + \frac{4f(z_n)}{f(x_n)} \right), \end{cases} \tag{35}$$

where φ is arbitrary real function satisfying the conditions

$$\varphi(0) = 1, \quad \varphi'(0) = 2, \quad \varphi''(0) = 10 - 4\beta, \quad \varphi'''(0) = 12\beta^2 - 72\beta + 72,$$

and a and β are real parameters.

5.4. Three-point methods of Petković

Using any optimal two-point method of the fourth order of the form (2) and a suitable approximation of a derivative in the third step, the following family of optimal three-point methods of the eight order has been derived in [14],

$$\begin{cases} w_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = \psi(x_n, w_n), \qquad (n = 0, 1, \ldots), \\ x_{n+1} = z_n - \frac{f(z_n)}{s_n}, \end{cases} \tag{36}$$

where ψ is a real function satisfying the condition (3) and

$$s_n = 2(f[x_n, z_n] - f[x_n, w_n]) + f[w_n, z_n] + \frac{w_n - z_n}{w_n - x_n}(f[x_n, w_n] - f'(x_n)).$$

In fact, s_n is the approximation of the derivative $f'(z_n)$ in Newton's formula $x_{n+1} = z_n - f(z_n)/f'(z_n)$, obtained by the Hermite interpolation polynomial of degree three.

We did not insert numerical results obtained by iterative methods of lower computational efficiency than that of the methods (15) and 32, (33)–(36) since the latter methods are obviously more efficient and possess (as expected) clear dominance.

We now turn to the four examples listed in Table 2 along with the initial guess used.

The absolute values $|x_n - \alpha|$ in the first three iterations for the tested examples are given in Tables 3–6, where $A(-t)$ means $A \times 10^{-t}$. The computational order of convergence, evaluated by the approximate formula (see [22])

$$r_c \approx \frac{\log |(x_{n+1} - \alpha)/(x_n - \alpha)|}{\log |(x_n - \alpha)/(x_{n-1} - \alpha)|}, \tag{37}$$

is included in the presented tables.

Table 2
List of experiments with initial guesses.

Example	Function	Root α	Initial guess x_0
1	$(x-2)(x^{10}+x+1)e^{-x-1}$	2	2.1
2	$x^2\sin^2 x + e^{x\cos x \sin x} - 18$	5.37643861 ...	5.9
3	$e^{-x^2+x+2} - \cos(x+1) + x^3 + 1$	−1	0
4	$x^2 - (1-x)^{25}$	0.14373925 ...	0.35

Table 3
Results of Examples 1, $f(x) = (x-2)(x^{10}+x+1)e^{-x-1}$.

Thee-point methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	r_c (37)
(15)–King's IM ($\beta = 0$)	3.75(−5)	1.08(−31)	5.13(−244)	7.999987
(15)–King's IM ($\beta = 1$)	9.67(−5)	9.37(−28)	7.28(−212)	7.999947
(15)–King's IM ($\beta = 2$)	1.31(−4)	1.87(−26)	3.32(−201)	7.999913
(15)–Maheshwari's IM	1.14(−4)	4.80(−27)	4.78(−206)	7.999930
(15)–(2) $\mu(t) = (1+t)^2$	1.17(−4)	6.13(−27)	3.38(−205)	7.999931
K-T (32), $\gamma = 0.01$	3.36(−4)	6.28(−23)	9.44(−173)	7.999784
K-T (33)	7.50(−5)	7.47(−29)	7.27(−221)	7.999912
(34), $h(t) = 1 + \frac{4t}{2-5t}$, $\beta = 3$	1.83(−5)	3.15(−34)	2.45(−264)	7.999863
(34), $h(t) = 1 + 2t + 5t^2 + t^3$, $\beta = 3$	1.64(−4)	9.83(−26)	1.58(−195)	8.000732
(35), $\varphi(t) = 12t^3 + 5t^2 + 2t + 1$, $a = 0$	1.50(−4)	8.13(−26)	6.15(−196)	7.999682
(35), $\varphi(t) = (1 + \frac{t}{1-2t})^2$, $a = 0$	6.12(−5)	1.11(−29)	1.34(−227)	7.999471
(36)–King's IM ($\beta = 0$)	1.45(−5)	1.19(−35)	2.41(−276)	7.999997
(36)–King's IM ($\beta = 1$)	8.39(−5)	2.68(−28)	2.98(−216)	7.999952
(36)–King's IM ($\beta = 2$)	1.50(−4)	8.93(−26)	1.39(−195)	7.999878

Table 4
Results of Examples 2, $f(x) = x^2 \sin^2 x + e^{x\cos x \sin x} - 18$.

