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Families of two-step fourth order P-stable methods for second order differential equations

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Abstract: This paper deals with a class of symmetric (hybrid) two-step fourth order P-stable methods for the numerical solution of special second order initial value problems. Such methods were proposed independently by Cash [1] and Chawla [3] and normally require three function evaluations per step. The purpose of this paper is to point out that there are some values of the (free) parameters available in the methods proposed which can reduce this work; we study two classes of such methods. The first is the class of 'economical' methods (see Definition 3.1) which reduce this work to two function evaluations per step, and the second is the class of 'efficient' methods (see Definition 3.2) which reduce this work with respect to implementation for nonlinear problems. We report numerical experiments to illustrate the order, acuracy and implementational aspects of these two classes of methods.

Keywords: Special second order initial value problems, two-step fourth order methods, P-stable methods, 'economical' methods, 'efficient' methods.

1. Introduction

Lambert and Watson [8] introduced P-stability for linear multistep methods for the numerical solution of second order initial value problems:

$$y'' = f(x, y), \qquad y(x_0) = y_0, \quad y'(x_0) = y_0'. \tag{1.1}$$

Consider a linear multistep method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}, \quad k \geqslant 2, \tag{1.2}$$

applied to the test equation:

$$y'' = -\lambda^2 y, \quad \lambda \text{ real}. \tag{1.3}$$

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As usual we set

$$\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j, \qquad \sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j. \tag{1.4}$$

Let r_s, $s = 1, 2, \ldots, k$, denote the zeros (assumed distinct) of the polynomial

$$\Omega(r;\ H^2) = \rho(r) + H^2\sigma(r), \quad H = \lambda h, \tag{1.5}$$

and let r_1, r_2 correspond to perturbations of the principal roots of $\rho(\zeta)$. Then a linear multistep method (1.2) is said to have an interval of periodicity $(0,\ H_0^2)$ if, for all H^2 in the interval, the roots r_s of (1.5) satisfy

$$r_1 = e^{i\theta(H)}, \quad r_2 = e^{-i\theta(H)}, \qquad |r_s| \leqslant 1, \quad s = 3, \ldots, k, \quad \theta(H) \text{ real}.$$

A linear multistep method is called P-stable if its interval of periodicity is $(0, \infty)$. For linear multistep methods, Lambert and Watson had shown that a P-stable method is nescessarily implicit and that the maximum order attainable by a P-stable method is at most two. We remark here that for the problem (1.1) having periodic solutions, P-stability is desirable if the period of the solution is not known; note also that this concept is also important in the case of 'periodic stiffness' as it is termed by Lambert and Watson, that is, when the solution consists of an oscillation of moderate frequency with a high frequency oscillation of small amplitude superimposed. If, however, the period or an estimate for it is known then one can use methods developed by Gautschi [6] and Neta and Ford [9].

In order to overcome the order-barrier on linear multistep P-stable methods, a class of symmetric hybrid two-step methods for (1.1) were introduced in [1,3] and the existence of fourth and sixth order P-stable methods was shown. However, the proposed fourth order P-stable methods normally require three function evaluations per step. The purpose of this paper is to show that there are some values of the (free) parameters available in the families of methods proposed in [1,3] which can reduce this work; we study two classes of such methods. The first is the class of 'economical' methods (see Definition 3.1) which reduces this work to two function evaluations per step, and the second is the class of 'efficient' methods (see Definition 3.2) which reduce this work with respect to implementation for nonlinear problems. Methods in both these classes are obtained. We report numerical experiments to illustrate the order, accuracy and implementational aspects of economical and efficient methods. We note that the two-step fourth order P-stable methods based on two function evaluations per step proposed in [2,4,5,7] are included as particular cases of our methods.

In the following we shall assume a familiarity with the notation and discussion in [3] and with the family $M_4^{(s)}(A_0;\ a_0,\ b_0,\ b^-)$ of symmetric hybrid two-step fourth order methods.

