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Large time behavior of solutions to a nonlinear integro-differential system

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ABSTRACT

Asymptotic behavior of solutions as $t \to \infty$ to the nonlinear integro-differential system associated with the penetration of a magnetic field into a substance is studied. Initial–boundary value problems with two kinds of boundary data are considered. The first with homogeneous conditions on whole boundary and the second with non-homogeneous boundary data on one side of lateral boundary. The rates of convergence are given too.

1. Introduction and statement of results

Integro-differential equations and systems arise in the study of various problems in physics, chemistry, technology, economics etc. (see, for example, [1–12]). The purpose of this paper is to study asymptotic behavior of solutions as $t \to \infty$ of initial–boundary value problems for the following nonlinear integro-differential system:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[\left(1 + \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau\right)\frac{\partial U}{\partial x}\right],$$
$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left[\left(1 + \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau\right)\frac{\partial V}{\partial x}\right]. \tag{1.1}$$

Integro-differential systems of (1.1) types, based on Maxwell's system [13], arise for mathematical modelling of the process of a magnetic field penetrating into a substance [14]. The existence and uniqueness properties of the solutions of the initial–boundary value problems for the equations and systems of (1.1) type were first studied in the works [14,15] and consequently in a number of other works as well (see, for example, [16–20]). The existence theorems, that are proved in [14–16], are based on a priori estimates, Galerkin's method and compactness arguments as in [21,22] for nonlinear parabolic equations.

Difference schemes for a certain nonlinear parabolic integro-differential model similar to (1.1) were studied in [23]. Neta [24] also discussed the finite element approximation of that nonlinear integro-differential equation.

It is important to investigate asymptotic behavior of solutions as $t \to \infty$ of the initial–boundary value problems for (1.1). In this direction research was made in the works [25–27]. In [26,27] investigations are made for the scalar equation of (1.1) type. In [25] the asymptotic behavior of solutions as $t \to \infty$ of (1.1) system for the homogeneous boundary conditions in the norm of the space $H^1(0,1)$ was given. Here and below we use usual Sobolev spaces $H^k(0,1)$.

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In this paper our interest is to continue study of the asymptotic behavior of solutions as $t \to \infty$ of the system (1.1).

In the domain $Q = [0,1] \times [0,\infty)$ initial–boundary value problems with the following two cases of boundary data are considered:

$$U(0,t) = U(1,t) = V(0,t) = V(1,t) = 0, \quad t \geqslant 0, \tag{1.2}$$

or

$$U(0,t) = V(0,t) = 0, \qquad U(1,t) = \psi_1, \qquad V(1,t) = \psi_2, \quad t \geqslant 0, \tag{1.3}$$

where $\psi_1 = Const \geqslant 0$, $\psi_2 = Const \geqslant 0$, $\psi_1^2 + \psi_2^2 \neq 0$. To complete the problem we include the initial conditions:

$$U(x,0) = U_0(x), \qquad V(x,0) = V_0(x), \quad x \in [0,1], \tag{1.4}$$

where $U_0 = U_0(x)$ and $V_0 = V_0(x)$ are given functions.

Everywhere in this paper the initial–boundary value problem for (1.1) with homogeneous boundary conditions (1.2) and initial data (1.4) will be referred to as Problem 1, while initial–boundary value problem for the same model with non-homogeneous boundary conditions (1.3) and initial data (1.4) will be referred to as Problem 2.

For Problems 1 and 2 we assume that $U = U(x,t)$, $V = V(x,t)$ is a solution on Q, such that $U(\cdot,t)$, $V(\cdot,t)$, $\frac{\partial U(\cdot,t)}{\partial x}$, $\frac{\partial V(\cdot,t)}{\partial x}$, $\frac{\partial U(\cdot,t)}{\partial t}$, $\frac{\partial V(\cdot,t)}{\partial t}$, $\frac{\partial^2 U(\cdot,t)}{\partial x^2}$, $\frac{\partial^2 V(\cdot,t)}{\partial x^2}$, $\frac{\partial^2 U(\cdot,t)}{\partial t\,\partial x}$, $\frac{\partial^2 V(\cdot,t)}{\partial t\,\partial x}$ are all in $C^0([0,\infty);\ L_2(0,1))$, while $\frac{\partial^2 U(\cdot,t)}{\partial t^2}$ and $\frac{\partial^2 V(\cdot,t)}{\partial t^2}$ are in $L_2((0,\infty);\ L_2(0,1))$.

