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An analysis of a family of Maheshwari-based optimal eighth order methods

Changbum Chun [a], Beny Neta [b,*]

[a] Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea
[b] Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA 93943, USA

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ABSTRACT

In this paper we analyze an optimal eighth-order family of methods based on Maheshwari's fourth order method. This family of methods uses a weight function. We analyze the family using the information on the extraneous fixed points. Two measures of closeness of an extraneous points set to the imaginary axis are considered and applied to the members of the family to find its best performer. The results are compared to a modified version of Wang–Liu method.

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1. Introduction

"Calculating zeros of a scalar function f ranks among the most significant problems in the theory and practice not only of applied mathematics, but also of many branches of engineering sciences, physics, computer science, finance, to mention only some fields" [1]. For example, to minimize a function $F(x)$ one has to find the points where the derivative vanishes, i.e. $F'(x) = 0$. There are many algorithms for the solution of nonlinear equations, see e.g. Traub [2], Neta [3] and the recent book by Petković et al. [1]. The methods can be classified as one step and multistep. One step methods are of the form

$$x_{n+1} = \phi(x_n).$$

The iteration function ϕ depends on the method used. For example, Newton's method is given by

$$x_{n+1} = \phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

Some one point methods allow the use of one or more previously found points, in such a case we have a one step method with memory. For example, the secant method uses one previous point and is given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n).$$

In order to increase the order of a one step method, one requires higher derivatives. For example, Halley's method is of third order and uses second derivatives [4]. In many cases the function is not smooth enough or the higher derivatives are too complicated. Another way to increase the order is by using multistep. The recent book by Petković et al. [1] is dedicated to multistep methods. A trivial example of a multistep method is a combination of two Newton steps, i.e.

* Corresponding author.
E-mail addresses: cbchun@skku.edu (C. Chun), bneta@nps.edu (B. Neta).

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$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \tag{2}$$

Of course this is too expensive. The cost of a method is defined by the number (ℓ) of function-evaluations per step. The method (2) requires four function-evaluations (including derivatives). The efficiency of a method is defined by

$$I = p^{1/\ell},$$

where p is the order of the method. Clearly one strives to find the most efficient methods. To this end, Kung and Traub [5] introduced the idea of optimality. A method using ℓ evaluations is optimal if the order is $2^{\ell-1}$. They have also developed optimal multistep methods of increasing order. See also Neta [6]. Newton's method (1) is optimal of order 2. King [7] has developed an optimal fourth order family of methods depending on a parameter β

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n)}\left[\frac{1+\beta r_n}{1+(\beta-2)r_n}\right], \tag{3}$$

where

$$r_n = \frac{f(w_n)}{f(x_n)}. \tag{4}$$

Maheshwari [8] has developed the following optimal fourth order method

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[r_n^2 - \frac{1}{1-r_n}\right], \tag{5}$$

Table 1
The eight cases for experimentation.

Case	Method	g	a
1	LQ	–	0.7
2	LQ	-	2.1
3	QQ	0.8	0.6
4	QQ	1.8	2
5	QC	−0.3	0.6
6	QC	−3.6	2
7	LQ	–	2
8	WLN	–	–

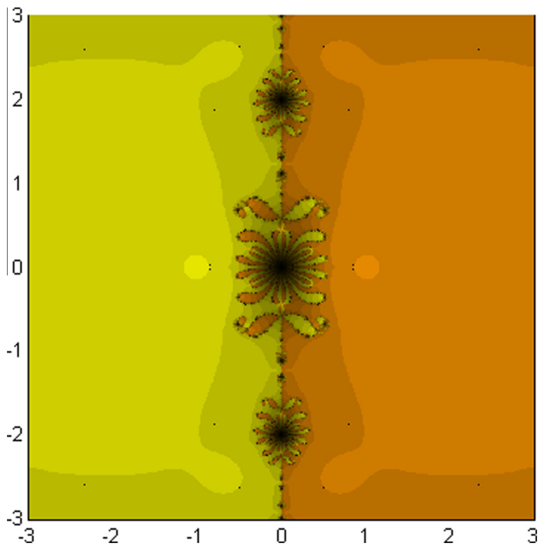

Fig. 1. LQ case 1 for the roots of the polynomial $z^2 - 1$.

where r_n is given by (4).

There are a number of ways to compare various techniques proposed for solving nonlinear equations. Comparisons of the various algorithms are based on the number of iterations required for convergence, number of function evaluations, and/or amount of CPU time. "The primary flaw in this type of comparison is that the starting point, although it may have been chosen at random, represents only one of an infinite number of other choices" [9]. In recent years the Basin of Attraction method was introduced to visually comprehend how an algorithm behaves as a function of the various starting points. The first comparative study using basin of attraction, to the best of our knowledge, is by Vrscay and Gilbert [10]. They analyzed Schröder and König rational iteration functions. Other work was done by Stewart [11], Amat et al. [12–16], Chicharro et al. [17], Magreñán [18], Chun et al. [19–21], Cordero et al. [22], Neta et al. [23,24] and Scott et al. [9]. There are also similar results for methods to find roots with multiplicity, see e.g. [25–28].

