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Basin attractors for various methods

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ABSTRACT

There are many methods for the solution of a nonlinear algebraic equation. The methods are classified by the order, informational efficiency and efficiency index. Here we consider other criteria, namely the basin of attraction of the method and its dependence on the order. We discuss several methods of various orders and present the basin of attraction for several examples. It can be seen that not all higher order methods were created equal. Newton's, Halley's, Murakami's and Neta–Johnson's methods are consistently better than the others. In two of the examples Neta's 16th order scheme was also as good.

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1. Introduction

There is a vast literature for the numerical solution of nonlinear equations. In general methods are classified as bracketing, fixed point or hybrid. In the first class one starts with an initial interval in which the function changes sign and at each iteration step the interval shrinks. In the fixed point methods one starts with an initial point and creates a sequence that should converge to the desired solution. The methods are also classified by their order of convergence, p, and the number of function-(and derivative-) evaluation per step, denoted by d. There are two efficiency measures defined as $I = p/d$ (informational efficiency) and $E = p^{1/d}$ (efficiency index). Methods for the approximation of multiple roots are also available in the literature. Some of these methods require the knowledge of the multiplicity in advance. We will not consider such methods here.

There are a number of ways to compare various techniques proposed for solving nonlinear equations. Frequently, authors will pick a collection of sample equations that include polynomials of various orders and/or transcendental components. Then a collection of algorithms is chosen for comparison which may include different orders of convergence. Then a starting point for each of the sample equations is chosen and the algorithms are allowed to iterate until a given level of convergence is achieved or until a maximum number of iterations has been completed without convergence. Then comparisons of the various algorithms are based on comparisons of the number of iterations required for convergence, number of function evaluations, and/or amount of CPU time. If a particular algorithm does not converge or if it converges to a different solution and it is not an algorithm developed by the author, then that particular algorithm is thought to be inferior to the others.

The primary flaw in this type of comparison is that the starting point, although it may have been chosen at random, represents only one of an infinite number of other choices. In order to improve on this, one could choose a number of other randomly chosen starting points. However, this is only an incrementally better method of comparison. With this in mind we began to discuss the following. How can a better methodology be developed for the comparison of algorithms for solving nonlinear equations? The basin of attraction is a method to visually comprehend how an algorithm behaves as a function of the various starting points. Natural questions then are:

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(1) How does the basin of attraction differ for algorithms with the same order of convergence.
(2) How does the basin of attraction differ for algorithms with different order of convergence.
(3) Can the differences be used to compare various algorithms?

In this paper we will discuss some qualitative issues using the basin of attraction as a criterion for comparison. To this end, we shall recall some preliminaries, see for example Milnor [9] and Amat et al. [1]. Let $R:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a rational map on the Riemann sphere.

Definition. For $z\in\widehat{\mathbb{C}}$ we define its orbit as the set

$$orb(z)=\{z,R(z),R^2(z),\dots,R^n(z),\dots\}.$$

Definition. A point z_0 is a fixed point of R if $R(z_0)=z_0$.

Definition. A periodic point z_0 of period m is such that $R^m(z_0)=z_0$ where m is the smallest such integer.

Remark 1. If z_0 is periodic of period m then it is a fixed point for R^m.

We classify the fixed points of a map based on the magnitude of the derivative.

Definition. A point z_0 is called attracting if $|R'(z_0)|<1$, repelling if $|R'(z_0)|>1$, and neutral if $|R'(z_0)|=1$. If the derivative is also zero then the point is called super-attracting.

Definition. The Julia set of a nonlinear map $R(z)$, denoted $J(R)$, is the closure of the set of its repelling periodic points. The complement of $J(R)$ is the Fatou set $\mathbb{F}(R)$.

By its definition, $J(R)$ is a closed subset of $\widehat{\mathbb{C}}$. A point z_0 belongs to the Julia set if and only if dynamics in a neighborhood of z_0 displays sensitive dependence on the initial conditions, so that nearby initial conditions lead to wildly different behavior after a number of iterations. As a simple example, consider the map $R(z)=z^2$ on $\widehat{\mathbb{C}}$. The entire open disk is contained in $\mathbb{F}(R)$, since successive iterates on any compact subset converge uniformly to zero. Similarly the exterior is contained in $\mathbb{F}(R)$. On the other hand if z_0 is on the unit circle then in any neighborhood of z_0 any limit of the iterates would necessarily have a jump discontinuity as we cross the unit circle. Therefore $J(R)$ is the unit circle. Such smooth Julia sets are exceptional.

Invariance Lemma [9]: The Julia set $J(R)$ of a holomorphic map $R:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is fully invariant under R. That is, z belongs to J if and only if $R(z)$ belongs to J.