Note that the existence of solutions of Problems 1 and 2 and the uniqueness for more general cases are proved in [14].

The rest of the paper is organized as follows. In Section 2 we discuss Problem 1. We show that stabilization is obtained in the norm of the space $C^1[0,1]$. In particular, we prove the following statement.

Theorem 1.1. *Suppose that $U_0, V_0 \in H^2(0,1)$, $U_0(0) = U_0(1) = V_0(0) = V_0(1) = 0$, then for the unique solution of Problem 1 the following relations hold*:

$$\left|\frac{\partial U(x,t)}{\partial x}\right| \leqslant C \exp\left(-\frac{t}{2}\right), \qquad \left|\frac{\partial V(x,t)}{\partial x}\right| \leqslant C \exp\left(-\frac{t}{2}\right), \quad t \geqslant 0.$$

Remark. Here and below C, C_i and c denote positive constants independent of t.

Section 3 is devoted to the study of the problem with non-zero boundary data on one side of lateral boundary. The asymptotic property for this case is also proved in the norm of the space $C^1[0,1]$. The main statement of this section has the following form.

Theorem 1.2. *Suppose that $U_0, V_0 \in H^2(0,1)$, $U_0(0) = V_0(0) = 0$, $U_0(1) = \psi_1 = Const \geqslant 0$, $V_0(1) = \psi_2 = Const \geqslant 0$, $\psi_1^2 + \psi_2^2 \neq 0$, then for the unique solution of Problem 2 the following estimates are true*:

$$\left|\frac{\partial U(x,t)}{\partial x} - \psi_1\right| \leqslant C(1+t)^{-2}, \qquad \left|\frac{\partial V(x,t)}{\partial x} - \psi_2\right| \leqslant C(1+t)^{-2}, \quad t \geqslant 0.$$

2. Proof of Theorem 1.1

In this section we investigate Problem 1.

First a word on notations. We will use usual L_2-inner product and the correspondence norm:

$$(u,v) = \int_0^1 u(x)v(x)\,dx, \qquad \|u\| = (u,u)^{1/2}.$$

For Problem 1 it is easy to get validity of the following estimates [25]:

$$\|U\| \leqslant C\exp(-t), \qquad \|V\| \leqslant C\exp(-t).$$

Note that these estimates give exponential stabilization of the solutions of Problem 1 in the norm of the space $L_2(0,1)$. The purpose of this section is to show that the stabilization is also achieved in the norm of the space $C^1[0,1]$. At first we formulate result of the stabilization for Problem 1 in the norm of the space $H^1(0,1)$ [25].

Theorem 2.1. *Suppose that $U_0, V_0 \in H^2(0,1)$, $U_0(0) = U_0(1) = V_0(0) = V_0(1) = 0$, then for the solution of Problem 1 the following estimate is true*:

$$\left\|\frac{\partial U}{\partial x}\right\| + \left\|\frac{\partial V}{\partial x}\right\| + \left\|\frac{\partial U}{\partial t}\right\| + \left\|\frac{\partial V}{\partial t}\right\| \leqslant C\exp\left(-\frac{t}{2}\right).$$

Now let us prove main result of this section, namely Theorem 1.1. For this we need some auxiliary estimates. We will prove the following estimates.

Lemma 2.1. *For Problem* 1 *the following estimates are true*:

$$c\varphi^{\frac{1}{3}}(t) \leqslant 1 + S(x,t) \leqslant C\varphi^{\frac{1}{3}}(t),$$

where

$$\varphi(t) = 1 + \int_0^t \int_0^1 \left(\sigma_1^2 + \sigma_2^2\right) dx\, d\tau, \tag{2.1}$$

$$S(x,t) = \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau, \tag{2.2}$$

and $\sigma_1 = (1+S)\partial U/\partial x$, $\sigma_2 = (1+S)\partial V/\partial x$.