In this paper we analyze a family of optimal eighth order methods based on Maheshwari's fourth order method (5). We will examine 3 families of weight functions and show how to choose the parameters involved in each family.

2. Optimal eighth-order family of methods

We analyze here the three-step method based on Maheshwari fourth order method ([1], p. 135) given by

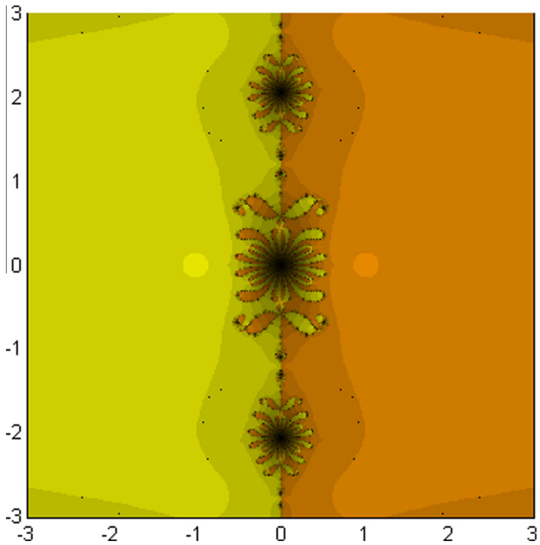

Fig. 2. LQ case 2 for the roots of the polynomial $z^2 - 1$.

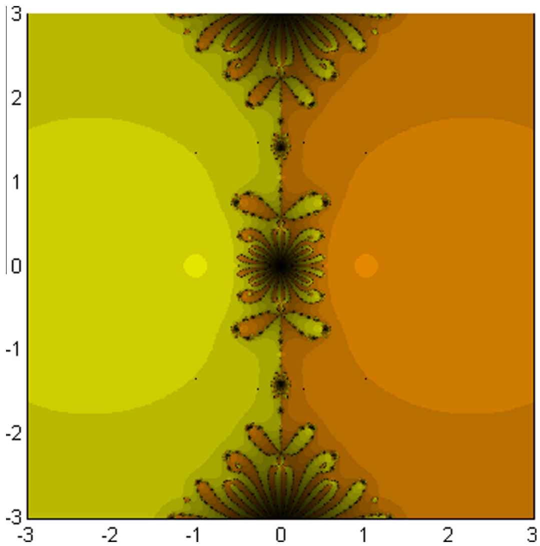

Fig. 3. QQ case 3 for the roots of the polynomial $z^2 - 1$.

$$\begin{cases} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ s_n &= x_n - \frac{f(x_n)}{f'(x_n)}\left[r_n^2 - \frac{1}{r_n - 1}\right], \\ x_{n+1} &= s_n - \frac{f(s_n)}{f'(x_n)}\left[\phi(r_n) + \frac{f(s_n)}{f(w_n) - af(s_n)} + \frac{4f(s_n)}{f(x_n)}\right], \end{cases} \tag{6}$$

where r_n is given by (4) and $\phi(r)$ is a real-valued weight function satisfying the conditions

$$\phi(0) = 1, \quad \phi'(0) = 2, \quad \phi''(0) = 4, \quad \phi'''(0) = -6, \quad \phi^{(4)}(0) = p. \tag{7}$$

The method defined by (6) has the error equation

$$e_{n+1} = \left[c_2(4c_2^2 - c_3)(39c_2^4 - 18c_2^2c_3 + c_3^2 + c_2c_4 + \frac{1}{6}pC_1 + aC_2\right]e_n^8 + O(e_n^9), \tag{8}$$

where $e_n = x_n - \xi, \xi$ is a simple zero of $f(x), c_i$ are given by

$$c_i = \frac{f^{(i)}(\xi)}{i!f'(\xi)}, \quad i \geqslant 1, \tag{9}$$

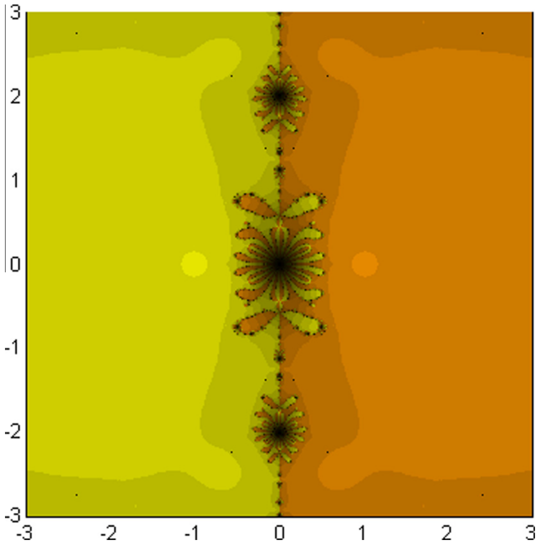

Fig. 4. QQ case 4 for the roots of the polynomial $z^2 - 1$.