2. Characterization of P-stable methods

As in [3], a method of the family $M_4^{(s)}(A_0;\ a_0,\ b_0,\ \mathrm{b}^-)$ is defined by

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left[2A_0 f_n + A_1(f_{n+1} + f_{n-1}) + B_s(\bar{f}_{n+s} + \bar{f}_{n-s})\right], \tag{2.1}$$

where

$$\bar{y}_{n\pm s} = a_0 y_n + a_1 y_{n\pm 1} + a_{-1} y_{n\mp 1} + h^2(b_0 f_n + b_1 f_{n\pm 1} + b_{-1} f_{n\mp 1}). \tag{2.2}$$

The local truncation error, $T_n(h)$, of this formula is geven by

$$T_n(h) = h^6\left[K_6 y_n^{(6)} + 2B_s\left(sd_3 y_n^{(3)}F_n' + d_4 y_n^{(4)}F_n\right)\right] + O(h^8), \tag{2.3}$$

where

$$K_6 = \tfrac{1}{360} - \tfrac{1}{12}\left(A_1 + s^4 B_s\right), \quad d_3 = \tfrac{1}{6}s^3 - \tfrac{1}{6}a^- - b^-, \quad d_4 = \tfrac{1}{24}s^4 - \tfrac{1}{24}a^+ - \tfrac{1}{2}b^+,$$

and we have set $a^\pm = a_1 \pm a_{-1}$, $b^\pm = b_1 \pm b_{-1}$, $F = \partial f/\partial y$, $F' = \mathrm{d}F/\mathrm{d}x$. Here, and in the following, we shall let $s \in [0,1]$. A method defined by (2.1)–(2.2) is fourth order if

$$\begin{aligned} &a_0 + a^+ = 1, \qquad a^- = s, \\ &\tfrac{1}{2}a^+ + b_0 + b^+ = \tfrac{1}{2}s^2, \qquad b^- = 0 \quad (\text{only if } s = 0), \\ &A_0 + A_1 + B_s = \tfrac{1}{2}, \qquad A_1 + s^2 B_s = \tfrac{1}{12}. \end{aligned} \tag{2.4}$$

We next characterize all P-stable methods of the family $M_4^{(s)}(A_0;\ a_0,\ b_0,\ b^-)$. Applying a method of the family to the test equation (1.3) we obtain the difference equation:

$$(1 + aH^2 - bH^4)y_{n+1} - 2(1 - cH^2 + dH^4)y_n + (1 + aH^2 - bH^4)y_{n-1} = 0, \tag{2.5}$$

where

$$a = A_1 + a^+ B_s, \quad b = b^+ B_s, \quad c = A_0 + a_0 B_s, \quad d = b_0 B_s. \tag{2.6}$$

It is clear that for the characteristic equation corresponding to (2.5), since the product of the roots is 1, the roots will be distinct complex conjugates and each of modulus 1 iff

$$-1 < (1 - cH^2 + dH^4)/(1 + aH^2 - bH^4) < 1. \tag{2.7}$$

It is convenient to set

$$p(H^2) = 2 + (a - c)H^2 + (d - b)H^4, \qquad q(H^2) = (a + c)H^2 - (b + d)H^4. \tag{2.8}$$

From (2.7) it now follows that a method of $M_4^{(s)}(A_0;\ a_0,\ b_0,\ b^-)$ is P-stable iff, for all $H^2 > 0$,

$$p(H^2) > 0 \quad \text{and} \quad q(H^2) > 0. \tag{2.9}$$

Theorem 2.1. *A method of the family $M_4^{(s)}(A_0;\ a_0,\ b_0,\ b^-)$ is P-stable if*

$$d = b \leqslant 0 \quad \textit{and} \quad a \geqslant \tfrac{1}{4}, \tag{2.10}$$

or

$$b < d \leqslant -b, \quad a > \tfrac{1}{4} \quad \textit{and} \quad 32(d - b) = (4a - 1)^2, \tag{2.11}$$

or

$$b < d \leqslant -b \quad \textit{and} \quad 32(d - b) > (4a - 1)^2. \tag{2.12}$$