Proof. From (2.2) it follows that:

$$\frac{\partial S}{\partial t} = \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2, \qquad S(x,0) = 0.$$

Let us multiply the first equality of the last relations by $(1+S)^2$:

$$\frac{1}{3}\frac{\partial (1+S)^3}{\partial t} = \left(\frac{\partial U}{\partial x}\right)^2 (1+S)^2 + \left(\frac{\partial V}{\partial x}\right)^2 (1+S)^2.$$

Since the system (1.1) can be rewritten as

$$\frac{\partial U}{\partial t} = \frac{\partial \sigma_1}{\partial x}, \qquad \frac{\partial V}{\partial t} = \frac{\partial \sigma_2}{\partial x}, \tag{2.3}$$

we have:

$$\frac{1}{3}\frac{\partial (1+S)^3}{\partial t} = \sigma_1^2 + \sigma_2^2, \tag{2.4}$$

$$\sigma_1^2(x,t) = \int_0^1 \sigma_1^2(y,t)\, dy + \int_0^1 \int_y^x \frac{\partial \sigma_1^2(\xi,t)}{\partial \xi}\, d\xi\, dy = \int_0^1 \sigma_1^2(y,t)\, dy + 2\int_0^1 \int_y^x \sigma_1(\xi,t)\frac{\partial U(\xi,t)}{\partial t}\, d\xi\, dy,$$

$$\sigma_2^2(x,t) = \int_0^1 \sigma_2^2(y,t)\, dy + \int_0^1 \int_y^x \frac{\partial \sigma_2^2(\xi,t)}{\partial \xi}\, d\xi\, dy = \int_0^1 \sigma_2^2(y,t)\, dy + 2\int_0^1 \int_y^x \sigma_2(\xi,t)\frac{\partial V(\xi,t)}{\partial t}\, d\xi\, dy. \tag{2.5}$$

In view of Theorem 2.1 and relations (2.1), (2.4), (2.5) we obtain

$$\begin{aligned}
\frac{1}{3}(1+S)^3 &= \int_0^t \left(\sigma_1^2 + \sigma_2^2\right) d\tau + \frac{1}{3} \\
&= \int_0^t \int_0^1 \left(\sigma_1^2(y,\tau) + \sigma_2^2(y,\tau)\right) dy\, d\tau + 2\int_0^t \int_0^1 \int_y^x \left(\sigma_1(\xi,\tau)\frac{\partial U(\xi,\tau)}{\partial \tau} + \sigma_2(\xi,\tau)\frac{\partial V(\xi,\tau)}{\partial \tau}\right) d\xi\, dy\, d\tau + \frac{1}{3} \\
&\leqslant 2\int_0^t \int_0^1 \left(\sigma_1^2(y,\tau) + \sigma_2^2(y,\tau)\right) dy\, d\tau + \int_0^t \int_0^1 \left[\left(\frac{\partial U(x,\tau)}{\partial \tau}\right)^2 + \left(\frac{\partial V(x,\tau)}{\partial \tau}\right)^2\right] dx\, d\tau + \frac{1}{3} \\
&\leqslant 2\int_0^t \int_0^1 \left(\sigma_1^2(y,\tau) + \sigma_2^2(y,\tau)\right) dy\, d\tau + C_1 \leqslant C_2\varphi(t),
\end{aligned}$$

i.e.,

$$1 + S(x,t) \leqslant C\varphi^{\frac{1}{3}}(t). \tag{2.6}$$

In an analogous way we deduce

$$\frac{1}{3}(1+S)^3 = \int_0^t\int_0^1 \left(\sigma_1^2(y,\tau)+\sigma_2^2(y,\tau)\right)dy\,d\tau + 2\int_0^t\int_0^1\int_y^x \left(\sigma_1(\xi,\tau)\frac{\partial U(\xi,\tau)}{\partial\tau}+\sigma_2(\xi,\tau)\frac{\partial V(\xi,\tau)}{\partial\tau}\right)d\xi\,dy\,d\tau + \frac{1}{3}$$
$$\geqslant \frac{1}{2}\int_0^t\int_0^1 \left(\sigma_1^2(y,\tau)+\sigma_2^2(y,\tau)\right)dy\,d\tau - C_1 = \frac{1}{2}\varphi(t) - C_2. \tag{2.7}$$

We have

$$C_2(1+S)^3 \geqslant C_2. \tag{2.8}$$

Thus, via relations (2.7) and (2.8) we obtain

$$\left(\frac{1}{3}+C_2\right)(1+S)^3 \geqslant \frac{1}{2}\varphi(t),$$

or

$$1+S(x,t) \geqslant c\varphi^{\frac{1}{3}}(t). \tag{2.9}$$

Finally, from (2.6) and (2.9) the validity of Lemma 2.1 follows.