Thee-point methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	r_c(37)
(15)–King's IM ($\beta = 0$)	2.00(−4)	7.87(−30)	4.46(−233)	7.999958
(15)–King's IM ($\beta = 1$)	2.14(−4)	2.59(−29)	1.78(−228)	7.999944
(15)–King's IM ($\beta = 2$)	2.32(−4)	7.08(−29)	5.46(−225)	7.999932
(15)–Maheshwari's IM	2.23(−4)	4.39(−29)	9.91(−227)	7.999938
(15)–(2) $\mu(t) = (1+t)^2$	2.22(−4)	4.25(−29)	7.63(−227)	7.999939
K-T (32), $\gamma = 0.01$	1.08(−5)	3.02(−40)	1.16(−316)	7.999998
K-T (33)	2.07(−4)	1.47(−29)	9.70(−231)	7.999951
(34), $h(t) = 1 + \frac{4t}{2-5t},\ \beta = 3$	2.65(−4)	7.11(−30)	1.89(−234)	7.999897
(34), $h(t) = 1 + 2t + 5t^2 + t^3$, $\beta = 3$	2.93(−4)	2.23(−28)	2.53(−221)	7.999947
(35), $\varphi(t) = 12t^3 + 5t^2 + 2t + 1$, $a = 0$	9.75(−4)	1.52(−23)	5.27(−182)	8.000351
(35), $\varphi(t) = \left(1 + \frac{t}{1-2t}\right)^2,\ a = 0$	9.31(−4)	5.48(−24)	7.81(−186)	8.000247
(36)–King's IM ($\beta = 0$)	1.24(−4)	1.87(−32)	5.08(−255)	7.999979
(36)–King's IM ($\beta = 1$)	1.48(−4)	4.32(−31)	2.23(−243)	7.999959
(36)–King's IM ($\beta = 2$)	1.81(−4)	5.17(−30)	2.27(−234)	7.999940

Table 5
Results of Examples 3, $f(x) = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1$.

Thee-point methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	r_c(37)
(15)–King's IM ($\beta = 0$)	6.96(−5)	2.57(−36)	9.02(−288)	7.999990
(15)–King's IM ($\beta = 1$)	7.02(−5)	2.39(−36)	4.26(−288)	7.999992
(15)–King's IM ($\beta = 2$)	7.09(−5)	2.14(−36)	1.48(−288)	7.999994
(15)–Maheshwari's IM	7.06(−5)	2.27(−36)	2.61(−288)	7.999993
(15)–(2) $\mu(t) = (1+t)^2$	7.05(−5)	2.27(−36)	2.64(−288)	7.999932
K-T (32), $\gamma = 0.01$	4.11(−6)	4.51(−46)	9.39(−366)	7.999999
K-T (33)	6.99(−5)	2.49(−36)	6.44(−288)	7.999991
(34), $h(t) = 1 + \frac{4t}{2-5t},\ \beta = 3$	7.69(−5)	3.72(−36)	1.13(−286)	7.999989
(34), $h(t) = 1 + 2t + 5t^2 + t^3$, $\beta = 3$	7.73(−5)	5.46(−36)	3.40(−285)	7.999985
(35), $\varphi(t) = 12t^3 + 5t^2 + 2t + 1$, $a = 0$	3.87(−4)	4.09(−31)	6.64(−247)	7.999489
(35), $\varphi(t) = \left(1 + \frac{t}{1-2t}\right)^2,\ a = 0$	3.86(−4)	2.90(−31)	2.74(−248)	8.001073
(36)–King's IM ($\beta = 0$)	1.22(−4)	9.15(−35)	9.30(−276)	7.999983
(36)–King's IM ($\beta = 1$)	1.22(−4)	7.75(−35)	2.02(−276)	7.999987
(36)–King's IM ($\beta = 2$)	1.23(−4)	6.35(−35)	3.20(−277)	7.999992

Table 6
Results of Examples 4, $f(x) = x^2 - (1-x)^{25}$.