Proof. Note that $a + c = A_0 + A_1 + (a_0 + a^+)B_s = \frac{1}{2}$. Therefore, $q(H^2) > 0$ for all $H^2 > 0$ only if

$$b + d \leqslant 0. \tag{2.13}$$

Since $a + c = \frac{1}{2}$, one has

$$p(H^2) = 2 + \left(2a - \tfrac{1}{2}\right)H^2 + (d - b)H^4. \tag{2.14}$$

In order that $p(H^2) > 0$ for all $H^2 > 0$, it is necessary that

$$d - b \geqslant 0. \tag{2.15}$$

Now, if $d = b$ then (2.15) is satisfied and one must have $b \leqslant 0$ and $2a - \frac{1}{2} \geqslant 0$. These yield (2.10). If $d > b$ then

$$p(H^2) = (1/16(d-b))\left[\{4(d-b)H^2 + 4a - 1\}^2 + 32(d-b) - (4a-1)^2\right]. \tag{2.16}$$

Thus for $p(H^2) > 0$ we have either $32(d-b) - (4a-1)^2 > 0$ or if $32(d-b) - (4a-1)^2 = 0$ then $a > \frac{1}{4}$. These conditions together with (2.13) and (2.15) yield (2.11) and (2.12). This completes the proof of Theorem 2.1. □

3. Efficiency and implementational aspects of the methods

First of all we note that, in general, a method of the family $M_4^{(s)}(A_0;\ a_0,\ b_0,\ b^-)$ requires three function evaluations per step. Towards finding methods which reduce this work to two function evaluations per step, we have

Definition 3.1. A method of the family $M_4^{(s)}(A_0;\ a_0,\ b_0,\ b^-)$ defined by (2.1) and (2.2) will be called *economical* if it involves only two function evaluations f_{n+1} and $\bar{f}_{n+s}$ during the step from x_n to x_{n+1} and that (for $s \neq 0$) $\bar{f}_{n-s}$ is used from the previous step(s) that determined y_n.

For $s = 0$, $\bar{y}_{n+s}$ and $\bar{y}_{n-s}$ are defined by the same expression and, therefore, all the methods corresponding to $s = 0$ are economical. On the other hand, for $s = \frac{1}{2}$ or 1, a method will be economical provided

$$\bar{f}_{n-s} - \bar{f}_{n-1+s} = \mathrm{O}(h^4),$$

or, equivalently, if

$$\bar{y}_{n-s} - \bar{y}_{n-1+s} = \mathrm{O}(h^4). \tag{3.1}$$

For nonlinear problems (1.1), a method of $M_4^{(s)}(A_0;\ a_0,\ b_0,\ b^-)$ results in an implicit equation for y_{n+1} and we shall need an iterative process to compute the solution at each step. The actual implementation of a class of these methods will be more efficient for nonlinear problems as defined in the following.

Definition 3.2. A method of $M_4^{(s)}(A_0;\ a_0,\ b_0,\ b^-)$ will be called *efficient* (*with respect to implementation for nonlinear problems*) if during the step from x_n to x_{n+1}, $\bar{y}_{n-s}$ does not depend on the not yet determined vector y_{n+1}.

A usual implementation of these schemes with $a_{-1} = b_{-1} = 0$ is to solve for y_{n+1} by means of a Newton-type iteration which for these methods reads:

$$\begin{aligned}&\left[I - h^2(A_1 + a_1 B_s)\partial f/\partial y - h^4 B_s b_1 (\partial f/\partial y)^2\right]\left(y_{n+1}^{(j+1)} - y_{n+1}^{(j)}\right)\\ &= -y_{n+1}^{(j)} + 2y_n - y_{n-1} + h^2\left[2A_0 f_n + A_1\left(f_{n+1}^{(j)} + f_{n-1}\right) + B_s\left(\bar{f}_{n+s}^{(j)} + \bar{f}_{n-s}\right)\right],\\ &\quad j = 0, 1, 2, \ldots,\end{aligned} \tag{3.2}$$

where the partial derivative $\partial f/\partial y$ is evaluated once during the iteration for $y = y_{n+1}^{(0)}$ for a suitable initial approximation $y_{n+1}^{(0)}$ for y_{n+1}. Now, if the problem is nonlinear, only $f_{n+1}^{(j)}$ and $\bar{f}_{n+s}^{(j)}$ have to be updated for each Newton-iteration and $\bar{f}_{n-s}$ is a constant vector for this iteration process. If $\bar{f}_{n-s}$ is calculated once in each step, the fourth order accuracy will be preserved.