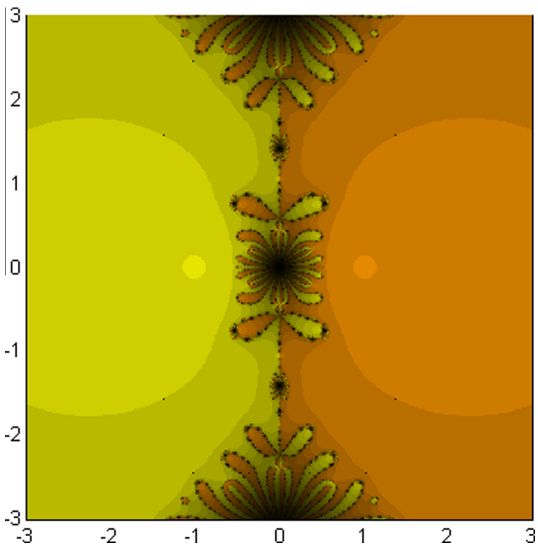

Fig. 5. QC case 5 for the roots of the polynomial $z^2 - 1$.

$$C_1 = \frac{1}{4} c_2^5 c_3 - c_2^7,$$

and

$$C_2 = 241920 c_2^7 - 120960 c_2^5 c_3 c_1 + 20160 c_2^3 c_3^2 c_1^2 - 1120 c_2 c_3^3 c_1^3.$$

We consider the three cases for the weight function $\phi(t)$:

- (LQ) Linear polynomial over quadratic

$$\phi(t) = \frac{\alpha + bt}{1 + dt + gt^2} \tag{10}$$

- (QQ) Quadratic polynomial over quadratic

$$\phi(t) = \frac{\alpha + bt + ct^2}{1 + dt + gt^2} \tag{11}$$

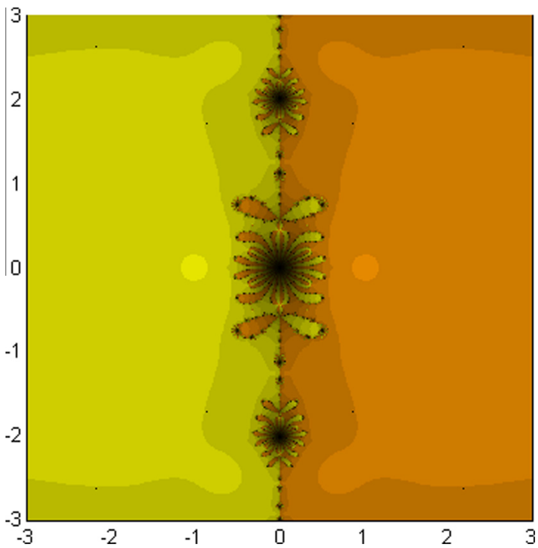

Fig. 6. QC case 6 for the roots of the polynomial $z^2 - 1$.

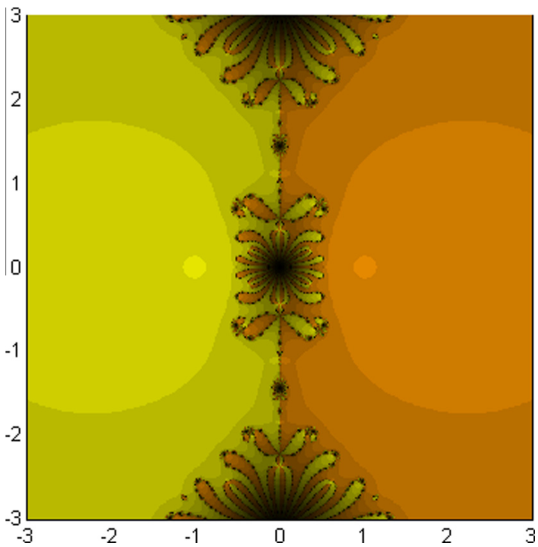

Fig. 7. LQ case 7 for the roots of the polynomial $z^2 - 1$.

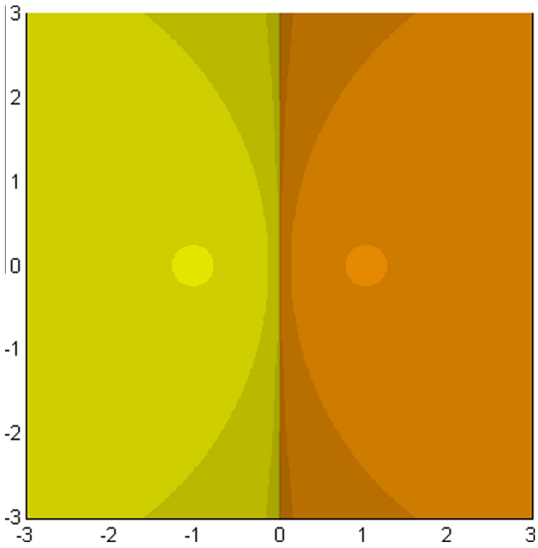

Fig. 8. WLN for the roots of the polynomial $z^2 - 1$.