Iteration Lemma: For any $k>0$, the Julia set $J(R^k)$ of the k-fold iterate coincides with $J(R)$.

Definition. If O is an attracting periodic orbit of period m, we define the basin of attraction to be the open set $A\in\widehat{\mathbb{C}}$ consisting of all points $z\in\widehat{\mathbb{C}}$ for which the successive iterates $R^m(z)$, $R^{2m}(z),\dots$ converge towards some point of O.

Lemma 1. *Every attracting periodic orbit is contained in the Fatou set of R. In fact the entire basin of attraction A for an attracting periodic orbit is contained in the Fatou set. However, every repelling periodic orbit is contained in the Julia set.*

The idea of basin of attraction of some root-finding methods was introduced by Stewart [17]. He compared Newton's method to the third order methods given by Halley [4], Popovski [5] and Laguerre [6]. In an ideal case, if a function has n distinct zeros, then the plane is divided to n basins. For example, if we have the polynomial z^3-1, then the roots are $z=1$ and $z=\frac{-1\pm\sqrt{3}i}{2}$, see Fig. 1. Ideally the basins boundaries are straight lines. Actually, depending on the numerical method, we find the basin boundaries are much more complex, see examples later.

Our study considers ten methods of various orders, two of which were considered by Stewart [17]. We include optimal methods of order $p=2,4,8,16$. Note that a method of order $p=2^n$ is optimal (see [8]) in the sense that it requires $n+1$ function-(and derivative-) evaluations per cycle. The methods we consider here with their order of convergence are:

(1) Newton's optimal method ($p=2$).
(2) Halley's method ($p=3$).
(3) King's family of optimal methods ($p=4$).
(4) Kung–Traub's optimal method ($p=4$).
(5) Murakami's method ($p=5$).
(6) Neta's family of methods ($p=6$).
(7) Chun–Neta's method ($p=6$).
(8) Neta–Johnson's method ($p=8$).
(9) Neta–Petkovic's optimal method ($p=8$).
(10) Neta's family of optimal methods ($p=16$).

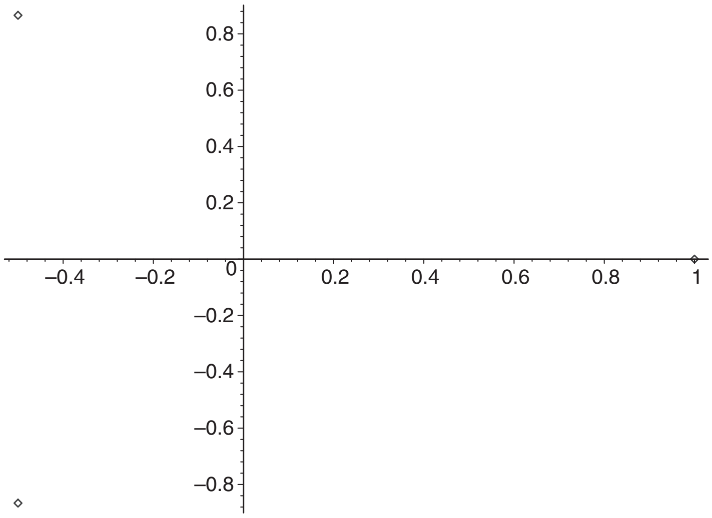

Fig. 1. Location of the roots of $z^3 - 1$.

The reason why we introduced more than one optimal fourth order method and more than one sixth order method will be clarified later.

Newton's optimal method (see e.g. Conte and deBoor [3]) is of second order for simple roots and given by

$$x_{n+1} = x_n - u_n, \tag{1}$$

where

$$u_n = \frac{f_n}{f'_n} \tag{2}$$

and $f_n = f(x_n)$ and similarly for the derivative. Halley's method [4] is of third order and given by

$$x_{n+1} = x_n - \frac{u_n}{1 - \frac{f''_n}{2f'_n} u_n}. \tag{3}$$

King's fourth order optimal family of methods [7] is given by

$$\begin{aligned} y_n &= x_n - u_n, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'_n} \frac{f_n + \beta f(y_n)}{f_n + (\beta - 2) f(y_n)}. \end{aligned} \tag{4}$$

For the case $\beta = 0$ the method is actually due to Ostrowski [16].