It is easy to see that, except at the first and at the last point of integration, J iterations for an efficient method at x_n to compute y_{n+1} will cost $2J+2$ f-evaluations and one f^y-evaluation. It may also be noted here that I iterations for an economical method at x_n to compute y_{n+1} will cost $2I+2$ f-evaluations and one f^y-evaluation, while a non-efficient or a non-economical method will cost $3I+1$ f-evaluations and one f^y-evaluation per I iterations per step. The interesting conclusion from these considerations is that for the same number of iterations, cost per step of an economical method and an efficient method is the same. However, either of these methods cost roughly $\frac{2}{3}$ times less than a non-economical or a non-efficient method.

4. Economical fourth order P-stable methods

Methods from the family $M_4^{(s)}(A_0;\ a_0,\ b_0,\ b^-)$ can be economical (see Definition 3.1) in the following four cases:

$$\text{(i)} \qquad A_1 = 0, \qquad b_1 = b_{-1} = 0, \qquad \tfrac{1}{6}s^3 = \tfrac{1}{6}a^- + b^-, \tag{4.1}$$

$$\text{(ii)} \qquad s = 0, \qquad b^- = 0, \tag{4.2}$$

$$\text{(iii)} \qquad s = \tfrac{1}{2}, \qquad a_{-1} = b_{-1} = 0, \qquad \tfrac{1}{48} = \tfrac{1}{6}a^- + b^-, \tag{4.3}$$

$$\text{(iv)} \qquad s = 1, \qquad a_{-1} = b_{-1} = 0, \qquad \tfrac{1}{6} = \tfrac{1}{6}a^- + b^-. \tag{4.4}$$

Because the first case yields explicit methods we have the following result.

Theorem 4.1. *There is no economical P-stable method from $M_4^{(s)}(A_0;\ a_0,\ b_0,\ b^-)$ in the first case.*

We next obtain economical P-stable methods for the second case.

Case (ii). $s = 0$, $b^- = 0$. In this case (2.2) defines just one $\bar{y}_n$ and thus (2.1) involves only two function evaluations f_{n+1} and $\bar{f}_n$ per step. From the order conditions (2.4) we obtain

$$a^+ = 1 - a_0, \qquad b^+ = -\tfrac{1}{2}(1 - a_0) - b_0, \qquad A_1 = \tfrac{1}{12}, \qquad A_0 = \tfrac{5}{12} - B_0, \tag{4.5}$$

where B_0, a_0 and b_0 are free parameters. Note that the case $B_0 = 0$ ($A_0 = \frac{5}{12}$) yields the well known Numerov's method which is not P-stable. We shall refer to this family of methods as $M_4^{(0)}(B_0;\ a_0,\ b_0)$.

Theorem 4.2. *All ecomonical methods from the family $M_4^{(0)}(B_0;\ a_0,\ b_0)$ are P-stable if any one of the following holds*:

$$B_0^* \leqslant -\tfrac{1}{24}, \qquad B_0^{**} = -4B_0^*, \tag{4.6}$$

or

$$-\tfrac{5}{48} \leqslant B_0^* < -\tfrac{1}{24}, \qquad B_0^{**} = \tfrac{2}{3} \pm \sqrt{4B_0^* + \tfrac{5}{12}}, \tag{4.7}$$

or

$$B_0^* > -\tfrac{5}{48}, \qquad B_0^{**} \geqslant 0, \qquad \tfrac{2}{3} - \sqrt{4B_0^* + \tfrac{5}{12}} < B_0^{**} < \tfrac{2}{3} + \sqrt{4B_0^* + \tfrac{5}{12}}, \tag{4.8}$$

where

$$B_0^* = b_0 B_0, \qquad B_0^{**} = (1 - a_0) B_0. \tag{4.9}$$

Proof. Substituting (4.5) in (2.6) one obtains

$$a = \tfrac{1}{12} + B_0^{**}, \qquad b = -B_0^* - \tfrac{1}{2}B_0^{**}, \qquad d = B_0^*. \tag{4.10}$$

