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Developing high order methods for the solution of systems of nonlinear equations

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ABSTRACT

Two families of order six for the solution of systems of nonlinear equations are developed and compared to existing schemes of order up to six. We have found that one of the methods in the literature has been rediscovered. The comparison is based on the total cost of an iteration and the performance on 14 examples of systems of dimensions 2–9.

1. Introduction

The solution of systems of nonlinear equations is required whenever a nonlinear partial differential equation is approximated. The most well known scheme is Newton's method given by (see e.g. [1,2] or [3])

$$x_{n+1} = x_n - \left[F'(x_n)\right]^{-1} F(x_n), \tag{1}$$

where $F(x) = 0$ is the system to be solved and $F'(x_n)$ is the Jacobian. Assuming one has a close enough initial vector x_0 and that the Jacobian never vanishes for any iterate x_n, the method will converge quadratically. This method requires the construction of the Jacobian and the solution of a system of linear equation at every step. To reduce the cost, one can keep the Jacobian fixed for say k iterates. In this case the order is $k+1$, e.g. if we keep the Jacobian for 3 iterates, we get a fourth order method. This is called modified Newton's method, denoted by MN, and given by

$$\begin{aligned}
y_n &= x_n - \left[F'(x_n)\right]^{-1} F(x_n), \\
z_n &= y_n - \left[F'(x_n)\right]^{-1} F(y_n), \\
x_{n+1} &= z_n - \left[F'(x_n)\right]^{-1} F(z_n).
\end{aligned} \tag{2}$$

There are other ways to modify the procedure, e.g. Steffensen method using divided difference to replace the Jacobian, see e.g. [4], Ezquerro et al. [5] and also a survey by Rheinboldt [6]. Artidiello et al. [7] have suggested the use of divided difference instead of one of the Jacobians.

Neta [8] has developed a fourth order method, denoted Neta4, based on his sixth order method for the solution of a single equation [9]. The method is given by

$$y_n = x_n - \left[F'(x_n)\right]^{-1} F(x_n),$$

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$$z_n = y_n - Q_1(x_n, y_n)\big[F'(x_n)\big]^{-1}F(y_n),$$
$$x_{n+1} = z_n - Q_2(x_n, y_n)\big[F'(x_n)\big]^{-1}F(z_n), \tag{3}$$

where the weight functions chosen here are

$$Q_1(x_n, y_n) = \frac{F^T(x_n)F(x_n) + 2F^T(x_n)F(y_n) - a(a-2)F^T(y_n)F(y_n)}{F^T(x_n)F(x_n) - (a-2)^2F^T(y_n)F(y_n)}, \tag{4}$$

and

$$Q_2(x_n, y_n) = \frac{F^T(x_n)F(x_n) + 2F^T(x_n)F(y_n) - 3F^T(y_n)F(y_n)}{F^T(x_n)F(x_n) - 9F^T(y_n)F(y_n)}, \tag{5}$$

and the parameter a was chosen as zero. The original idea is to have the weight function chosen in such a way that the method will be of higher order than 4. This was not successful as the numerical experiments will show.

Methods of higher order than 4 were developed in the literature and we will quote several methods of order five and six. Cordero et al. [10] have developed a fifth order method, denoted here by CHMT, given by

$$y_n = x_n - \big[F'(x_n)\big]^{-1}F(x_n),$$
$$z_n = x_n - 2\big[F'(x_n) + F'(y_n)\big]^{-1}F(x_n),$$
$$x_{n+1} = z_n - \big[F'(y_n)\big]^{-1}F(z_n). \tag{6}$$

Another fifth order family of methods due to Sharma et al. [11] is given by

$$y_n = x_n - \theta\big[F'(x_n)\big]^{-1}F(x_n),$$
$$z_n = x_n - \left[\left(1 + \frac{1}{2\theta}\right)I - \frac{1}{2\theta}\big[F'(x_n)\big]^{-1}F'(y_n)\right]\big[F'(x_n)\big]^{-1}F(x_n),$$
$$x_{n+1} = z_n - \left[\left(1 + \frac{1}{\theta}\right)I - \frac{1}{\theta}\big[F'(x_n)\big]^{-1}F'(y_n)\right]\big[F'(x_n)\big]^{-1}F(z_n). \tag{7}$$

The case $\theta = 1$ was shown to be the best and we will use that here and denote it SSK. We also used $\theta = 2/3$ to match with the other schemes by [12,13].

The first family of methods of order six is found in Hueso et al. [12]

$$y_n = x_n - \frac{2}{3}\big[F'(x_n)\big]^{-1}F(x_n),$$
$$z_n = x_n - \left[\frac{5-8a_2}{8}I + a_2\big[F'(y_n)\big]^{-1}F'(x_n) + \frac{a_2}{3}\big[F'(x_n)\big]^{-1}F'(y_n) + \frac{9-8a_2}{24}\left(\big[F'(y_n)\big]^{-1}F'(x_n)\right)^2\right]\big[F'(x_n)\big]^{-1}F(x_n),$$
$$x_{n+1} = z_n - \left[b_1 I - \frac{3+8b_1}{8}\big[F'(y_n)\big]^{-1}F'(x_n) + \frac{15-8b_1}{24}\big[F'(x_n)\big]^{-1}F'(y_n) + \frac{9+4b_1}{12}\left(\big[F'(y_n)\big]^{-1}F'(x_n)\right)^2\right]\big[F'(y_n)\big]^{-1}F(z_n). \tag{8}$$

Two members were experimented with in [12] and chosen because of their computational efficiency. These are