Taking into account definition (2.1), Lemma 2.1 and Theorem 2.1 we arrive at

$$\frac{d\varphi(t)}{dt} = \int_0^1 (1+S)^2\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]dx \leqslant C\varphi^{\frac{2}{3}}(t)\exp(-t),$$

or

$$\frac{d}{dt}\left(\varphi^{\frac{1}{3}}(t)\right) \leqslant C\exp(-t).$$

After integrating from 0 to t, keeping in mind definition (2.1), we get

$$1 \leqslant \varphi(t) \leqslant C.$$

From this, using Lemma 2.1, for the function S we have

$$1 \leqslant 1+S(x,t) \leqslant C. \tag{2.10}$$

Using (2.10) and Theorem 2.1, the equalities (2.5) give

$$\sigma_1^2(x,t)+\sigma_2^2(x,t) \leqslant 2\int_0^1 (1+S)^2\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]dx + \int_0^1\left[\left(\frac{\partial U}{\partial t}\right)^2+\left(\frac{\partial V}{\partial t}\right)^2\right]dx \leqslant C\exp(-t),$$

or

$$\left|\sigma_1(x,t)\right| \leqslant C\exp\left(-\frac{t}{2}\right), \qquad \left|\sigma_2(x,t)\right| \leqslant C\exp\left(-\frac{t}{2}\right).$$

These estimates, taking into account (2.10) and the relations

$$\sigma_1 = (1+S)\partial U/\partial x, \qquad \sigma_2 = (1+S)\partial V/\partial x,$$

complete the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

We open this section by proving some auxiliary lemmas.

Lemma 3.1. *For the solution of Problem* 2 *the following estimates hold*:

$$\int_0^t\int_0^1\left(\frac{\partial U}{\partial\tau}\right)^2 dx\,d\tau \leqslant C, \qquad \int_0^t\int_0^1\left(\frac{\partial V}{\partial\tau}\right)^2 dx\,d\tau \leqslant C,$$
$$\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx \leqslant C, \qquad \int_0^1\left(\frac{\partial V}{\partial t}\right)^2 dx \leqslant C.$$

Proof. Let us differentiate the first equation of the system (1.1) with respect to t:

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial x}\left\{\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right]\frac{\partial U}{\partial x} + (1+S)\frac{\partial^2 U}{\partial t \partial x}\right\} = 0. \tag{3.1}$$

Multiplying (3.1) by $\partial U/\partial t$ and using integration by parts we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \int_0^1 (1+S)\left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx + \int_0^1 \left(\frac{\partial U}{\partial x}\right)^3 \frac{\partial^2 U}{\partial t \partial x} dx + \int_0^1 \frac{\partial U}{\partial x}\left(\frac{\partial V}{\partial x}\right)^2 \frac{\partial^2 U}{\partial t \partial x} dx = 0. \tag{3.2}$$

In an analogous way we deduce

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{\partial V}{\partial t}\right)^2 dx + \int_0^1 (1+S)\left(\frac{\partial^2 V}{\partial t \partial x}\right)^2 dx + \int_0^1 \left(\frac{\partial V}{\partial x}\right)^3 \frac{\partial^2 V}{\partial t \partial x} dx + \int_0^1 \frac{\partial V}{\partial x}\left(\frac{\partial U}{\partial x}\right)^2 \frac{\partial^2 V}{\partial t \partial x} dx = 0. \tag{3.3}$$

Combining (3.2), (3.3) and taking into account the nonnegativity of the function S, we obtain

$$\frac{d}{dt}\left[\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \int_0^1 \left(\frac{\partial V}{\partial t}\right)^2 dx\right] + 2\left[\int_0^1 \left(\frac{\partial^2 U}{\partial x \partial t}\right)^2 dx + \int_0^1 \left(\frac{\partial^2 V}{\partial x \partial t}\right)^2 dx\right]$$
$$+ \frac{1}{2}\int_0^1 \frac{\partial}{\partial t}\left[\left(\frac{\partial U}{\partial x}\right)^4 + \left(\frac{\partial V}{\partial x}\right)^4\right] dx + \int_0^1 \frac{\partial}{\partial t}\left[\left(\frac{\partial U}{\partial x}\right)^2 \left(\frac{\partial V}{\partial x}\right)^2\right] dx \leqslant 0,$$

or

$$\int_0^1 \left[\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial t}\right)^2\right] dx + 2\int_0^t \left[\int_0^1 \left(\frac{\partial^2 U}{\partial x \partial \tau}\right)^2 dx + \int_0^1 \left(\frac{\partial^2 V}{\partial x \partial \tau}\right)^2 dx\right] d\tau$$
$$+ \frac{1}{2}\int_0^t \int_0^1 \frac{\partial}{\partial \tau}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right]^2 dx\, d\tau \leqslant C.$$