Table 2
Average number of iterations per point for each example (1–5) and each case.

Case	Ex1	Ex2	Ex3	Ex4	Ex5	Average
1	3.7846	7.6753	5.508	13.5639	16.1492	9.3362
2	2.8003	5.8498	4.5572	9.2212	11.6864	6.82298
3	3.7570	7.7573	5.4770	13.6376	16.0767	9.34112
4	2.8166	6.0908	4.2825	9.0854	10.8273	6.62052
5	3.7468	7.6661	5.4823	13.536	16.0151	9.28926
6	2.7996	6.076	4.2564	8.9523	10.6627	6.5494
7	2.8466	6.0881	4.3114	9.0903	10.8122	6.62972
8	2.2676	2.7084	2.5306	3.7191	4.7871	3.20256

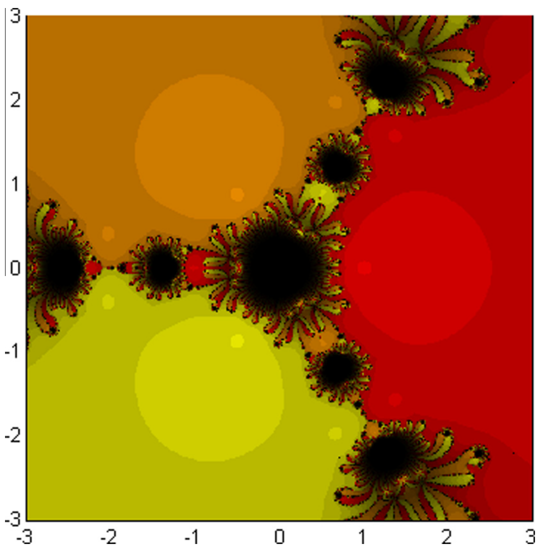

Fig. 9. LQ case 1 for the roots of the polynomial $z^3 - 1$.

- (QC) Quadratic polynomial over cubic

$$\phi(t) = \frac{\alpha + bt + ct^2}{1 + dt + gt^2 + ht^3}. \tag{12}$$

In order for the conditions (7) to be satisfied, these functions are given by

- (LQ) Linear polynomial over quadratic

$$\phi(t) = \frac{-t + 2}{6t^2 - 5t + 2} \tag{13}$$

- (QQ) Quadratic polynomial over quadratic

$$\phi(t) = \frac{2(3-g)t^2 + (5-2g)t + 2}{2gt^2 + (1-2g)t + 2} \tag{14}$$

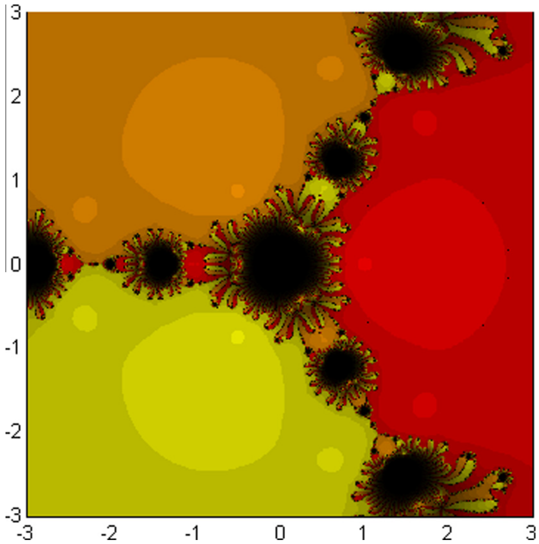

Fig. 10. LQ case 2 for the roots of the polynomial $z^3 - 1$.

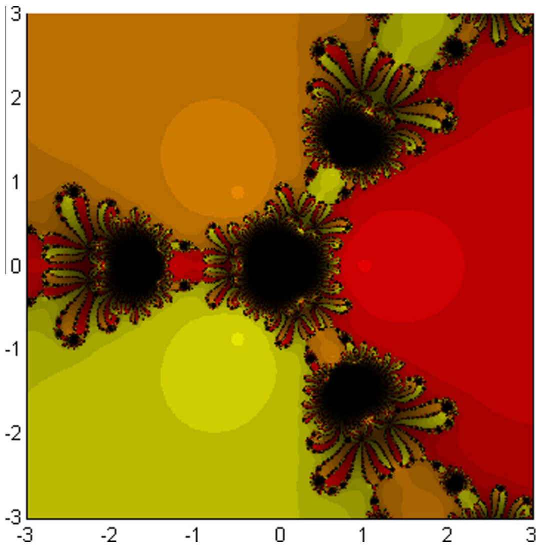

Fig. 11. QQ case 3 for the roots of the polynomial $z^3 - 1$.