Thee-point methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	r_c(37)
(15)–King's IM ($\beta = 0$)	1.34(−4)	9.19(−27)	4.14(−204)	8.000685
(15)–King's IM ($\beta = 1$)	2.85(−4)	3.26(−23)	8.96(−175)	8.001402
(15)–King's IM ($\beta = 2$)	3.28(−4)	2.12(−22)	5.98(−168)	8.001756
(15)–Maheshwari's IM	3.12(−4)	1.04(−22)	1.46(−170)	8.001621
(15)–(2) $\mu(t) = (1+t)^2$	3.15(−4)	1.14(−22)	3.12(−170)	*8.001620*
K-T (32), $\gamma = 0.01$	2.66(−4)	8.45(−24)	8.27(−180)	8.001179
K-T (33)	2.39(−4)	3.45(−24)	6.20(−183)	8.001042
(34), $h(t) = 1 + \frac{4t}{2-5t},\ \beta = 3$	8.94(−3)	7.78(−10)	2.11(−66)	8.012016
(34), $h(t) = 1 + 2t + 5t^2 + t^3$, $\beta = 3$	3.76(−3)	2.64(−12)	2.24(−85)	7.982736
(35), $\varphi(t) = 12t^3 + 5t^2 + 2t + 1$, $a = 0$	1.97(−2)	3.10(−8)	1.32(−53)	7.817654
(35), $\varphi(t) = \left(1 + \frac{t}{1-2t}\right)^2,\ a = 0$	1.81(−2)	1.44(−9)	3.23(−65)	7.841149
(36)–King's IM ($\beta = 0$)	9.94(−4)	4.24(−19)	4.60(−142)	8.000171
(36)–King's IM ($\beta = 1$)	1.08(−3)	2.49(−17)	2.08(−126)	7.999431
(36)–King's IM ($\beta = 2$)	1.04(−3)	7.07(−17)	3.19(−122)	8.000159

From the results shown in Tables 3–6 and a number of numerical experiments we conclude that the proposed three-point methods (15) are remarkably fast. This class of methods is competitive to other existing eight order methods. Applying the iterative formula (15) we could not find a specific two-point method among optimal fourth order methods, used at the first two steps, which would be best for all tested examples. If initial approximations are sufficiently close to the wanted roots, then only two iterations are necessary for most practical problems. Also, the computational order of convergence r_c, defined by (37), matches very well the theoretical results given in Theorem 1. Finally, from Table 6 (Example 1) we observe that the methods (34) and (35) show relatively slower convergence rate compared with other tested methods. Note that solving

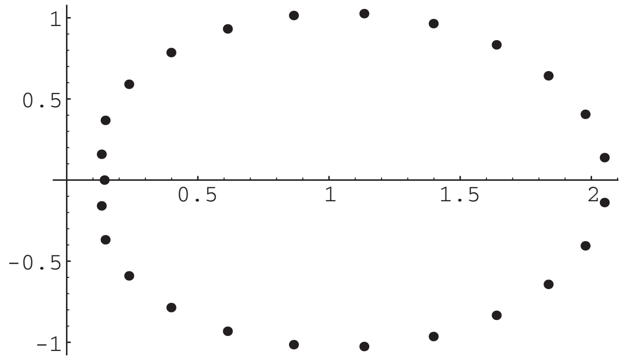

Fig. 1. Distribution of the roots of the polynomial $f(x) = x^2 - (1 - x^{25})$.

tested polynomial $f(x) = x^2 - (1 - x)^{25}$ is not an easy task because this polynomial has very large coefficients in magnitude and roots grouped in the rectangle $[0,2] \times [-1,1]$ making a cluster, see Fig. 1.

The optimal four-point methods (29) were also tested on a number of various numerical examples. These methods produce spectacularly fast convergence which perfectly coincides with the theoretical order of convergence equal to 16. We omitted numerical results to save the space.

We note that, applying iterative methods for solving nonlinear equations, an important problem of determining good initial guesses appears. An efficient method for finding sufficiently accurate initial guesses was recently proposed in [23]. To demonstrate, we give a simple program in the programming package *Mathematica*, applied to the function from Example 3 and a rather wide interval $[-4, 3]$,

```
f[x_] = Exp[-x^2 + x + 2] - Cos[x + 1] + x^3 + 1; a = -4; b = 3; m = 5;
x0 = 0.5 * (a + b + Sign[f[a]] * NIntegrate[Tanh[m * f[x]], {x, a, b}])
```

The outcome $x_0 = -1.00015$ is a very close approximation to the root $\alpha = -1$. To analyze the convergence behavior of the tested methods, we have chosen considerably cruder initial guess $x_0 = 0$ to decelerate (!) the convergence rate, see Table 5.

6. Conclusions

We gave a general way to construct m-point methods by using any optimal two-point method at the first two steps and inverse interpolation at the next steps. These methods have the order 2^m and require $m + 1$ function evaluations per iteration, which means that they are optimal in Kung and Traub's sense [4]. Their efficiency index $I_m = 2^{m/(m+1)}$ is equal or greater compared to existing methods. The presented multipoint methods are tested on several nonlinear functions and demonstrate remarkably fast convergence. A comparison analysis shows that the proposed methods are competitive with existing optimal methods.

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