For the last term on the left-hand side of this inequality we have

$$\frac{1}{2}\int_0^t \int_0^1 \frac{\partial}{\partial \tau}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right]^2 dx\, d\tau = \frac{1}{2}\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right]^2 dx - C.$$

So, taking into account Poincare's inequality we get

$$\int_0^1 \left[\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial t}\right)^2\right] dx + 2\int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial \tau}\right)^2 + \left(\frac{\partial V}{\partial \tau}\right)^2\right] dx\, d\tau \leqslant C.$$

This completes the proof of Lemma 3.1. □

Note that from Lemma 3.1, according to the scheme applied in the second section, we get validity of Lemma 2.1 for Problem 2 too.

Lemma 3.2. *For Problem* 2 *the following estimates are true*:

$$c\varphi^{\frac{1}{3}}(t) \leqslant 1 + S(x,t) \leqslant C\varphi^{\frac{1}{3}}(t).$$

Now let us estimate functions $\sigma_1(x,t)$ and $\sigma_2(x,t)$ in the norm of the space $L_2(0,1)$.

Lemma 3.3. *For the solution of Problem* 2 *the following estimates are true*:

$$c\varphi^{\frac{2}{3}}(t) \leqslant \int_0^1 \left(\sigma_1^2(x,t) + \sigma_2^2(x,t)\right) dx \leqslant C\varphi^{\frac{2}{3}}(t).$$

Proof. Taking into account Lemma 3.2 we get

$$\int_0^1 \left(\sigma_1^2+\sigma_2^2\right)dx=\int_0^1 (1+S)^2\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]dx \geqslant c\varphi^{\frac{2}{3}}(t)\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]dx$$

$$\geqslant c\varphi^{\frac{2}{3}}(t)\left[\left(\int_0^1 \frac{\partial U}{\partial x}\,dx\right)^2+\left(\int_0^1 \frac{\partial V}{\partial x}\,dx\right)^2\right]=\left(\psi_1^2+\psi_2^2\right)c\varphi^{\frac{2}{3}}(t),$$

or

$$\int_0^1 \left(\sigma_1^2(x,t)+\sigma_2^2(x,t)\right)dx \geqslant c\varphi^{\frac{2}{3}}(t). \tag{3.4}$$

Using again Lemma 3.2 and definition of σ_1 and σ_2 we have

$$\left\{\int_0^1 \left[\sigma_1^2(x,t)+\sigma_2^2(x,t)\right]dx\right\}^2 \leqslant 2\left[\int_0^1 \sigma_1^2(x,t)\,dx\right]^2+2\left[\int_0^1 \sigma_2^2(x,t)\,dx\right]^2$$

$$\leqslant 2C\varphi^{\frac{2}{3}}(t)\left\{\left[\int_0^1 (1+S)\left(\frac{\partial U}{\partial x}\right)^2 dx\right]^2+\left[\int_0^1 (1+S)\left(\frac{\partial V}{\partial x}\right)^2 dx\right]^2\right\}. \tag{3.5}$$

Let us multiply Eqs. (1.1) scalarly by U and V, respectively. Using the boundary conditions (1.3) we have:

$$\int_0^1 U\frac{\partial U}{\partial t}\,dx+\int_0^1 (1+S)\left(\frac{\partial U}{\partial x}\right)^2 dx=\psi_1\sigma_1(1,t),$$

$$\int_0^1 V\frac{\partial V}{\partial t}\,dx+\int_0^1 (1+S)\left(\frac{\partial V}{\partial x}\right)^2 dx=\psi_2\sigma_2(1,t).$$

Using these equalities, Schwarz's inequality and Lemma 3.1, from (3.5) we get

$$\left\{\int_0^1 \left[\sigma_1^2(x,t)+\sigma_2^2(x,t)\right]dx\right\}^2 \leqslant 2C_1\varphi^{\frac{2}{3}}(t)\left[\left(\psi_1\sigma_1(1,t)-\int_0^1 U\frac{\partial U}{\partial t}\,dx\right)^2+\left(\psi_2\sigma_2(1,t)-\int_0^1 V\frac{\partial V}{\partial t}\,dx\right)^2\right]$$

$$\leqslant 4C_1\varphi^{\frac{2}{3}}(t)\left[\psi_1^2\sigma_1^2(1,t)+\int_0^1 U^2\,dx\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx+\psi_2^2\sigma_2^2(1,t)+\int_0^1 V^2\,dx\int_0^1\left(\frac{\partial V}{\partial t}\right)^2 dx\right]$$

$$\leqslant 4C_1\varphi^{\frac{2}{3}}(t)\left[\left(\psi_1^2+\psi_2^2\right)\left(\sigma_1^2(1,t)+\sigma_2^2(1,t)\right)+C_2\left(\int_0^1 U^2\,dx+\int_0^1 V^2\,dx\right)\right].$$