- HMT1, when $a_2 = 9/8$ and $b_1 = -9/4$

$$y_n = x_n - \frac{2}{3}\big[F'(x_n)\big]^{-1}F(x_n),$$
$$z_n = x_n - \left[-\frac{1}{2}I + \frac{9}{8}\big[F'(y_n)\big]^{-1}F'(x_n) + \frac{3}{8}\big[F'(x_n)\big]^{-1}F'(y_n)\right]\big[F'(x_n)\big]^{-1}F(x_n),$$
$$x_{n+1} = z_n - \left[-\frac{9}{4}I + \frac{15}{8}\big[F'(y_n)\big]^{-1}F'(x_n) + \frac{11}{8}\big[F'(x_n)\big]^{-1}F'(y_n)\right]\big[F'(y_n)\big]^{-1}F(z_n). \tag{9}$$

- HMT2, when $a_2 = 0$ and $b_1 = -9/4$

$$y_n = x_n - \frac{2}{3}\big[F'(x_n)\big]^{-1}F(x_n),$$
$$z_n = x_n - \left[\frac{5}{8}I + \frac{3}{8}\left(\big[F'(y_n)\big]^{-1}F'(x_n)\right)^2\right]\big[F'(x_n)\big]^{-1}F(x_n),$$
$$x_{n+1} = z_n - \left[-\frac{9}{4}I + \frac{15}{8}\big[F'(y_n)\big]^{-1}F'(x_n) + \frac{11}{8}\big[F'(x_n)\big]^{-1}F'(y_n)\right]\big[F'(y_n)\big]^{-1}F(z_n). \tag{10}$$

Table 1
Weight functions.

Method	w_1	$W_1(x_n, y_n)$	$W_2(x_n, y_n)$
CHMT	1	$2\big[F'(x_n)+F'(y_n)\big]^{-1}F'(x_n)$	s_n
SSK	θ	$\big(1+\frac{1}{2\theta}\big)I-\frac{1}{2\theta}t_n$	$\big(1+\frac{1}{\theta}\big)I-\frac{t_n}{\theta}$
HMT1	2/3	$-\frac{1}{2}I+\frac{9}{8}s_n+\frac{3}{8}t_n$	$\frac{11}{8}I-\frac{9}{4}s_n+\frac{15}{8}s_n^2$
HMT2	2/3	$\frac{5}{8}I+\frac{3}{8}s_n^2$	$\frac{11}{8}I-\frac{9}{4}s_n+\frac{15}{8}s_n^2$
MSSM	2/3	$\frac{23}{8}I-3t_n+\frac{9}{8}t_n^2$	$\frac{5}{2}I-\frac{3}{2}t_n$
ABCTL	2/3	$I+\frac{21}{8}t_n-\frac{9}{2}t_n^2+\frac{15}{8}t_n^3$	$3I-\frac{5}{2}t_n+\frac{1}{2}t_n^2$

Another sixth order by Montazeri et al. [13] denoted by MSSM is given by

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}\big[F'(x_n)\big]^{-1}F(x_n),\\
z_n &= x_n - \left[\frac{23}{8}I - 3\big[F'(x_n)\big]^{-1}F'(y_n) + \frac{9}{8}\Big(\big[F'(x_n)\big]^{-1}F'(y_n)\Big)^2\right]\big[F'(x_n)\big]^{-1}F(x_n),\\
x_{n+1} &= z_n - \left[\frac{5}{2}I - \frac{3}{2}\big[F'(x_n)\big]^{-1}F'(y_n)\right]\big[F'(x_n)\big]^{-1}F(z_n).
\end{aligned}
\tag{11}
$$

This method was rediscovered by Sharma and Arora [14].

Abbasbandy et al. [15] has developed a sixth order method denoted by ABCTL and given by

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}\big[F'(x_n)\big]^{-1}F(x_n),\\
z_n &= x_n - \left[I + \frac{21}{8}\big[F'(x_n)\big]^{-1}F'(y_n) - \frac{9}{2}\Big(\big[F'(x_n)\big]^{-1}F'(y_n)\Big)^2\right.\\
&\quad\left. + \frac{15}{8}\Big(\big[F'(x_n)\big]^{-1}F'(y_n)\Big)^3\right]\big[F'(x_n)\big]^{-1}F(x_n),\\
x_{n+1} &= z_n - \left[3I - \frac{5}{2}\big[F'(x_n)\big]^{-1}F'(y_n) + \frac{1}{2}\Big(\big[F'(x_n)\big]^{-1}F'(y_n)\Big)^2\right]\big[F'(x_n)\big]^{-1}F(z_n).
\end{aligned}
\tag{12}
$$

2. Development of high order methods

One of the techniques to develop high order methods for the solution of a single nonlinear equation is the weight function approach, see e.g. Chapter 4 of Petković et al. [16]. One of the early attempts to use this idea is due to Neta [8] which generalizes the sixth order method using the weight function

$$\frac{1 + af(y_n)/f(x_n)}{1 + (a-2)f(y_n)/f(x_n)}.$$

We have experimented with several ways to generalize this to systems of equations. Neta [8] have suggested to use a diagonal matrix as a weight function with diagonal elements being

$$\frac{1 + aF_i(y_n)/F_i(x_n)}{1 + (a-2)F_i(y_n)/F_i(x_n)}.$$

Other ways were considered to get a scalar weight function as in (4) or

$$\frac{1 + aF^T(x_n)F(y_n)/F^T(x_n)F(x_n)}{1 + (a-2)F^T(x_n)F(y_n)/F^T(x_n)F(x_n)}.$$

All these choices did not allow the method to be of order higher than 4 as we have seen in the examples.