Condition (2.10) for P-stability becomes

$$B_0^{**} \geqslant \tfrac{1}{6}, \qquad B_0^* \leqslant 0, \qquad B_0^{**} = -4B_0^*. \tag{4.11}$$

These yield (4.6). Condition (2.11) becomes

$$B_0^{**} = \tfrac{2}{3} \pm \sqrt{4B_0^* + \tfrac{5}{12}}, \qquad B_0^{**} > \tfrac{1}{6}, \qquad 2B_0^* + \tfrac{1}{2}B_0^{**} > 0, \qquad B_0^{**} \geqslant 0. \tag{4.12}$$

These yield (4.7). Condition (2.12) becomes

$$\tfrac{2}{3} - \sqrt{4B_0^* + \tfrac{5}{12}} < B_0^{**} < \tfrac{2}{3} + \sqrt{4B_0^* + \tfrac{5}{12}}, \qquad 2B_0^* + \tfrac{1}{2}B_0^{**} > 0, \qquad B_0^{**} \geqslant 0. \tag{4.13}$$

These yield (4.8). □

Remark. For methods of the family $M_4^{(0)}(B_0;\ a_0,\ b_0)$ the coefficient of the leading term in the local truncation error is given (see (2.3)) by

$$-\tfrac{1}{240}y_n^{(6)} + \tfrac{1}{12}B_0\left[5(1-a_0) + 12b_0\right]y_n^{(4)}F_n. \tag{4.14}$$

A particularly simple class of methods corresponds to $B_0 = \frac{5}{12}$ ($A_0 = 0$) and $a_0 = 1$ which give a one-parameter family of economical P-stable methods:

$$\begin{aligned} \bar{y}_n &= y_n - \tfrac{1}{2}b_0 h^2(f_{n+1} - 2f_n + f_{n-1}), \\ y_{n+1} - 2y_n + y_{n-1} &= \tfrac{1}{12}h^2(f_{n+1} + 10\bar{f}_n + f_{n-1}), \end{aligned} \tag{4.15}$$

where $b_0 > \frac{1}{60}$ and the leading coefficient of the truncation error is

$$-\tfrac{1}{240}y_n^{(6)} + \tfrac{5}{12}b_0 y_n^{(4)}F_n. \tag{4.16}$$

This one-parameter family was obtained by Chawla [4].

A particular formula of the family $M_4^{(0)}(B_0;\ a_0,\ b_0)$ with $B_0 = \frac{1}{3}$ $a_0 = 1$ and $b_0 = \frac{1}{47}$ has also been given by Cash [2] (with his equation (2.10) corrected as reported by Cash [2, p. 400].

From the discussion in Section 5 it will follow that no economical P-stable methods can be found for the Case (iv), while economical P-stable methods for the Case (iii) are precisely the class of methods obtained by Costabile and Costabile [5].

5. Efficient fourth order P-stable methods

Efficient methods (see Definition 3.2) can be found from the family $M_4^{(s)}(A_0;\ a_0,\ b_0,\ b^-)$ in the following two cases:

$$\text{(i)} \qquad s=\tfrac{1}{2}, \qquad a_{-1}=b_{-1}=0, \tag{5.1}$$

$$\text{(ii)} \qquad s=1, \qquad a_{-1}=b_{-1}=0. \tag{5.2}$$

Case (i). $s=\frac{1}{2}$, $a_{-1}=b_{-1}=0$. In this case one obtains from (2.4) the following relations:

$$a_0=a^+=a^-=\tfrac{1}{2}, \qquad b^+=b^-=-\tfrac{1}{8}-b_0,$$
$$A_1=\tfrac{1}{18}(6A_0-1), \qquad B_{1/2}=\tfrac{1}{9}(5-12A_0), \tag{5.3}$$

where A_0, b_0 are now free parameters. We shall refer to this family as $M_4^{(1/2)}(A_0;\ b_0)$.