- (QC) Quadratic polynomial over cubic

$$\phi(t)=\frac{2(12g+168+p)t^2+(288-48g+p)t+120}{2(12-72g-p)t^3+120gt^2-(48g-48-p)t+120} \tag{15}$$

Particularly when $p=0$ (15) becomes

$$\phi(t)=\frac{(g+14)t^2+2(6-g)t+5}{(1-6g)t^3+5gt^2+2(1-g)t+5}. \tag{16}$$

3. Extraneous fixed points

In solving a nonlinear equation iteratively we are looking for fixed points which are zeros of the given nonlinear function. Many multipoint iterative methods have fixed points that are not zeros of the function of interest. Thus, it is imperative to

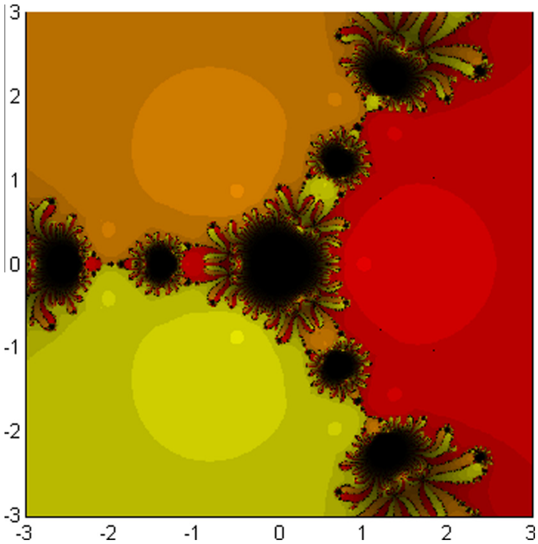

Fig. 12. QQ case 4 for the roots of the polynomial z^3-1.

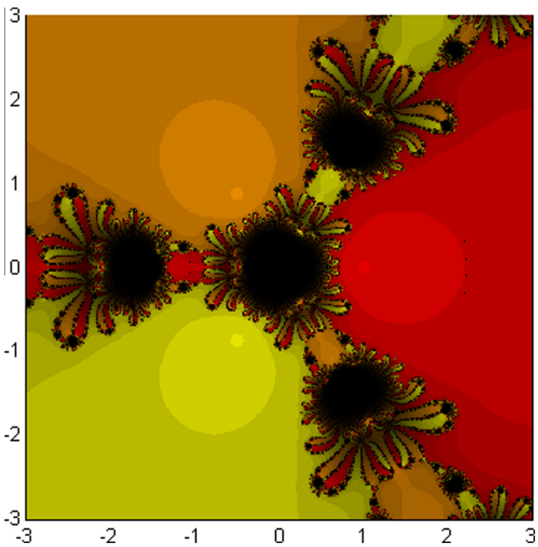

Fig. 13. QC case 5 for the roots of the polynomial z^3-1.

investigate the number of extraneous fixed points, their location and their properties. In the family of methods studied in this paper, the parameters a and g can be chosen to position the extraneous points on the imaginary axis or, at least, close to that axis.

In order to find the extraneous fixed point, we rewrite the methods of interest in the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n, w_n, s_n), \tag{17}$$

where the function H_f for Maheshwari-based method is given by

$$H_f(x_n, w_n, s_n) = r_n^2 - \frac{1}{r_n - 1} + \frac{f(s_n)}{f(x_n)}\left[\phi(r_n) + \frac{f(s_n)}{f(w_n) - af(s_n)} + \frac{4f(s_n)}{f(x_n)}\right]. \tag{18}$$

We have searched the parameter spaces (a in the case of LQ, g, a in the cases of QQ and QC) and found that the extraneous fixed points are not on the imaginary axis. We have considered two measures of closeness to the imaginary axis and experimented with those members from the parameter space.

Let $E = \{z_1, z_2, \ldots, z_{n_{g,a}}\}$ be the set of the extraneous fixed points corresponding to the values given to g and a. We define

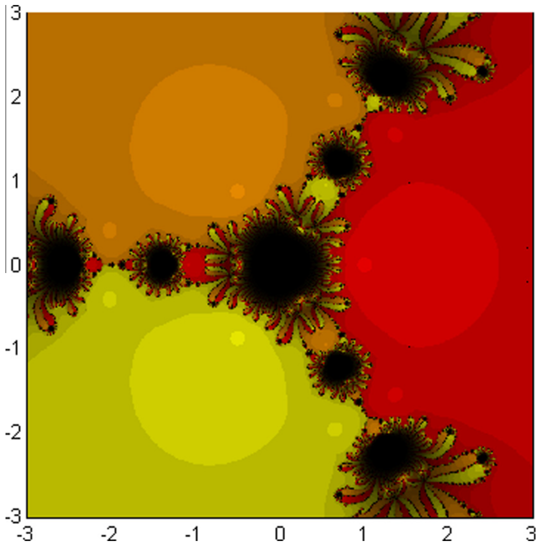

Fig. 14. QC case 6 for the roots of the polynomial $z^3 - 1$.