Another optimal fourth order method is due to Kung and Traub [8] given by

$$\begin{aligned} y_n &= x_n - u_n, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'_n} \frac{1}{[1 - f(y_n)/f_n]^2}. \end{aligned} \tag{5}$$

Murakami's fifth order method [10] is given by

$$x_{n+1} = x_n - a_1 u_n - a_2 w_2(x_n) - a_3 w_3(x_n) - \psi(x_n), \tag{6}$$

where u_n is given by (2) and

$$\begin{aligned} w_2(x_n) &= \frac{f_n}{f'(x_n - u_n)}, \\ w_3(x_n) &= \frac{f_n}{f'(x_n + \beta u_n + \gamma w_2(x_n))}, \\ \psi(x_n) &= \frac{f_n}{b_1 f'_n + b_2 f'(x_n - u_n)}. \end{aligned} \tag{7}$$

To get fifth order, Murakami suggested several possibilities and we picked the following

$$\begin{aligned} &\gamma = 0, \quad a_1 = .3, \quad a_2 = -.5, \quad a_3 = \frac{2}{3}, \\ &b_1 = -\frac{15}{32}, \quad b_2 = \frac{75}{32}, \quad \beta = -\frac{1}{2}. \end{aligned} \tag{8}$$

Neta's sixth order family of methods [11] is given by

$$\begin{aligned} y_n &= x_n - u_n, \\ z_n &= y_n - \frac{f(y_n)}{f'_n} \frac{f_n + \beta f(y_n)}{f_n + (\beta - 2) f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'_n} \frac{f_n - f(y_n)}{f_n - 3f(y_n)}. \end{aligned} \tag{9}$$

Note that the first two substeps are King's method. Several choices for the parameter β were discussed. Chun and Neta [2] show that $\beta = -\frac{1}{2}$ is best.

Another sixth order method due to Chun and Neta [2] is based on Kung and Traub scheme [8],

$$\begin{aligned} y_n &= x_n - u_n, \\ z_n &= y_n - \frac{f(y_n)}{f'_n} \frac{1}{[1 - f(y_n)/f_n]^2}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'_n} \frac{1}{[1 - f(y_n)/f_n - f(z_n)/f_n]^2}. \end{aligned} \tag{10}$$

Neta and Johnson [12] have developed an eighth order method based on Jarratt's method [13]

$$\begin{aligned} y_n &= x_n - u_n, \\ z_n &= x_n - \frac{f_n}{\frac{1}{6} f'_n + \frac{1}{6} f'(y_n) + \frac{2}{3} f'(\eta_n)}, \\ \eta_n &= x_n - \frac{1}{8} u_n - \frac{3}{8} \frac{f_n}{f'(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'_n} \frac{f'_n + f'(y_n) + a_2 f'(\eta_n)}{(-1 - a_2) f'_n + (3 + a_2) f'(y_n) + a_2 f'(\eta_n)}. \end{aligned} \tag{11}$$

In our experiments we have used $a_2 = -1$. This is not an optimal method since it requires 2 function- and 3 derivative-evaluation per cycle.

Another eighth order method is the optimal scheme due to Neta and Petković [14]. It is based on Kung and Traub's optimal fourth order method [8] and inverse interpolation.

$$\begin{aligned} y_n &= x_n - u_n, \\ z_n &= x_n - \frac{f(y_n)}{f'_n} \frac{1}{[1 - f(y_n)/f_n]^2}, \\ x_{n+1} &= x_n - \frac{f_n}{f'_n} + c_n f_n^2 - d_n f_n^3, \end{aligned} \tag{12}$$

where

$$\begin{aligned} d_n &= \frac{1}{[f(y_n) - f_n][f(y_n) - f(z_n)]} \left[\frac{y_n - x_n}{f(y_n) - f_n} - \frac{1}{f'_n} \right] - \frac{1}{[f(y_n) - f(z_n)][f(z_n) - f_n]} \left[\frac{z_n - x_n}{f(z_n) - f_n} - \frac{1}{f'_n} \right], \\ c_n &= \frac{1}{f(y_n) - f_n} \left[\frac{y_n - x_n}{f(y_n) - f_n} - \frac{1}{f'_n} \right] - d_n [f(y_n) - f_n]. \end{aligned} \tag{13}$$

Neta's 16th order family of optimal methods [15] is given by

$$\begin{aligned} y_n &= x_n - u_n, \\ z_n &= y_n - \frac{f(y_n)}{f'_n} \frac{f_n + \beta f(y_n)}{f_n + (\beta - 2) f(y_n)}, \\ t_n &= x_n - \frac{f_n}{f'_n} + c_n f_n^2 - d_n f_n^3, \\ x_{n+1} &= x_n - \frac{f_n}{f'_n} + \rho_n f_n^2 - \gamma_n f_n^3 + q_n f_n^4, \end{aligned} \tag{14}$$