The only other possibility to have a weight function in form of a matrix depending on a second Jacobian. This is the idea found in the methods (6)–(12). We will write those methods in terms of weight functions as follows:

$$
\begin{aligned}
y_n &= x_n - w_1\big[F'(x_n)\big]^{-1}F(x_n),\\
z_n &= x_n - W_1(x_n, y_n)\big[F'(x_n)\big]^{-1}F(x_n).\\
x_{n+1} &= z_n - W_2(x_n, y_n)\big[F'(x_n)\big]^{-1}F(z_n).
\end{aligned}
\tag{13}
$$

The weights for each method are given in Table 1, where we used (see also [12]) the following notations:

$$s_n = \big[F'(y_n)\big]^{-1}F'(x_n),$$

and

$$t_n = \left[F'(x_n)\right]^{-1} F'(y_n).$$

Based on this table, we suggest the following general family (13) with

$$w_1 = 2/3, \tag{14}$$

$$W_1(x_n, y_n) = a_1 I + a_2 s_n + a_3 t_n + a_4 s_n^2 + a_5 t_n^2 + a_6 t_n^3, \tag{15}$$

$$W_2(x_n, y_n) = b_1 I + b_2 s_n + b_3 t_n + b_4 s_n^2 + b_5 t_n^2. \tag{16}$$

Clearly this family of methods includes HMT1, HMT2, MSSM and ABCTL as special cases. For the family, we have the following convergence analysis.

Theorem 2.1. *Let the function $F : D \subset \mathbb{R}^m \to \mathbb{R}^m$ be sufficiently differentiable in a convex set D containing a zero α of $F(x)$. Suppose that $F'(x)$ is continuous and nonsingular in α. Then for all $a_i, 1 \le i \le 6$ and $b_j, 1 \le j \le 5$ satisfying*

$$\begin{aligned} a_1 &= -1/2 + 3a_4 + 3a_5 + 8a_6, \\ a_2 &= 9/8 - 3a_4 - a_5 - 3a_6, \\ a_3 &= 3/8 - a_4 - 3a_5 - 6a_6, \\ b_1 &= -1/2 - 2b_3 + b_4 - 3b_5, \\ b_2 &= 3/2 + b_3 - 2b_4 + 2b_5, \end{aligned} \tag{17}$$

the local convergence order of the family (13)–(16) is at least six, and the error constant is given by

$$\frac{1024}{243}\left(K_1 c_2^2 + \frac{9}{16}c_3\right)\left(K_2 c_2^3 + \frac{27}{64}c_2 c_3 - \frac{3}{64}c_4\right),$$

where

$$\begin{aligned} K_1 &= b_3 + b_4 + 3b_5 - \frac{15}{8}, \\ K_2 &= a_4 - a_5 - 4a_6 - \frac{63}{64}, \end{aligned} \tag{18}$$

where $e = x_n - \alpha \in \mathbb{R}^m$, $e^i = \underbrace{(e, e, \ldots, e)}_{i-times}$, $c_j = (1/j!)[F'(\alpha)]^{-1} F^{(j)}(\alpha) \in L_i(\mathbb{R}^m, \mathbb{R}^m)$, $F^{(j)} \in L(\mathbb{R}^m \times \cdots \times \mathbb{R}^m, \mathbb{R}^m)$ *and* $[F'(\alpha)]^{-1} \in L(\mathbb{R}^m)$.

Proof. By the Taylor expansion of $F(x_n)$ around α we have

$$F(x_n) = F'(\alpha)\left[e + c_2 e^2 + c_3 e^3 + c_4 e^4 + c_5 e^5 + c_6 e^6 + O(e^7)\right] \tag{19}$$

and

$$F'(x_n) = F'(\alpha)\left[I + 2c_2 e + 3c_3 e^2 + 4c_4 e^3 + 5c_5 e^4 + 6c_6 e^5 + O(e^6)\right]. \tag{20}$$

Inversion of $F'(x_n)$ yields

$$\begin{aligned} F'(x_n)^{-1} = &\left[I - 2c_2 e + \left(4c_2^2 - 3c_3\right)e^2 - \left(8c_2^3 - 12c_2 c_3 + 4c_4\right)e^3 \right. \\ &\left. + \left(16c_2^4 - 36c_2^2 c_3 + 16c_2 c_4 + 9c_3^2 - 5c_5\right)e^4\right]F'(\alpha)^{-1} + O(e^5). \end{aligned} \tag{21}$$

Let us denote $E = y_n - \alpha$. From (19) and (21), we get

$$\begin{aligned} E = &\frac{1}{3}e + \frac{2}{3}c_2 e^2 + \frac{4}{3}\left(c_3 - c_2^2\right)e^3 + \left(\frac{8}{3}c_2^3 + 2c_4 - \frac{14}{3}c_2 c_3\right)e^4 \\ &+ \left(\frac{40}{3}c_2^2 c_3 - \frac{20}{3}c_2 c_4 - \frac{16}{3}c_2^4 - 4c_3^2 + \frac{8}{3}c_5\right)e^5 + O(e^6). \end{aligned} \tag{22}$$