Theorem 5.1. *All efficient methods from the family $M_4^{(1/2)}(A_0;\ b_0)$ are P-stable if any one of the following holds*:

$$b_0^*=0, \qquad A_0^*\geqslant 1, \tag{5.4}$$

or

$$b_0^*>0, \qquad \rho-\sqrt{\rho^2-1}<A_0^*\leqslant\rho+\sqrt{\rho^2-1}, \tag{5.5}$$

where

$$b_0^*=1+16b_0, \qquad A_0^*=\tfrac{1}{6}(5-12A_0), \qquad \rho=1+3b_0^*. \tag{5.6}$$

Proof. Combining (2.6) with (5.3) yields

$$a=\tfrac{1}{12}(1+2A_0^*), \qquad b=-\tfrac{1}{24}A_0^*(1+b_0^*), \qquad d=\tfrac{1}{24}A_0^*(b_0^*-1). \tag{5.7}$$

Condition (2.10) for P-stability then becomes $b_0^*=0$ and $A_0^*\geqslant 1$ which is (5.4). Condition (2.11) becomes

$$b_0^*>0, \qquad A_0^*=\rho+\sqrt{\rho^2-1}. \tag{5.8}$$

Finally from (2.12) it follows that a method is P-stable if

$$b_0^*>0 \quad\text{and}\quad \rho-\sqrt{\rho^2-1}<A_0^*<\rho+\sqrt{\rho^2-1}. \tag{5.9}$$

(5.8) and (5.9) together establish (5.5). □

Remark. The family of P-stable methods developed by Costabile and Costabile [5] is a special case corresponding to (5.4). Note also that the last condition in (4.3) is satisfied by methods of $M_4^{(1/2)}(A_0;\ b_0)$ for which $b_0=-\frac{1}{16}$, that is, $b_0^*=0$. Interestingly, therefore, the P-stable methods of Costabile and Costabile are both economical and efficient.

The coefficient of the leading term in the local truncation error for the family $M_4^{(1/2)}(A_0;\ b_0)$ is given by

$$\tfrac{1}{480}(5A_0^*-2)y_n^{(6)}+\tfrac{1}{24}A_0^*\left[b_0^*y_n^{(3)}F_n'+\left(b_0^*+\tfrac{5}{12}\right)y_n^{(4)}F_n\right]. \tag{5.10}$$

We construct some P-stable methods from $M_4^{(1/2)}(A_0;\ b_0)$ satisfying (5.5). A particularly simple method corresponding to $b_0^* = 1$ ($b_0 = 0$) and $A_0^* = \frac{2}{5}$ ($A_0 = \frac{13}{60}$) is

$$\bar{y}_{n\pm 1/2} = \tfrac{1}{2}(y_n + y_{n\pm 1}) - \tfrac{1}{8}h^2 f_{n\pm 1},$$
$$y_{n+1} = 2y_n - y_{n-1} + \tfrac{1}{60}h^2\left[f_{n+1} + 26f_n + f_{n-1} + 16(\bar{f}_{n+1/2} + \bar{f}_{n-1/2})\right], \tag{5.11}$$

with truncation error

$$T_n(h) = \tfrac{1}{720}(12y_n^{(3)}F_n' + 17y_n^{(4)}F_n)h^6 + O(h^8). \tag{5.12}$$