where c_n and d_n are given by (13) and

$$q_n = \frac{\frac{\phi(t_n)-\phi(z_n)}{F(t_n)-F(z_n)} - \frac{\phi(y_n)-\phi(z_n)}{F(y_n)-F(z_n)}}{F(t_n)-F(y_n)},$$
$$\gamma_n = \frac{\phi(t_n)-\phi(z_n)}{F(t_n)-F(z_n)} - q_n(F(t_n)+F(z_n)), \tag{15}$$
$$\rho_n = \phi(t_n) - \gamma_n F(t_n) - q_n F^2(t_n)$$

and for $\delta_n = y_n, z_n, t_n$

$$F(\delta_n) = f(\delta_n) - f_n,$$
$$\phi(\delta_n) = \frac{(\delta_n - x_n)}{F^2(\delta_n)} - \frac{1}{f_n^4 F(\delta_n)}. \tag{16}$$

In our experiments we have used $\beta = 2$.

2. Numerical experiments

We have used the above methods for 6 different polynomials. Some have real and some have complex coefficients. One example have only real roots and the rest have a combination of real and complex ones. All the roots are simple. In the first case we have taken the cubic polynomial

$$x^3 + 4x^2 - 10. \tag{17}$$

Clearly, one root is real (1.365230013) and the other two are complex conjugate.

Note that the basin of attraction of each root is larger for Halley's method than Newton's, see Fig. 2.

We have shown the results for King's method using the parameter $\beta = -\frac{1}{2}$. The results are not much better for several other values of β we tried. The basins of attraction for the optimal Kung–Traub method is better than any of the King's method (notice the second quadrant of Fig. 3).

Murakami's fifth order method gives basins of attraction similar to Newton's, see Fig. 4. On the other hand, Neta's sixth order method which is based on King's method and uses $\beta = -1/2$ shows some chaotic behavior in the second quadrant (Fig. 4). One sees too many points there that converge to the root in the third quadrant. This is similar to the results for King's method.

Neta–Johnson's eighth order method has basins of attraction similar to Newton's method, see Fig. 5. On the other hand, the optimal eighth order method (12) is much more chaotic (see Fig. 6). The 16th order optimal method (see Fig. 6) has a smaller basin of attraction for the real root, but it does not show the chaotic behavior of King's, Neta's sixth order and Neta–Petkovic eighth order. In general, one cannot say that increasing the order of the method will adversely affect the basins of attraction very much.

In our next example, we took a quintic polynomial with real simple roots. It is clear that the best methods are Newton's (Fig. 7), Halley's (Fig. 7), Murakami's (Fig. 9) and Neta–Johnson's (Fig. 10) schemes. See Figs. 7–11.

$$x^5 - 5x^3 + 4x. \tag{18}$$

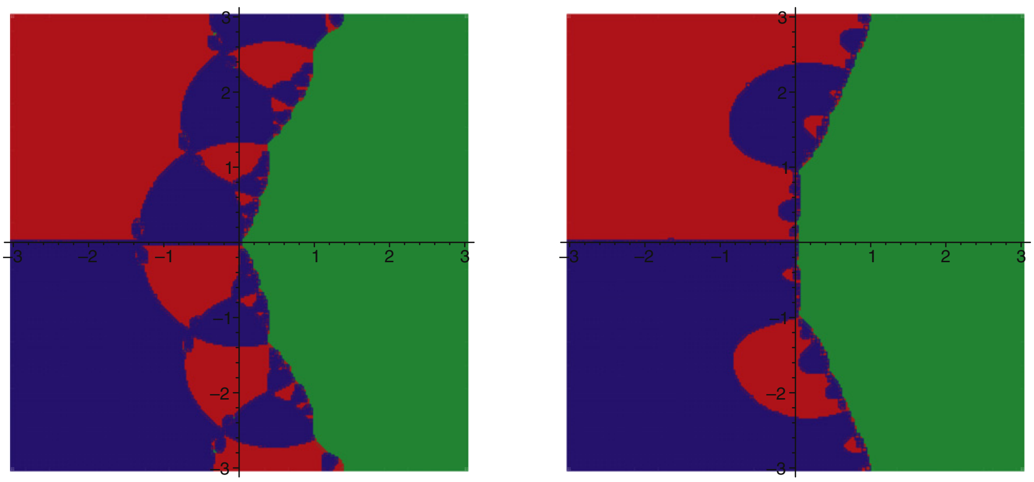

Fig. 2. Newton's (left) and Halley's method (right) for the cubic polynomial whose roots are: 1.365230013, −2.682615007+.3582593602*i*, −2.682615007−.3582593602*i*.