We then obtain

$$F'(y_n) = F'(\alpha)[I + 2c_2 E + 3c_3 E^2 + 4c_4 E^3 + 5c_5 E^4 + O(E^5)] \tag{23}$$

and its inverse as

$$\begin{aligned} F'(y_n)^{-1} = &\left[I - \frac{2}{3}c_2 e - \frac{1}{9}\left(8c_2^2 + 3c_3\right)e^2 + \frac{4}{27}\left(28c_2^3 - 24c_2 c_3 - c_4\right)e^3 \right. \\ &\left. - \frac{1}{81}\left(704c_2^4 - 1332c_2^2 c_3 + 380c_2 c_4 + 207c_3^2 + 5c_5\right)e^4\right]F'(\alpha)^{-1} + O(e^5) \end{aligned} \tag{24}$$

which lead to

$$\begin{aligned} s_n &= I + \frac{4}{3}c_2 e + \left(\frac{8}{3}c_3 - \frac{20}{9}c_2^2\right)e^2 + \left(\frac{104}{27}c_4 - \frac{56}{9}c_2c_3 + \frac{64}{27}c_2^3\right)e^3 \\ &\quad + \left(\frac{400}{81}c_5 - \frac{620}{81}c_2c_4 + \frac{20}{3}c_2^2c_3 - \frac{32}{9}c_3^2 - \frac{32}{81}c_2^4\right)e^4 + O(e^5) \end{aligned} \tag{25}$$

and

$$\begin{aligned} t_n &= I - \frac{4}{3}c_2 e + \left(4c_2^2 - \frac{8}{3}c_3\right)e^2 + \left(\frac{40}{3}c_2c_3 - \frac{32}{3}c_2^3 - \frac{104}{27}c_4\right)e^3 \\ &\quad - \left(\frac{148}{3}c_2^2c_3 - \frac{484}{27}c_2c_4 - \frac{80}{3}c_2^4 - \frac{32}{3}c_3^2 + \frac{400}{81}c_5\right)e^4 + O(e^5). \end{aligned} \tag{26}$$

We denote $\epsilon = z_n - \alpha$. Using (19), (21), (25) and (26) in the second step of the family, we obtain

$$\epsilon = A_1 e + A_2 e^2 + A_3 e^3 + A_4 e^4 + O(e^5), \tag{27}$$

where

$$\begin{aligned} A_1 &= 1 - a_1 - a_2 - a_3 - a_4 - a_5 - a_6, \\ A_2 &= \left(a_1 - \frac{1}{3}a_2 + \frac{7}{3}a_3 - \frac{5}{3}a_4 + \frac{11}{3}a_5 + 5a_6\right)c_2, \\ A_3 &= \left(2a_1 - \frac{2}{3}a_2 + \frac{14}{3}a_3 - \frac{10}{3}a_4 + \frac{22}{3}a_5 + 10a_6\right)c_3 + \left(-2a_1 + \frac{14}{9}a_2 - \frac{22}{3}a_3 + \frac{10}{3}a_4 - \frac{130}{9}a_5 - \frac{70}{3}a_6\right)c_2^2, \\ A_4 &= \left(4a_1 - \frac{88}{27}a_2 + \frac{64}{3}a_3 - \frac{76}{27}a_4 + \frac{460}{9}a_5 + \frac{2584}{27}a_6\right)c_2^3 - 7\left(a_1 - \frac{41}{63}a_2 + \frac{11}{3}a_3 - \frac{9}{7}a_4 + \frac{463}{63}a_5 + \frac{253}{21}a_6\right)c_3c_2 \\ &\quad + 3\left(a_1 - \frac{23}{81}a_2 + \frac{185}{81}a_3 - \frac{127}{81}a_4 + \frac{289}{81}a_5 + \frac{131}{27}a_6\right)c_4. \end{aligned} \tag{28}$$

We now find conditions on the a_i to make the first two substeps of the family fourth-order by requiring $A_1 = A_2 = A_3 = 0$. They are given by

$$\begin{aligned} a_1 &= -\frac{1}{2} + 3a_4 + 3a_5 + 8a_6, \\ a_2 &= \frac{9}{8} - 3a_4 - a_5 - 3a_6, \\ a_3 &= \frac{3}{8} - a_4 - 3a_5 - 6a_6, \end{aligned} \tag{29}$$

in this case,

$$\epsilon = \left[\frac{1}{9}c_4 - c_2c_3 + \left(\frac{7}{3} - \frac{64}{27}a_4 + \frac{64}{27}a_5 + \frac{256}{27}a_6\right)c_2^3\right]e^4 + O(e^5). \tag{30}$$

Using Taylor series of $F(z_n)$ about α gives

$$F(z_n) = F'(\alpha)[\epsilon + c_2\epsilon^2 + O(\epsilon^3)]. \tag{31}$$

Using (21), (25), (26), (30), (31) in third substep of the family we get

$$\begin{aligned} x_{n+1} - \alpha &= \epsilon - W_2(x_n, y_n)\left[F'(x_n)\right]^{-1}F(z_n) \\ &= B_4e^4 + B_5e^5 + B_6e^6 + O(e^7), \end{aligned} \tag{32}$$

where

$$\begin{aligned} B_4 &= (b_1 + b_2 + b_3 + b_4 + b_5 - 1)\left[c_3c_2 - \frac{1}{9}c_4 + \frac{64}{27}\left(a_4 - a_5 - 4a_6 - \frac{63}{64}\right)c_2^3\right], \\ B_5 &= \frac{1}{81}D_1c_2^4 + \frac{128}{9}D_2c_3c_2^2 + \frac{22}{9}D_3c_4c_2 + 2(b_1 + b_2 + b_3 + b_4 + b_5 - 1)\left(c_3^2 - \frac{4}{27}c_5\right), \\ B_6 &= \frac{1}{243}G_1c_2^5 - \frac{12992}{81}G_2c_3c_2^3 + \frac{1664}{81}G_3c_4c_2^2 \\ &\quad + \frac{1}{243}c_2(G_4c_3^2 + 954G_5c_5) + \frac{23}{3}G_6c_3c_4 - \frac{14}{27}G_7c_6, \\ D_1 &= (-1792b_1 - 1536b_2 - 2048b_3 - 1280b_4 - 2304b_5 + 1408)a_4 \\ &\quad + (2048b_1 + 1792b_2 + 2304b_3 + 1536b_4 + 2560b_5 - 1664)a_5 \\ &\quad + (8448b_1 + 7424b_2 + 9472b_3 + 6400b_4 + 10496b_5 - 6912)a_6 \\ &\quad + 1422b_1 + 1170b_2 + 1674b_3 + 918b_4 + 1926b_5 - 1044, \end{aligned}$$