Another particularly simple method corresponding to $b_0^* = 1$ ($b_0 = 0$) and $A_0^* = \frac{1}{2}$ ($A_0 = \frac{1}{6}$, $A_1 = 0$) may be noted:

$$\bar{y}_{n\pm 1/2} = \tfrac{1}{2}(y_n + y_{n\pm 1}) - \tfrac{1}{8}h^2 f_{n\pm 1},$$
$$y_{n+1} = 2y_n - y_{n-1} + \tfrac{1}{3}h^2(\bar{f}_{n+1/2} + f_n + \bar{f}_{n-1/2}), \tag{5.13}$$

with truncation error

$$T_n(h) = \tfrac{1}{2880}\left[3y_n^{(6)} + 5(12y_n^{(3)}F_n' + 17y_n^{(4)}F_n)\right]h^6 + O(h^8). \tag{5.14}$$

Case (ii). $s = 1$, $a_{-1} = b_{-1} = 0$. In this case one obtains from the order conditions (2.4) the following relations:

$$a_0 = 0, \qquad a_1 = 1, \qquad b_1 = -b_0,$$
$$A_0 = \tfrac{5}{12}, \qquad A_1 = \tfrac{1}{12} - B_1, \tag{5.15}$$

where B_1 and b_0 are free parameters. We shall denote this family by $M_4^{(1)}(B_1;\ b_0)$. The case $B_1 = 0$ gives Numerov's method which is not P-stable.

Theorem 5.2. *All efficient methods from the family $M_4^{(1)}(B_1;\ b_0)$ are P-stable if*

$$b_0 B_1 > \tfrac{1}{144}. \tag{5.16}$$

Proof. Relations (2.6) combined with (5.15) yield

$$a = \tfrac{1}{12}, \qquad b = -d = -b_0 B_1, \qquad c = \tfrac{5}{12}. \tag{5.17}$$

Since $a = \frac{1}{12}$, one can not obtain P-stable methods unless (2.12) is satisfied. The first condition of (2.12) is satisfied if $b_0 B_1 > 0$ and the second is exactly (5.16). □

Remark. It can be shown that no method of $M_4^{(1)}(B_1;\ b_0)$ can be economical in the sense of Definition 3.1.

The leading coefficient of the truncation error for methods of $M_4^{(1)}(B_1;\ b_0)$ is given by

$$-\tfrac{1}{240}y_n^{(6)} + b_0 B_1(2y_n^{(3)}F_n' + y_n^{(4)}F_n). \tag{5.18}$$

A particularly simple one-parameter family of efficient P-stable methods from $M_4^{(1)}(B_1;\ b_0)$ is obtained when $B_1 = \frac{1}{12}$ ($A_1 = 0$). The method is

$$\bar{y}_{n\pm 1} = y_{n\pm 1} + b_0 h^2(f_n - f_{n\pm 1}),$$
$$y_{n+1} = 2y_n - y_{n-1} + \tfrac{1}{12}h^2(\bar{f}_{n+1} + 10f_n + \bar{f}_{n-1}), \tag{5.19}$$

where $b_0 > \frac{1}{12}$. The truncation error is then

$$T_n(h) = \left[-\tfrac{1}{240} y_n^{(6)} + \tfrac{1}{12} b_0 \left(2 y_n^{(3)} F_n' + y_n^{(4)} F_n \right) \right] h^6 + O(h^8). \tag{5.20}$$

6. Numerical experiments

To illustrate the order, accuracy and implementational aspects of economical and efficient methods obtained in Sections 4 and 5, we discuss numerical experiments with these methods by considering the following nonlinear example due to Van Dooren [10].

We consider the Duffing equation forced by a harmonic function:

$$y'' + y + y^3 = \delta \cos(\mu x), \tag{6.1}$$

with the values of the parameters $\delta = 0.002$ and $\mu = 1.01$, and with the initial conditions $y(0) = A$, $y'(0) = 0$, taking for A the value of the Galerkin approximation y_G at $x = 0$. By Urabe's method applied to Galerkin's procedure, Van Dooren has computed the Galerkin's approximation of order nine to a periodic solution having the same period as the forcing term, with a precision of the coefficients of 10^{-12}:

$$y_G = \sum_{i=0}^{4} a_{2i+1} \cos((2i+1)\mu x), \tag{6.2}$$

where

$$a_1 = 0.200179477536, \quad a_3 = 0.246946143\ (-3),$$
$$a_5 = 0.304014\ (-6), \quad a_7 = 0.374\ (-9), \quad a_9 = 0.0.$$