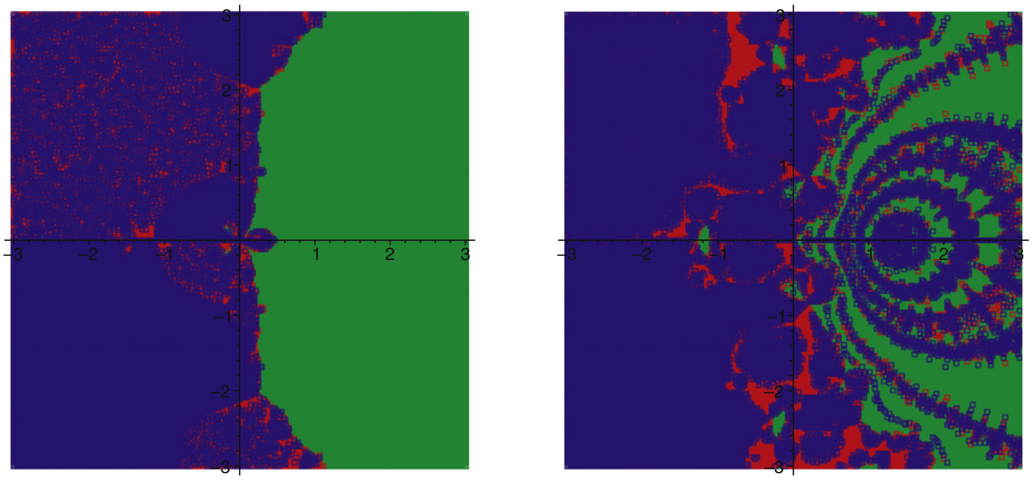

Fig. 3. King's fourth order method with $\beta = -1/2$ (left) and Kung and Traub's fourth order method (right) for the cubic polynomial whose roots are: 1.365230013, −2.682615007+.3582593602*i*, −2.682615007−.3582593602*i*.

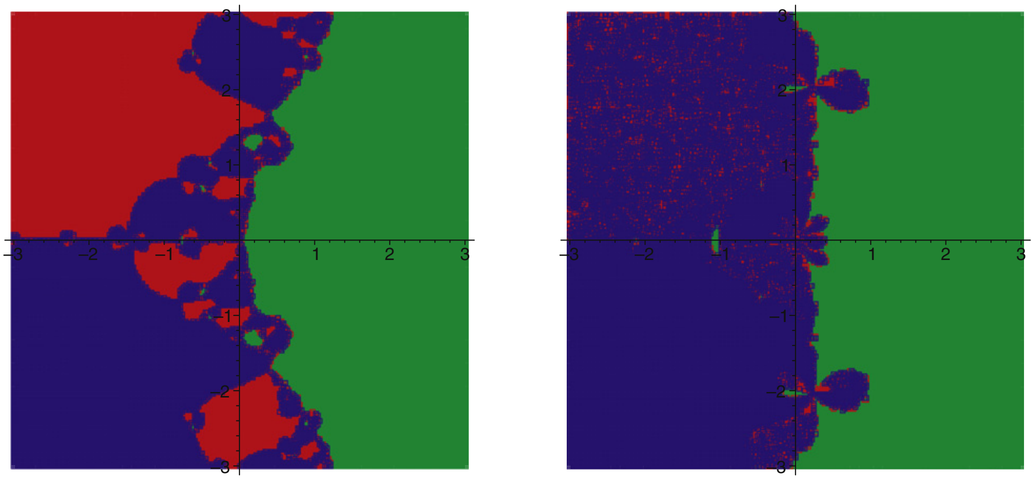

Fig. 4. Murakami's fifth order method (left) and Neta's sixth order method with $\beta = -\frac{1}{2}$ (right) for the cubic polynomial whose roots are: 1.365230013, −2.682615007+.3582593602*i*, −2.682615007−.3582593602*i*.