$$\begin{aligned}
D_2 &= (b_1+b_2+b_3+b_4+b_5-1)(a_4-a_5-4a_6) \\
&\quad -\frac{81}{64}b_1-\frac{75}{64}b_2-\frac{87}{64}b_3-\frac{69}{64}b_4-\frac{93}{64}b_5+\frac{9}{8}, \\
D_3 &= b_1+\frac{31}{33}b_2+\frac{35}{33}b_3+\frac{29}{33}b_4+\frac{37}{33}b_5-\frac{10}{11}, \\
G_1 &= (29568b_1+21120b_2+39040b_3+13696b_4+49536b_5-18816)a_4 \\
&\quad -(39040b_1+29568b_2+49536b_3+21120b_4+61056b_5-26752)a_5 \\
&\quad -(166656b_1+127744b_2+209664b_3+92928b_4+256768b_5-115968)a_6 \\
&\quad -19746b_1-12798b_2-27702b_3-6858b_4-36666b_5+11214, \\
G_2 &= \left(b_1+\frac{171}{203}b_2+\frac{235}{203}b_3+\frac{139}{203}b_4+\frac{267}{203}b_5-\frac{158}{203}\right)a_4 \\
&\quad -\left(\frac{235}{203}b_1+b_2-\frac{267}{203}b_3+\frac{171}{203}b_4+\frac{299}{203}b_5-\frac{190}{203}\right)a_5 \\
&\quad -\left(\frac{972}{203}b_1+\frac{844}{203}b_2+\frac{1100}{203}b_3+\frac{716}{203}b_4+\frac{1228}{203}b_5-\frac{792}{203}\right)a_6 \\
&\quad -\frac{1575}{1856}b_1-\frac{8397}{12992}b_2-\frac{1971}{1856}b_3-\frac{5913}{12992}b_4-\frac{16713}{12992}b_5+\frac{3771}{6496}, \\
G_3 &= (b_1+b_2+b_3+b_4+b_5-1)(a_4-a_5-4a_6) \\
&\quad -\frac{1161}{832}b_1-\frac{1019}{832}b_2-\frac{1311}{832}b_3-\frac{885}{832}b_4-\frac{113}{64}b_5+\frac{963}{832}, \\
G_4 &= 6912(b_1+b_2+b_3+b_4+b_5-1)(a_4-a_5-4a_6) \\
&\quad -9963b_1-8667b_2-11259b_3-7371b_4-12555b_5+8262, \\
G_5 &= b_1+\frac{143}{159}b_2+\frac{175}{159}b_3+\frac{127}{159}b_4+\frac{191}{159}b_5-\frac{45}{53}, \\
G_6 &= b_1+\frac{199}{207}b_2+\frac{215}{207}b_3+\frac{191}{207}b_4+\frac{223}{207}b_5-\frac{22}{23}, \\
G_7 &= b_1+b_2+b_3+b_4+b_5-1.
\end{aligned} \tag{33}$$

We find conditions on the b_i to make the family sixth-order by requiring $B_4 = B_5 = 0$. They are given by

$$\begin{aligned}
b_1 &= -\frac{1}{2}-2b_3+b_4-3b_5, \\
b_2 &= \frac{3}{2}+b_3-2b_4+2b_5,
\end{aligned} \tag{34}$$

in this case,

$$x_{n+1}-\alpha = \frac{1024}{243}\left(K_1c_2^2+\frac{9}{16}c_3\right)\left(K_2c_2^3+\frac{27}{64}c_2c_3-\frac{3}{64}c_4\right)e^6+O(e^7), \tag{35}$$

where

$$\begin{aligned}
K_1 &= b_3+b_4+3b_5-\frac{15}{8}, \\
K_2 &= a_4-a_5-4a_6-\frac{63}{64}.
\end{aligned} \tag{36}$$

This implies that the family (13)–(16) under the conditions given by (17) is of sixth-order convergence. This completes the proof. □

It is **not** possible to increase the order by adding more terms to the weights. We may choose the 6 parameters to simplify the forms of W_1 and W_2. One choice is

$$\begin{aligned}
a_4 &= 0, \\
a_5 &= 9/8, \\
a_6 &= 0, \\
b_3 &= -3/2-2b_5, \\
b_4 &= 0.
\end{aligned} \tag{37}$$

This gives a one parameter family of methods (13), denoted CN1, with the weights

$$W_1(x_n, y_n) = \frac{23}{8}I - 3t_n + \frac{9}{8}t_n^2,$$
$$W_2(x_n, y_n) = \left(\frac{5}{2} + b_5\right)I - \left(\frac{3}{2} + 2b_5\right)t_n + b_5 t_n^2. \quad (38)$$

This family uses only one Jacobian (since only t_n appears) as with MSSM, which is the case of $b_5 = 0$. In fact, if we choose $a_6 \neq 0$ we still have only one Jacobian.