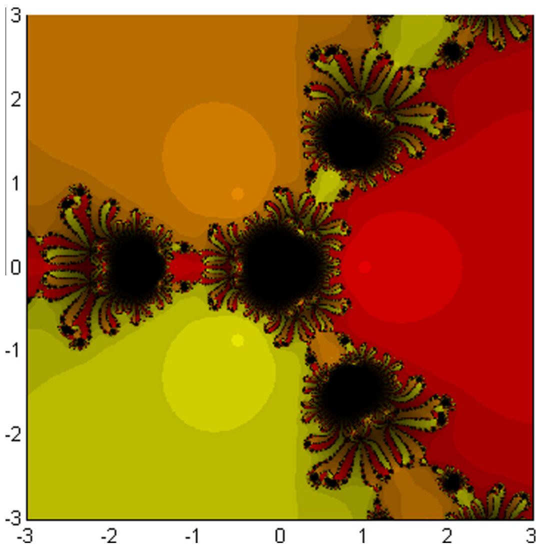

Fig. 15. LQ case 7 for the roots of the polynomial $z^3 - 1$.

$$d(g,a) = \max_{z_i \in E} |Re(z_i)|. \tag{19}$$

We look for the parameters g and a which attain the minimum of $d(g,a)$. For the family LQ, the minimum of $d(a)$ occurs at $a = 0.7$. For the QQ family, the minimum of $d(g,a)$ occurs at $g = 0.8$ and $a = 0.6$. For the QC family, the minimum of $d(g,a)$ occurs at $g = -0.3$ and $a = 0.6$.

Another method to choose the parameters is by considering the stability of $z \in E$ defined by

$$dq(z) = \frac{dq}{dz}(z), \tag{20}$$

where q is the iteration function of (17). We define a function called the averaged stability value of the set E by

$$A(g,a) = \frac{\sum_{z_i \in E} |dq(z_i)|}{n_{g,a}}. \tag{21}$$

The smaller A becomes, the less chaotic the basin of attraction tends to.

For the family LQ, the minimum of $A(a)$ occurs at $a = 2.1$. For the family QQ, the minimum of $A(g,a)$ occurs at $g = 1.8$ and $a = 2$. For the family QC, the minimum of $A(g,a)$ occurs at $g = -3.6$ and $a = 2$.

In the next section we plot the basins of attraction for these seven cases to find the best performer.

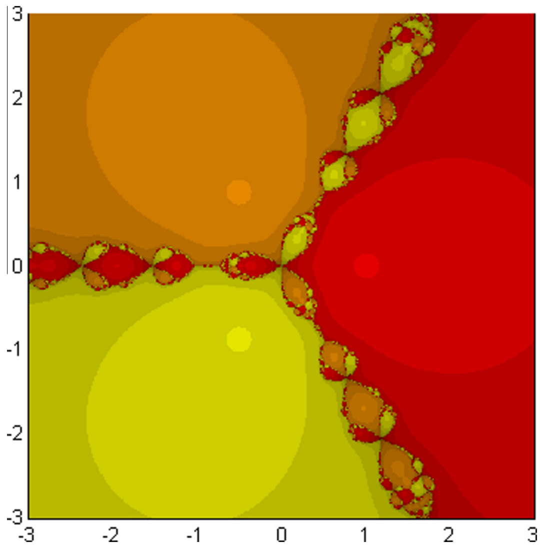

Fig. 16. WLN for the roots of the polynomial $z^3 - 1$.

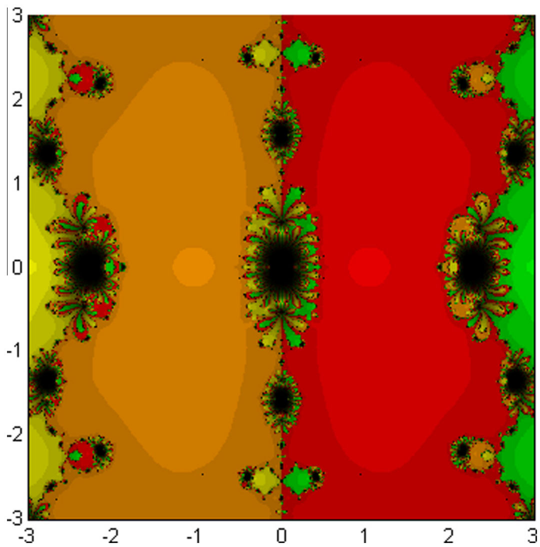

Fig. 17. LQ case 2 for the roots of the polynomial $z^4 - 10z^2 + 9$.

4. Numerical experiments

In this section, we give the results of using the 8 cases described in Table 1 on five different polynomial equations.