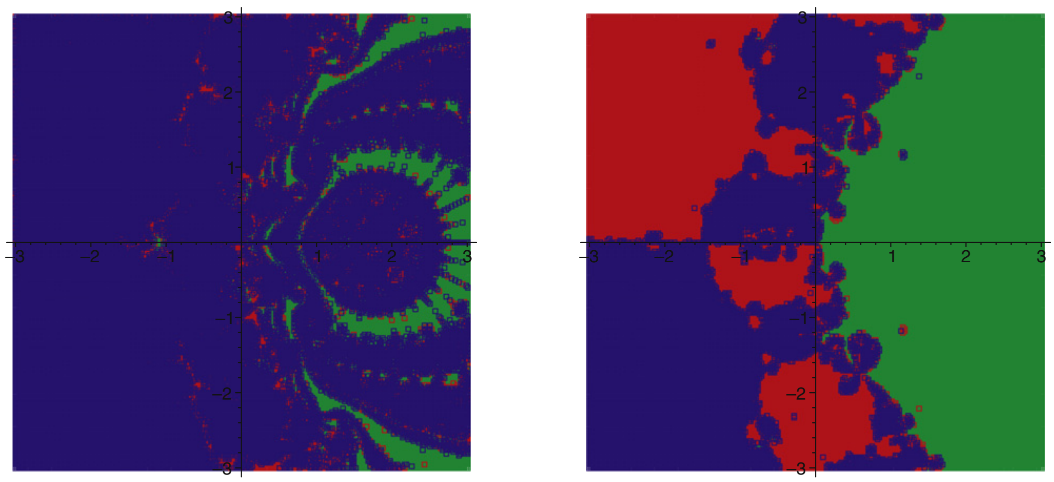

Fig. 5. Neta's sixth order method based on Kung–Traub scheme (left) and Neta and Johnson's eighth order (right) for the cubic polynomial whose roots are: 1.365230013, −2.682615007+.3582593602*i*, −2.682615007−.3582593602*i*.

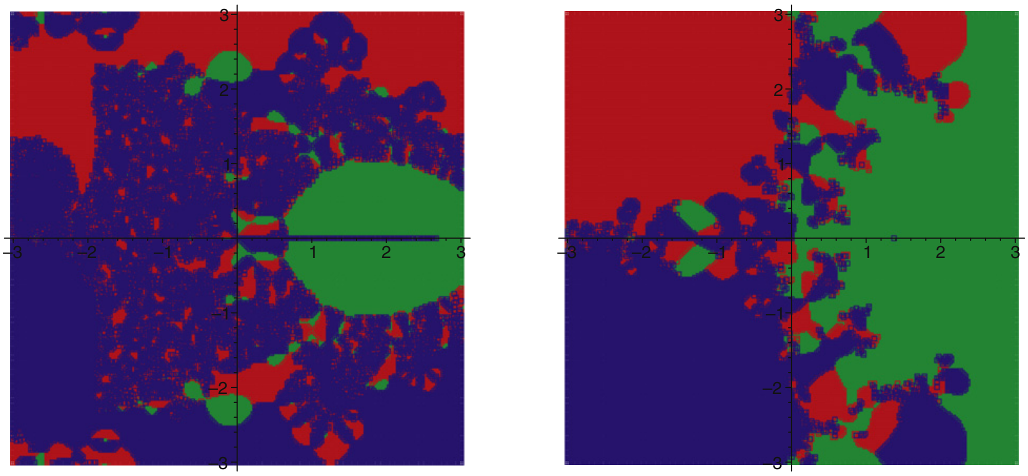

Fig. 6. Neta and Petkovic's optimal eighth order method (left) and Neta's 16th order method with $\beta = 2$ (right) for the cubic polynomial whose roots are: 1.365230013, −2.682615007+.3582593602*i*, −2.682615007−.3582593602*i*.

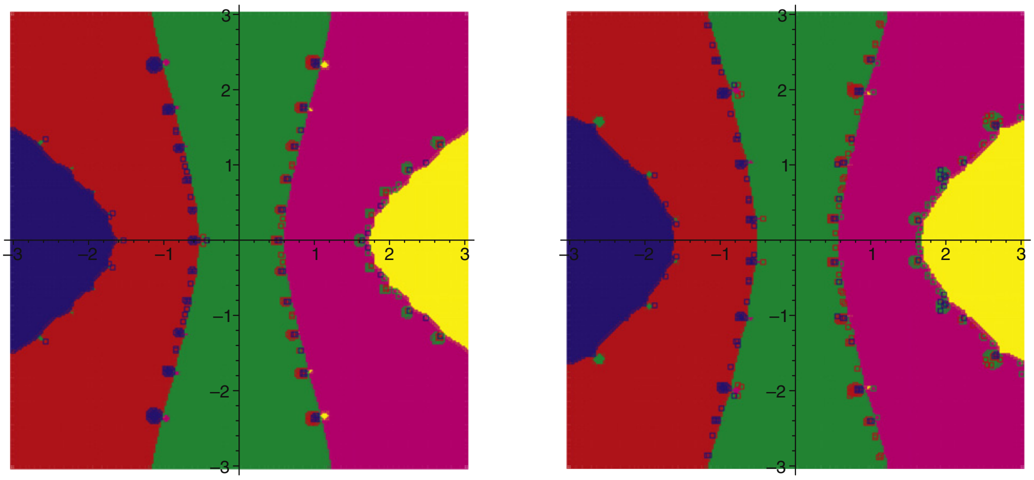

Fig. 7. Newton's (left) and Halley's method (right) for the quintic polynomial whose roots are: −2, −1, 0, 1, 2.