Another possibility is to choose the parameters to annihilate the coefficients K_1 of c_2^2 and K_2 of c_2^3, e.g.

$$a_4 = 63/64,$$
$$a_5 = 0,$$
$$a_6 = 0,$$
$$b_3 = 15/8 - 3b_5,$$
$$b_4 = 0. \quad (39)$$

This gives a one parameter family of methods (13), denoted CN2, with the weights

$$W_1(x_n, y_n) = \frac{157}{64}I - \frac{117}{64}s_n - \frac{39}{64}t_n + \frac{63}{64}s_n^2$$
$$W_2(x_n, y_n) = -\left(\frac{17}{4} + 3b_5\right)I + \left(\frac{27}{8} + 2b_5\right)s_n + \left(\frac{15}{8} - 3b_5\right)t_n + b_5 t_n^2. \quad (40)$$

Remark: If we take the first two sub-steps of (13) we get a three-parameter fourth-order family of methods with a_i satisfying (17). It is **not** possible to use the parameters to increase the order beyond four.

3. Numerical experiments

We have experimented with these methods using several systems of 2, 3, 4, 5 and 9 equations given here. There are 5 examples of systems of 2 equations, 6 examples of systems of 3 equations and one each of a system of 4, 5 and 9 equations. In each case we listed the initial iterate x_0 and the exact solution(s) α. In case there is more than one solution, we will first list the solution to which the methods converged to.

- Example 1

$$x_1 + e^{x_2} - \cos x_2 = 0$$
$$3x_1 - x_2 - \sin x_2 = 0 \quad (41)$$

$$x_0 = (.5, .5)^T$$

$$\alpha = (0, 0)^T$$

- Example 2

$$x_1 + 3\log x_1 - x_2^2 = 0$$
$$2x_1^2 - x_1 x_2 - 5x_1 + 1 = 0 \quad (42)$$

$$x_0 = (1, -2)^T$$

$$\alpha = (1.3734783533, -1.524964837)^T$$

$$\alpha = (3.756834008, 2.779849593)^T$$

- Example 3

$$x_1^2 + x_1 x_2^3 - 9 = 0$$
$$3x_1^2 x_2 - x_2^3 - 4 = 0 \quad (43)$$

$$x_0 = (-1.2, -2.5)^T$$

$$\alpha = (-.9012661905, -2.086587595)^T$$

$$\alpha = (9.985950982, -2.086587595)^T$$

$$\alpha = (2.998375993, 0.1481079950)^T$$

$$\alpha = (-3.001624887, 0.1481079950)^T$$

$$\alpha = (1.336355377, 1.754235198)^T$$

$$\alpha = (-6.734735503, 1.754235198)^T$$

- Example 4

$$\begin{aligned} &3x_1^2 + 4x_2^2 - 1 = 0 \\ &x_2^3 - 8x_1^3 - 1 = 0 \end{aligned} \tag{44}$$

$$x_0 = (-.7, .2)^T$$

$$\alpha = (-0.49725120256, 0.254078592490)^T$$

- Example 5

$$\begin{aligned} &4x_1^2 + x_2^2 - 4 = 0 \\ &x_1 + x_2 - \sin(x_1 - x_2) = 0 \end{aligned} \tag{45}$$

$$x_0 = (1.2, 0.3)^T$$

$$\alpha = (0.998606944097, -.105530492)^T$$

- Example 6

$$\begin{aligned} &\cos x_2 - \sin x_1 = 0 \\ &x_3^{x_1} - 1/x_2 = 0 \\ &e^{x_1} - x_3^2 = 0 \end{aligned} \tag{46}$$

$$x_0 = (1.2, .5, 1.5)^T$$

$$\alpha = (.9095694944, .6612268323, 1.575834144)^T$$

$$\alpha = (-.9095694944, .6612268323, .6345845493)^T$$

- Example 7

$$\begin{aligned} &x_i x_{i+1} - 1 = 0, \quad i = 1, 2, \dots, n-1 \\ &x_n x_1 - 1 = 0 \end{aligned} \tag{47}$$

$$x_0 = (2, 2, \dots, 2)^T$$

If n is odd there are two solutions:

$$\alpha = (1, 1, \dots, 1)^T$$

$$\alpha = (-1, -1, \dots, -1)^T$$

If n is even, then choose x_n

$$x_1 = x_3 = \cdots = x_{n-1} = \frac{1}{x_n}$$

$$x_2 = x_4 = \cdots = x_{n-2} = x_n$$

We have used this example for $n = 3$.

- Example 8

$$\begin{aligned}&(x_1-1)x_2x_3=0\\&x_1(x_2-1)(x_2+2)x_3=0\\&(x_3+1)(x_3-1/2)=0\end{aligned}\tag{48}$$

$$x_0=(1,2,2)^T$$

$$\alpha=(1,1,1/2)^T$$

$$\alpha=(0,0,-1)^T$$

$$\alpha=(0,0,1/2)^T$$

$$\alpha=(1,-2,-1)^T$$

$$\alpha=(1,-2,1/2)^T$$

$$\alpha=(1,1,-1)^T$$

- Example 9

$$\begin{aligned}&x_1^5+x_2^3x_3^4+1=0\\&x_1^2x_2x_3=0\\&x_3^4-1=0\end{aligned}\tag{49}$$

$$x_0=(-100,0,100)^T$$

$$\alpha=(-1,0,1)^T$$

- Example 10

$$\begin{aligned}&6x_1^2+x_2-\frac{37}{6}=0\\&x_1-6x_2^2-\frac{5}{6}=0\\&x_1+x_2+x_3-\frac{1}{2}=0\end{aligned}\tag{50}$$

$$x_0=(3,0,-1)^T$$

$$\alpha=(1,1/6,-2/3)^T$$

$$\alpha=(1.028512437,-.1803603357,-.3481521018)^T$$

and two other complex conjugate solutions.