Now taking into account relations (2.3), (2.5), (3.4), Lemma 3.1 and the maximum principle [28]

$$\left|U(x,t)\right| \leqslant \max_{0\leqslant y\leqslant 1}\left|U_0(y)\right|, \qquad \left|V(x,t)\right| \leqslant \max_{0\leqslant y\leqslant 1}\left|V_0(y)\right|, \quad 0\leqslant x\leqslant 1,\ t\geqslant 0,$$

we get

$$\left\{\int_0^1 \left[\sigma_1^2(x,t)+\sigma_2^2(x,t)\right]dx\right\}^2 \leqslant 4C_1\varphi^{\frac{2}{3}}(t)\left\{\left(\psi_1^2+\psi_2^2\right)\left(2\int_0^1 \sigma_1^2\,dx+\int_0^1\left(\frac{\partial\sigma_1}{\partial x}\right)^2 dx+2\int_0^1 \sigma_2^2\,dx+\int_0^1\left(\frac{\partial\sigma_2}{\partial x}\right)^2 dx\right)\right.$$

$$\left.+C_2\left[\left(\max_{0\leqslant y\leqslant 1}\left|U_0(y)\right|\right)^2+\left(\max_{0\leqslant y\leqslant 1}\left|V_0(y)\right|\right)^2\right]\right\}$$

$$\leqslant 4C_1\varphi^{\frac{2}{3}}(t)\left[\left(\psi_1^2+\psi_2^2\right)\left(2\int_0^1 \sigma_1^2\,dx+\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx+2\int_0^1 \sigma_2^2\,dx+\int_0^1\left(\frac{\partial V}{\partial t}\right)^2 dx\right)+C_3\right]$$

$$\leqslant 4C_1\varphi^{\frac{2}{3}}(t)\left[C_4\int_0^1\left(\sigma_1^2+\sigma_2^2\right)dx+\frac{C_5}{\varphi^{\frac{2}{3}}(t)}\int_0^1\left(\sigma_1^2+\sigma_2^2\right)dx\right].$$

From this, taking into account relation $\varphi(t) \geqslant 1$, we get

$$\int_0^1 \left(\sigma_1^2(x,t) + \sigma_2^2(x,t)\right) dx \leqslant C\varphi^{\frac{2}{3}}(t). \tag{3.6}$$

Finally, using (3.4) and (3.6) the proof of Lemma 3.3 is complete. □

From Lemma 3.3 and relation (2.1) we get the following estimates:

$$c\varphi^{\frac{2}{3}}(t) \leqslant \frac{d\varphi(t)}{dt} \leqslant C\varphi^{\frac{2}{3}}(t).$$

Integrating these inequalities one can easily get

$$\left(1 + \frac{c}{3}t\right)^3 \leqslant \varphi(t) \leqslant \left(1 + \frac{C}{3}t\right)^3,$$

or

$$c(1+t)^3 \leqslant \varphi(t) \leqslant C(1+t)^3.$$

From this, taking into account Lemmas 3.2 and 3.3 we get the following estimates:

$$c(1+t) \leqslant 1 + S(x,t) \leqslant C(1+t), \quad t \geqslant 0, \tag{3.7}$$

$$c(1+t)^2 \leqslant \int_0^1 \left(\sigma_1^2(x,t) + \sigma_2^2(x,t)\right) dx \leqslant C(1+t)^2, \quad t \geqslant 0. \tag{3.8}$$

Lemma 3.4. *The derivatives $\partial U/\partial t$ and $\partial V/\partial t$ satisfy the inequality*

$$\int_0^1 \left[\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial t}\right)^2\right] dx \leqslant C(1+t)^{-2}, \quad t \geqslant 0.$$

Proof. Using the inequality $ab \leqslant a^2/4 + b^2$, equality (3.2) yields

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \int_0^1 (1+S)\left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx \leqslant 2\int_0^1 (1+S)^{-1}\left(\frac{\partial U}{\partial x}\right)^6 dx + 2\int_0^1 (1+S)^{-1}\left(\frac{\partial U}{\partial x}\right)^2\left(\frac{\partial V}{\partial x}\right)^4 dx. \tag{3.9}$$