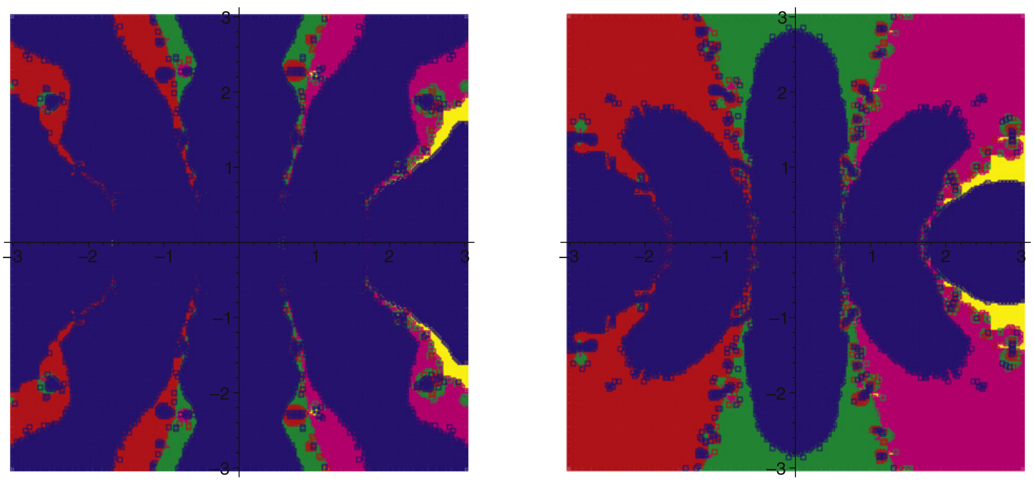

Fig. 8. King's fourth order method with $\beta = -1/2$ (left) and Kung and Traub's fourth order method (right) for the quintic polynomial whose roots are: −2, −1, 0, 1, 2.

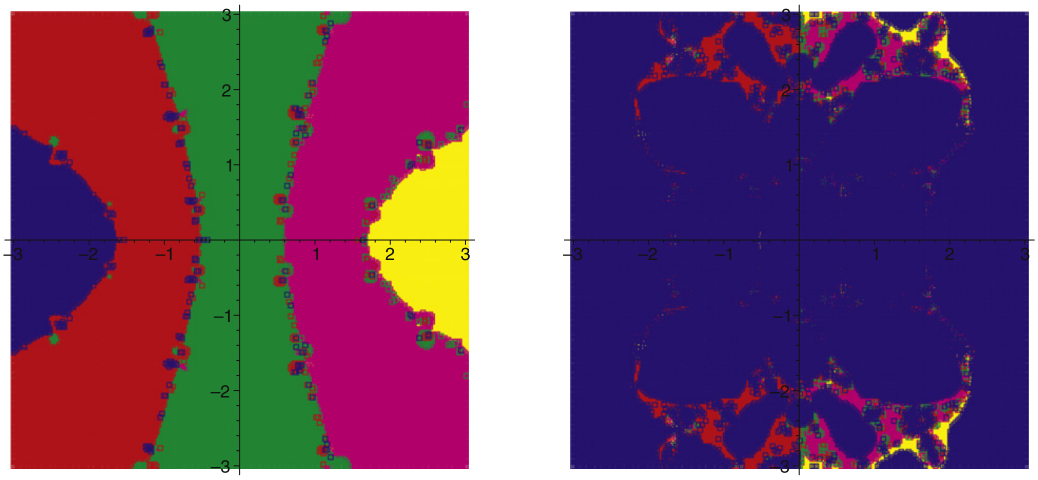

Fig. 9. Murakami's fifth order method (left) and Neta's sixth order method with $\beta = -\frac{1}{2}$ (right) for the quintic polynomial whose roots are: −2, −1, 0, 1, 2.