- Example 11

$$\begin{aligned}&12x_1-3x_2^2-4x_3-7.17=0\\&x_1^2+10x_2-x_3-11.54=0\\&x_2^3+7x_3-7.631=0\end{aligned}\tag{51}$$

$$x_0=(3,0,1)^T$$

$$\alpha=(1.2,1.1,.9)^T$$

$$\alpha=(7.809384276,-3.953119569,9.915287083)^T$$

and two other pair of complex conjugate solutions.

Table 2
Computational order of convergence.

Example	Newton	MN	Neta4	CHMT	SSK ($\theta = 1$)	SSK ($\theta = 2/3$)
1	2	4	3.137	6.03	3.479	5.04
2	1.994	3.945	3.965	5.119	4.502	4.402
3	2.004	3.969	4.037	2.641	3.845	3.69
4	2.001	1.567	4.007	3.443	5	4.913
5	2.001	3.947	4.024	4.993	1.26	5.038
6	2.001	4.025	4.026	5.014	4.24	div
7	2	4	7	4.998	5.0	5.0
8	2	4	3.993	5.001	5.003	5.003
9	div	4	div	4.997	5.0	4.996
10	2	div	4.04	2.339	div	div
11	2.02	4	4.065	4.952	4.252	5.039
12	1.993	3.965	3.982	2.897	5.024	5.024
13	2	4	7	4.998	5.0	5.0
14	2	4	7	4.998	5.0	5.0

Note that, for example 10, the method Neta4 converged to the second solution listed there.

Table 3
Computational order of convergence.

Example	HMT1	HMT2	MSSM	ABCTL	CN1 ($b_5 = -53/4$)	CN2 ($b_5 = -1/4$)
1	6.073	6.069	3.905	5.994	5.957	5.997
2	6.163	6.141	6.146	2.807	5.986	5.981
3	6.046	6.041	1.634	6.103	2.659	5.992
4	5.928	6.015	2.366	6.002	5.842	1.342
5	6.021	6.012	5.981	5.997	5.915	6.023
6	2.866	4.25	div	div	div	5.992
7	6.999	6.999	5.995	5.993	3.999	7.0
8	6.431	6.375	6.0	6.001	6.0	6.474
9	5.847	5.947	5.964	div	6.0	5.99
10	2.475	6.992	div	div	div	div
11	6.481	6.557	6.308	3.864	5.788	6.210
12	6.009	5.930	2.360	3.008	6.035	6.002
13	6.999	6.999	5.995	5.993	3.999	7.0
14	6.999	6.999	5.995	5.993	3.999	7.0

Note that, for example 10, the method HMT2 converged to the second solution listed there.

- Example 12

$$\begin{aligned} x_2x_3 + x_4(x_2 + x_3) &= 0 \\ x_1x_3 + x_4(x_1 + x_3) &= 0 \\ x_1x_2 + x_4(x_1 + x_2) &= 0 \\ x_1x_2 + x_1x_3 + x_2x_3 - 1 &= 0 \end{aligned} \tag{52}$$

$$x_0 = (1.7, .7, 1.8, .8)^T$$

$$\alpha = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, -1/(2\sqrt{3}))^T$$

$$\alpha = (-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}, 1/(2\sqrt{3}))^T$$

- Example 13
  This is the same as Example 7 with $n = 5$.
- Example 14
  This is the same as Example 7 with $n = 9$.

In Tables 2 and 3, we have listed the computational order of convergence (COC) and in Tables 4 and 5 the number of iterations required for convergence.

$$COC = \frac{ln(||x_{i+1} - x_i||/||x_i - x_{i-1}||)}{ln(||x_i - x_{i-1}||/||x_{i-1} - x_{i-2}||)}. \tag{53}$$

We separated the sixth order methods HMT1, HMT2, MSSM, ABCTL, CN1, and CN2 from the lower order schemes Table 3.

Table 4
Number of iterations.

Example	Newton	MN	Neta4	CHMT	SSK ($\theta = 1$)	SSK ($\theta = 2/3$)
1	9	5	5	5	5	5
2	8	5	5	5	5	5
3	9	5	5	5	5	5
4	9	6	5	5	5	5
5	8	5	5	4	5	5
6	10	6	7	5	8	–
7	9	6	4	5	5	5
8	11	6	5	6	6	6
9	–	18	–	20	16	16
10	19	–	8	8	–	–
11	9	5	5	5	5	5
12	11	7	6	6	6	6
13	9	6	4	5	5	5
14	9	6	4	5	5	5
Average	10	6.615	5.23	6.35	6.23	5.615

Table 5
Number of iterations.

Example	HMT1	HMT2	MSSM	ABCTL	CN1 ($b_5 = -53/4$)	CN2 ($b_5 = -1/4$)
1	4	4	5	5	5	4
2	4	4	5	5	5	4
3	4	4	5	5	5	4
4	4	5	5	5	5	5
5	4	4	4	4	4	4
6	6	5	–	–	–	5
7	4	4	5	5	5	4
8	5	5	5	5	6	5
9	19	19	20	–	18	13
10	19	19	–	–	–	–
11	4	4	5	5	5	4
12	5	5	6	6	6	5
13	4	4	5	5	5	4
14	4	4	5	5	5	4
Average	6.43	6.43	6.25	5.0	6.17	5.0

Notice that examples 6, 9 and 10 were the most demanding (see Tables 4 and 5). For example 6, the methods SSK ($\theta = 2/3$) , MSSM, ABCTL and CN1 did not converge within 21 iterations. For example 9, Newton's method Table 4, Neta4 and ABCTL (Table 5) did not converge. For example 10, modified Newton's method, SSK (with both values of θ), MSSM, ABCTL, CN1 and CN2 did not converge. In summary, Newton's method, modified Newton, Neta4, SSK ($\theta = 1$) and CN2 had diverged for one example, SSK ($\theta = 2/3$) , MSSM, and CN1 had diverged for two examples and ABCTL diverged for 3 examples. The only methods that performed well in all examples are CHMT, HMT1 and HMT2. We have computed the average number of iterations over the convergent examples and found that ABCTL and CN2 have the lowest average (5.0) followed by Neta4 (5.23). The difference, of course, is that CN2 has only one divergent case and ABCTL has 3 of those. Amongst the three methods that always converged, CHMT has a slightly lower average (6.35 iterations versus 6.43).