For all the methods tested on the problem (6.1), we computed the absolute error in the computation of $y(40\pi)$ using step sizes $h = \frac{1}{5}\pi, \frac{1}{10}\pi, \frac{1}{20}\pi, \frac{1}{40}\pi$; corresponding results are shown in Tables 1 and 2. Also shown in the tables alongside the errors are the number of iterations a method took to converge to the accuracy shown and its rate of convergence. To test the two classes of methods proposed in the present paper, we took three representative methods from each class for the purpose of illustration. From the class of 'economical' methods, we selected the three methods: Chawla's method $M_4^{(0)}(\frac{5}{12}; 1, \frac{1}{59})$, Cash's [2] recently given method $M_4^{(0)}(\frac{1}{3}; 1, \frac{1}{47})$ and the method $M_4^{(1/2)}(-\frac{1}{12}; -\frac{1}{16})$ of Costabile and Costabile. To illustrate the accuracy and rate of convergence of 'efficient' methods on the nonlinear problem (6.1), we took the three representative methods as described in (5.11), (5.13) and (5.19) (with $b_0 = \frac{1}{10}$).

Table 1
'Economical' methods

h	Chawla's method (4.15) with $b_0 = \frac{1}{59}$			Cash's method [2]			Costabile and Costabile method [5] with $b_0 = -\frac{1}{16}$		
$\frac{1}{5}\pi$	4.50 (−3)	#2		4.54 (−3)	#2		3.10 (−2)	#2	
$\frac{1}{10}\pi$	2.85 (−4)	#1	3.98	2.88 (−4)	#1	3.98	2.05 (−3)	#2	3.92
$\frac{1}{20}\pi$	1.79 (−5)	#1	4.00	1.81 (−5)	#1	4.00	1.30 (−4)	#2	3.98
$\frac{1}{40}\pi$	1.12 (−6)	#1	4.00	1.13 (−6)	#1	4.00	8.15 (−6)	#2	3.99

Table 2
'Efficient' methods

h	Method (5.11)			Method (5.13)			Method (5.19) with $b_0 = \frac{1}{10}$		
$\frac{1}{5}\pi$	3.18 (−2)	#2		4.09 (−2)	#2		1.64 (−2)	#2	
$\frac{1}{10}\pi$	2.02 (−3)	#2	3.97	2.61 (−3)	#2	3.97	1.02 (−3)	#2	4.01
$\frac{1}{20}\pi$	1.27 (−4)	#2	3.99	1.65 (−4)	#1	3.98	6.35 (−5)	#2	4.00
$\frac{1}{40}\pi$	7.97 (−6)	#2	4.00	1.03 (−5)	#1	4.00	3.97 (−6)	#1	4.00

The resulting implicit equations giving y_{n+1} at each step were solved by using the Newton-type iteration (3.2). As an initial approximation for use with the Newton-type iteration (3.2) at each step, we took

$$y_{n+1}^{(0)} = 2y_n - y_{n-1} + h^2 f_n. \tag{6.3}$$

The conclusions that can be drawn from the above numerical experiments are the following. First of all, since the method of Costabile and Costabile is both economical and efficient, as expected, this method implemented either as an economical or as an efficient method produced precisely the same numerical results which are shown in the third column of Table 1. (For this reason, results of the method of Costabile and Costabile run as an efficient method are not shown in Table 2.)

Most interestingly, all the representative efficient methods implemented as suggested in the Definition 3.2 of 'efficient' methods show fourth order rate of convergence; the method (5.19), in fact, exhibits a slight over-rate of convergence. Note also that the method of Costabile and Costabile is less accurate, costs more in terms of function evaluations for a lesser accuracy and has a slower rate of convergence; with the efficient method (5.11) being comparable with it. Finally, from these experiments we can conclude that Chawla's method (4.15), with $b_0 = \frac{1}{59}$, is most accurate and costs the least for fourth order convergence, with Cash's method being comparable.

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