The first two cases are of type LQ. For case 1 the parameter a is obtained using the first measure of closeness and the second case is using the second measure. The next two cases are of QQ type, the first of which when using the first measure of closeness and the second when using the second measure. Cases 5 and 6 are of type QC using the first measure of closeness to get the parameters for case 5 and the second measure to get the parameters for case 6. Case 7 is of type LQ with $a = 2$, since the second measure always gave this parameter as best. The last case (WLN) is the best eighth order method as modified by Neta et al. [23] and given by

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ s_n &= w_n - \frac{f(w_n)}{f'(x_n)} \frac{f(x_n)}{f(x_n) - 2f(w_n)}, \\ x_{n+1} &= s_n - \frac{H_3(s_n)}{f'(s_n)}, \end{aligned} \tag{22}$$

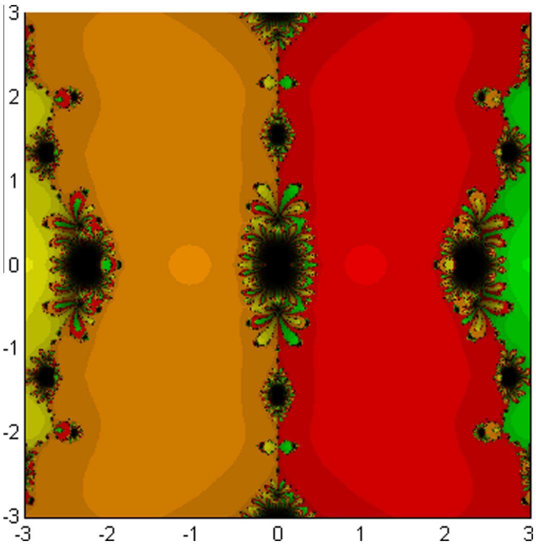

Fig. 18. QQ case 4 for the roots of the polynomial $z^4 - 10z^2 + 9$.

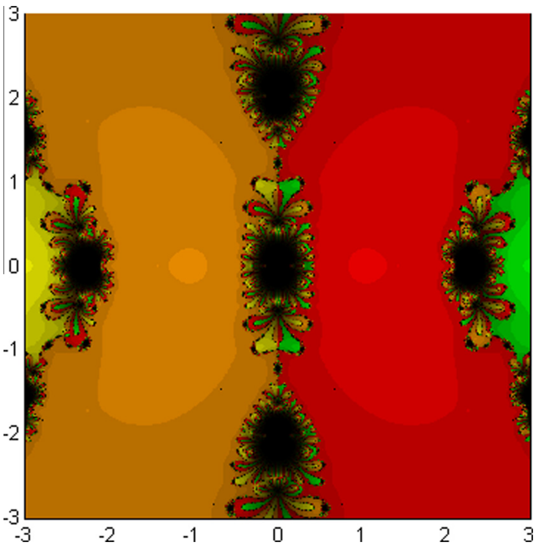

Fig. 19. QC case 5 for the roots of the polynomial $z^4 - 10z^2 + 9$.

where

$$H_3(s_n) = f(x_n) + f'(x_n)\frac{(s_n - w_n)^2(s_n - x_n)}{(w_n - x_n)(x_n + 2w_n - 3s_n)} + f'(s_n)\frac{(s_n - w_n)(x_n - s_n)}{x_n + 2w_n - 3s_n} - \frac{f(x_n) - f(w_n)}{x_n - w_n}\frac{(s_n - x_n)^3}{(w_n - x_n)(x_n + 2w_n - 3s_n)}. \quad (23)$$

We have ran our code for each case and each example on a 6 by 6 square centered at the origin. We have taken 360,000 equally spaced points in the square as initial points for the algorithms. We have recorded the root the method converged to and the number of iterations it took. We chose a color for each root and the intensity of the color gives information on the number of iterations. The slower the convergence the darker the shade. If the scheme did not converge in 40 iterations to one of the roots, we color the point black.

Example 1. In our first example, we have taken the polynomial to be

$$p_1(z) = z^2 - 1, \quad (24)$$

whose roots $z = \pm 1$ are both real. The results are presented in Figs. 1–8. It is clear that WLN outperforms all the others. There are no black points in Fig. 8. To get a more quantitative comparison, we have computed the average number of iterations

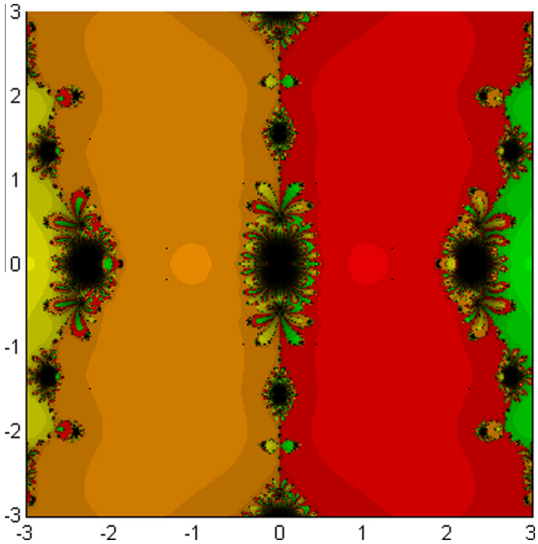

Fig. 20. QC case 6 for the roots of the polynomial $z^4 - 10z^2 + 9$.