As can be seen in Tables 6 and 7, the most expensive method is CHMT for which the total cost is n^3 (not including lower powers of the dimension n of the system). Three methods (namely, HMT1, HMT2 and CN2) cost $2n^3/3$. All other methods cost $n^3/3$.

Where n is the system dimension, $\alpha = \frac{n(n-1)(2n-1)}{6}$, $\beta = n(n-1)$, μ_0 and μ_1 are relative cost of evaluation of F and Jacobian, respectively, in terms of multiplications and ℓ is the relative cost of division in terms of multiplications.

4. Conclusions

We have developed two families of order six and one can create even more in the same fashion. Two methods, one from each family, were experimented with and compared their performance to existing methods. One of the methods is cheapest but did not converge in two examples, the other one costs more but diverged only in one example.

Table 6
The cost of each iteration.

Method	Evaluation of F and Jacobian	Scalar vector multiply	Matrix vector multiply	Linear solve	Total
Newton	$n\mu_0 + n^2\mu_1$	n	0	$\alpha+\beta$ $+\left(\frac{\beta}{2}+n\right)\ell$	$n^3/3$ $+\left(\mu_1+\frac{1+\ell}{2}\right)n^2$ $+\left(\mu_0+\frac{1+3\ell}{6}\right)n$
MN	$3n\mu_0 + n^2\mu_1$	$3n$	0	$\alpha+3\beta$ $+\left(\frac{\beta}{2}+3n\right)\ell$	$n^3/3$ $+\left(\mu_1+\frac{5+\ell}{2}\right)n^2$ $+\left(3\mu_0+\frac{1+15\ell}{6}\right)n$
Neta4	$3n\mu_0 + n^2\mu_1$	$3n$	0	$\alpha+3\beta$ $+\left(\frac{\beta}{2}+3n\right)\ell$	$n^3/3$ $+\left(\mu_1+\frac{5+\ell}{2}\right)n^2$ $+\left(3\mu_0+\frac{1+15\ell}{6}\right)n$
CHMT	$2n\mu_0 + 2n^2\mu_1$	$3n$	0	$3\alpha+3\beta$ $+\left(\frac{3\beta}{2}+3n\right)\ell$	n^3 $+\left(2\mu_1+\frac{3+3\ell}{2}\right)n^2$ $+\left(2\mu_0+\frac{1+3\ell}{2}\right)n$
SSK	$2n\mu_0 + 2n^2\mu_1$	$5n$	$2n^2$	$\alpha+4\beta$ $+\left(\frac{\beta}{2}+4n\right)\ell$	$n^3/3$ $+\left(2\mu_1+\frac{11+\ell}{2}\right)n^2$ $+\left(2\mu_0+\frac{7+21\ell}{6}\right)n$

Table 7
The cost of each iteration.

Method	Evaluation of F and Jacobian	Scalar vector multiply	Matrix vector multiply	Linear solve	Total
HMT1	$2n\mu_0 + 2n^2\mu_1$	$7n$	$2n^2$	$2\alpha+6\beta$ $+(\beta+6n)\ell$	$2n^3/3$ $+(2\mu_1+7+\ell)n^2$ $+\left(2\mu_0+\frac{4+15\ell}{3}\right)n$
HMT2	$2n\mu_0 + 2n^2\mu_1$	$6n$	$2n^2$	$2\alpha+6\beta$ $+(\beta+6n)\ell$	$2n^3/3$ $+(2\mu_1+7+\ell)n^2$ $+\left(2\mu_0+\frac{1+15\ell}{3}\right)n$
MSSM	$2n\mu_0 + 2n^2\mu_1$	$6n$	$3n^2$	$\alpha+5\beta$ $+\left(\frac{\beta}{2}+5n\right)\ell$	$n^3/3$ $+\left(2\mu_1+\frac{15+\ell}{2}\right)n^2$ $+\left(2\mu_0+\frac{7+27\ell}{6}\right)n$
ABCTL	$2n\mu_0 + 2n^2\mu_1$	$8n$	$5n^2$	$\alpha+7\beta$ $+\left(\frac{\beta}{2}+7n\right)\ell$	$n^3/3$ $+\left(2\mu_1+\frac{23+\ell}{2}\right)n^2$ $+\left(2\mu_0+\frac{7+39\ell}{6}\right)n$
CN1	$2n\mu_0 + 2n^2\mu_1$	$7n$	$4n^2$	$\alpha+6\beta$ $+\left(\frac{\beta}{2}+6n\right)\ell$	$n^3/3$ $+\left(2\mu_1+\frac{19+\ell}{2}\right)n^2$ $+\left(2\mu_0+\frac{7+33\ell}{6}\right)n$
CN2	$2n\mu_0 + 2n^2\mu_1$	$9n$	$4n^2$	$2\alpha+8\beta$ $+(\beta+8n)\ell$	$2n^3/3$ $+(2\mu_1+11+\ell)n^2$ $+\left(2\mu_0+\frac{4+21\ell}{3}\right)n$

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