Now using Lemma 3.1, keeping in mind definitions of σ_1, σ_2, relations (2.5), (3.7), (3.8), we get from (3.9)

$$\begin{aligned}
\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + c(1+t)\int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx &\leqslant C_1(1+t)^{-7}\int_0^1 \left(\sigma_1^6 + \sigma_1^2\sigma_2^4\right) dx \\
&\leqslant C_1(1+t)^{-7}\int_0^1 \sigma_1^2(x,t)\, dx\left\{\left[\max_{0\leqslant x\leqslant 1}\sigma_1^2(x,t)\right]^2 + \left[\max_{0\leqslant x\leqslant 1}\sigma_2^2(x,t)\right]^2\right\} \\
&\leqslant C_2(1+t)^{-5}\Bigg(\left\{\int_0^1 \sigma_1^2\, dx + 2\left[\int_0^1 \sigma_1^2\, dx\right]^{1/2}\left[\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx\right]^{1/2}\right\}^2 \\
&\quad + \left\{\int_0^1 \sigma_2^2\, dx + 2\left[\int_0^1 \sigma_2^2\, dx\right]^{1/2}\left[\int_0^1 \left(\frac{\partial V}{\partial t}\right)^2 dx\right]^{1/2}\right\}^2\Bigg) \\
&\leqslant C_2(1+t)^{-5}\left(C_3(1+t)^4 + C_4(1+t)^2\right) \leqslant C(1+t)^{-1}.
\end{aligned}$$

Similarly,

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial V}{\partial t}\right)^2 dx + c(1+t)\int_0^1 \left(\frac{\partial^2 V}{\partial t \partial x}\right)^2 dx \leqslant C(1+t)^{-1}.$$

Thanks to Poincare's inequality we arrive at

$$\frac{d}{dt}\int_0^1\left[\left(\frac{\partial U}{\partial t}\right)^2+\left(\frac{\partial V}{\partial t}\right)^2\right]dx+c(1+t)\int_0^1\left[\left(\frac{\partial U}{\partial t}\right)^2+\left(\frac{\partial V}{\partial t}\right)^2\right]dx\leqslant C(1+t)^{-1}. \tag{3.10}$$

From (3.10), using Grönwall's inequality we get

$$\begin{aligned}&\int_0^1\left[\left(\frac{\partial U}{\partial t}\right)^2+\left(\frac{\partial V}{\partial t}\right)^2\right]dx\\ &\quad\leqslant \exp\left(-c\int_0^t(1+\tau)\,d\tau\right)\left\{\int_0^1\left[\left(\frac{\partial U}{\partial t}\right)^2+\left(\frac{\partial V}{\partial t}\right)^2\right]dx\bigg|_{t=0}+C\int_0^t\exp\left(c\int_0^\tau(1+\xi)\,d\xi\right)(1+\tau)^{-1}\,d\tau\right\}\\ &\quad=C_1\exp\left(-\frac{c(1+t)^2}{2}\right)\left[C_2+C_3\int_0^t\exp\left(\frac{c(1+\tau)^2}{2}\right)(1+\tau)^{-1}\,d\tau\right].\end{aligned} \tag{3.11}$$

Applying L'Hopital's rule we obtain

$$\lim_{t\to\infty}\frac{\int_0^t\exp(\frac{c(1+\tau)^2}{2})(1+\tau)^{-1}\,d\tau}{\exp(\frac{c(1+t)^2}{2})(1+t)^{-2}}=\lim_{t\to\infty}\frac{\exp(\frac{c(1+t)^2}{2})(1+t)^{-1}}{\exp(\frac{c(1+t)^2}{2})(1+t)^{-1}[c-2(1+t)^{-2}]}=\lim_{t\to\infty}\frac{1}{c-2(1+t)^{-2}}=C. \tag{3.12}$$

So, the validity of Lemma 3.4 follows from (3.11) and (3.12). □

Our next step is to estimate $\partial S/\partial x$ in $L_1(0,1)$.