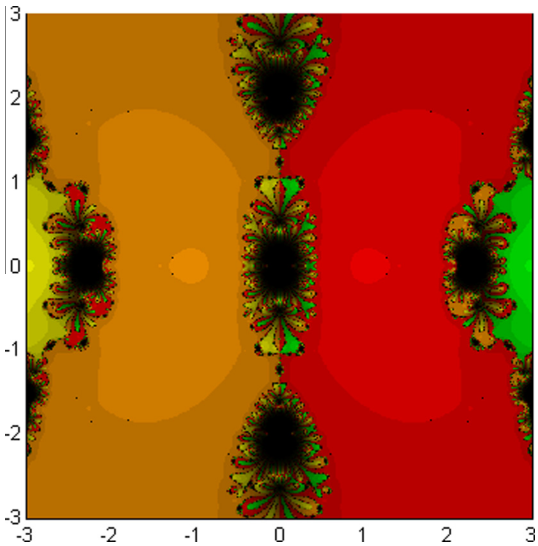

Fig. 21. LQ case 7 for the roots of the polynomial $z^4 - 10z^2 + 9$.

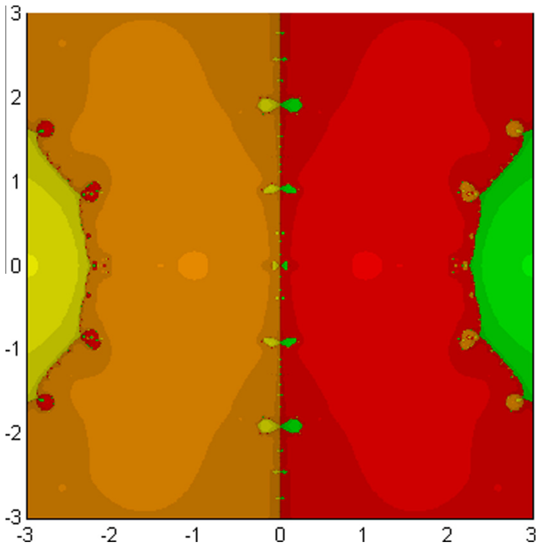

Fig. 22. WLN for the roots of the polynomial $z^4 - 10z^2 + 9$.

used when starting at each of the 360,000 initial points in the 6 by 6 square. These results are presented in Table 2. It can be seen that case 6 is the closest to case 8. The worst are cases 1 and 3.

Example 2. In the second example we have taken a cubic polynomial with the 3 roots of unity, i.e.

$$p_2(z) = z^3 - 1. \tag{25}$$

The results are presented in Figs. 9–16. Again cases 1 and 3 are worst, followed by cases 5, 4, and 7. Case 2 requires more than double the number used by case 8. As a result of this, we will not show the plots for cases 1 and 3 for the rest of the examples.

Example 3. In the third example we have taken a polynomial of degree 4 with 4 real roots at $\pm 1, \pm 3$, i.e.

$$p_3(z) = z^4 - 10z^2 + 9. \tag{26}$$

The results are displayed in Figs. 17–22. Again, the only Figure without black points is Fig. 22 (WLN). One can conclude that getting the extraneous fixed point close to the imaginary axis in some sense is not enough. Methods that have extraneous fixed points on the imaginary axis (such as WLN) can perform better. The results of the last two experiments are not presented graphically.

Example 4. In the next example we have taken a polynomial of degree 5 with the 5 roots of unity, i.e.

$$p_4(z) = z^5 - 1. \tag{27}$$

The average number of iterations per initial point is the smallest for case 8 (WLN), followed by cases 6, 4, and 7. Notice that $a = 2$ for cases 6, 4 and 7, see Table 1. If we take a different value of a, the results are even worse.

Example 5. In the last example we took a polynomial of degree 7 having the 7 roots of unity, i.e.

$$p_5(z) = z^7 - 1. \tag{28}$$

The conclusion from Table 2 is almost the same as before. The best cases are 8 and 6 as before and the worst are cases 1, 3, and 5.

In the last column of the table, we have averaged those results and it is not surprising that case 8 (WLN) has the smallest average. The next best are cases 6, 4, and 7 (all with $a = 2$). Notice that the parameter for cases 1, 3, and 5 are almost the same and the averages are close. Cases 4 and 6 performed better than cases 3 and 5. Notice that except for LQ, the methods based on the measure $A(g, a)$ performed better than those based on the measure d.

5. Conclusion

We have analyzed the Maheshwari-based eighth order family of methods. We have discussed 3 possible families of weight functions as rational functions and chose the parameters of the families (denoted LQ, QQ, QC) to get the best basins of attraction. We have compared our results to the basin of the modified Wang–Liu method ([23]). The best Maheshwari-based method is case 6 which using QC and the choice of the parameters is based on the measure $A(g, a)$. In fact all QQ and QC methods based on this measure performed better than those based on the measure *d*. But close to the imaginary axis is not as good as being on the imaginary axis (as in the case WLN). We can conclude that WLN performed better than any of the possible version of Maheshwari-based eighth order method.

Acknowledgements

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