Lemma 3.5. *For Problem* 2 *the following estimate is true*:

$$\int_0^1\left|\frac{\partial S}{\partial x}\right|dx\leqslant C(1+t)^{-1},\quad t\geqslant 0.$$

Proof. Let us differentiate (2.4) with respect to x:

$$\frac{\partial}{\partial t}\left[(1+S)^2\frac{\partial S}{\partial x}\right]=2\sigma_1\frac{\partial \sigma_1}{\partial x}+2\sigma_2\frac{\partial \sigma_2}{\partial x}. \tag{3.13}$$

Using Schwarz's inequality, Lemma 3.4 and estimate (3.8) we have

$$\begin{aligned}&\int_0^1\left|\sigma_1\frac{\partial U}{\partial t}\right|dx\leqslant C(1+t)^1(1+t)^{-1}=C,\\ &\int_0^1\left|\sigma_2\frac{\partial V}{\partial t}\right|dx\leqslant C(1+t)^1(1+t)^{-1}=C.\end{aligned} \tag{3.14}$$

From relations (2.3), (3.7), (3.13), (3.14), we receive

$$\begin{aligned}&(1+S)^2\frac{\partial S}{\partial x}=\int_0^t\left(2\sigma_1\frac{\partial U}{\partial \tau}+2\sigma_2\frac{\partial V}{\partial \tau}\right)d\tau,\\ &\int_0^1\left|\frac{\partial S}{\partial x}\right|dx\leqslant C_1(1+t)^{-2}\int_0^t C_2\,d\tau\leqslant C(1+t)^{-1}.\end{aligned} \tag{3.15}$$

So, Lemma 3.5 has been proven. □

Using relations (2.5), (3.8), (3.14), we obtain

$$\sigma_1^2(x,t)\leqslant\int_0^1\sigma_1^2(y,t)\,dy+2\int_0^1\left|\sigma_1(y,t)\frac{\partial U(y,t)}{\partial t}\right|dy\leqslant C_1(1+t)^2+C_2\leqslant C(1+t)^2,$$

or

$$\left|\sigma_1(x,t)\right| \leqslant C(1+t).$$

Taking into account Lemmas 3.4, 3.5, relations (2.3), (3.7), the last estimate and the identity

$$\frac{\partial U}{\partial x} = \sigma_1 (1+S)^{-1},$$

we derive

$$\begin{aligned}
\int_0^1 \left|\frac{\partial^2 U(x,t)}{\partial x^2}\right| dx &\leqslant \int_0^1 \left|\frac{\partial \sigma_1}{\partial x}(1+S)^{-1}\right| dx + \int_0^1 \left|\sigma_1 (1+S)^{-2}\frac{\partial S}{\partial x}\right| dx \\
&\leqslant \left[\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx\right]^{1/2} \left[\int_0^1 (1+S)^{-2}\, dx\right]^{1/2} + \int_0^1 \left|\sigma_1 (1+S)^{-2}\frac{\partial S}{\partial x}\right| dx \\
&\leqslant C_1(1+t)^{-1}(1+t)^{-1} + C_2(1+t)(1+t)^{-2}\int_0^1 \left|\frac{\partial S}{\partial x}\right| dx \leqslant C(1+t)^{-2}.
\end{aligned}$$

Hence, we have

$$\int_0^1 \left|\frac{\partial^2 U(x,t)}{\partial x^2}\right| dx \leqslant C(1+t)^{-2}, \quad t \geqslant 0.$$

From this, taking into account the relation

$$\frac{\partial U(x,t)}{\partial x} = \int_0^1 \frac{\partial U(y,t)}{\partial y}\, dy + \int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}\, d\xi\, dy$$

and the boundary conditions (1.3), it follows that

$$\left|\frac{\partial U(x,t)}{\partial x} - \psi_1\right| = \left|\int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}\, d\xi\, dy\right| \leqslant \int_0^1 \left|\frac{\partial^2 U(y,t)}{\partial y^2}\right| dy \leqslant C(1+t)^{-2}, \quad t \geqslant 0.$$

The same estimate is valid for $\partial V/\partial x$:

$$\left|\frac{\partial V(x,t)}{\partial x} - \psi_2\right| \leqslant C(1+t)^{-2}, \quad t \geqslant 0.$$

Thus, Theorem 1.2 has been proven.

Remarks.

1. The existence of a globally defined solutions of Problems 1 and 2 can now be obtained by a routine procedure, proving first the existence of the local solutions on a maximal time interval and then using the derived a-priori estimates to show that these solutions cannot escape in a finite time [14–16,21,22].
2. Let us mention that in Section 3 we used the scheme of [29] in which the adiabatic shearing of incompressible fluids with temperature-dependent viscosity is studied. Note also that boundary conditions (1.3) are used here taking into account the physical problem considered in [30].

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