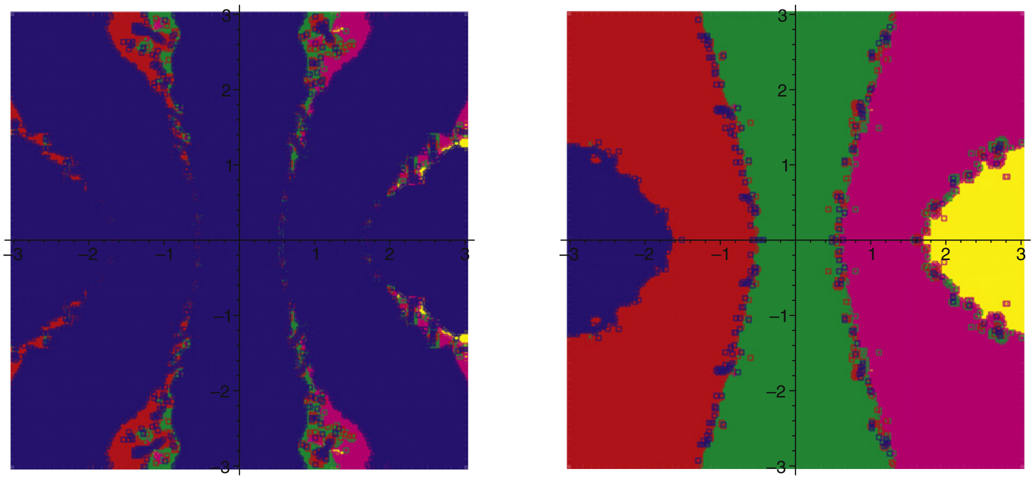

Fig. 10. Neta's sixth order method based on Kung–Traub scheme (left) and Neta and Johnson's eighth order (right) for the quintic polynomial whose roots are: −2, −1, 0, 1, 2.

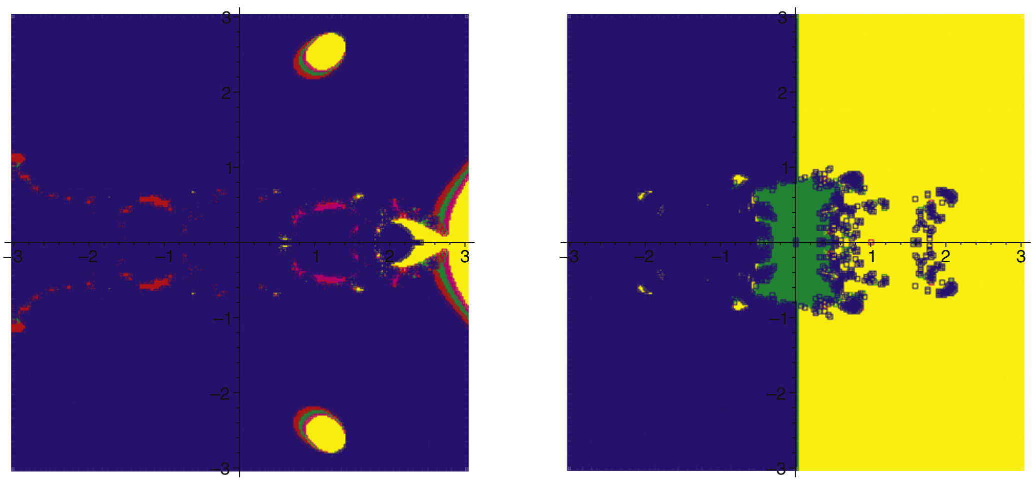

Fig. 11. Neta and Petkovic's optimal eighth order method (left) and Neta's 16th order method with $\beta = 2$ (right) for the quintic polynomial whose roots are: −2, −1, 0, 1, 2.

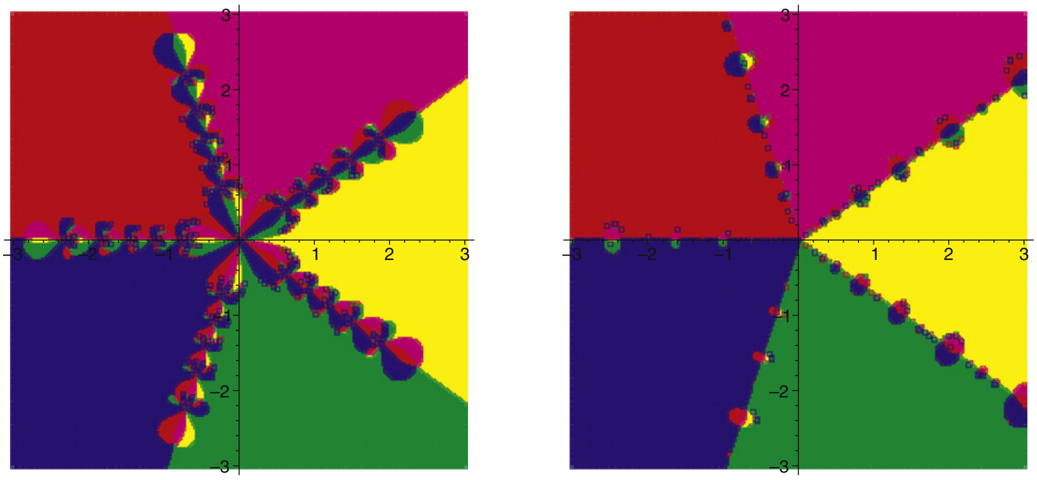

Fig. 12. Newton's (left) and Halley's method (right) for the five roots of unity.

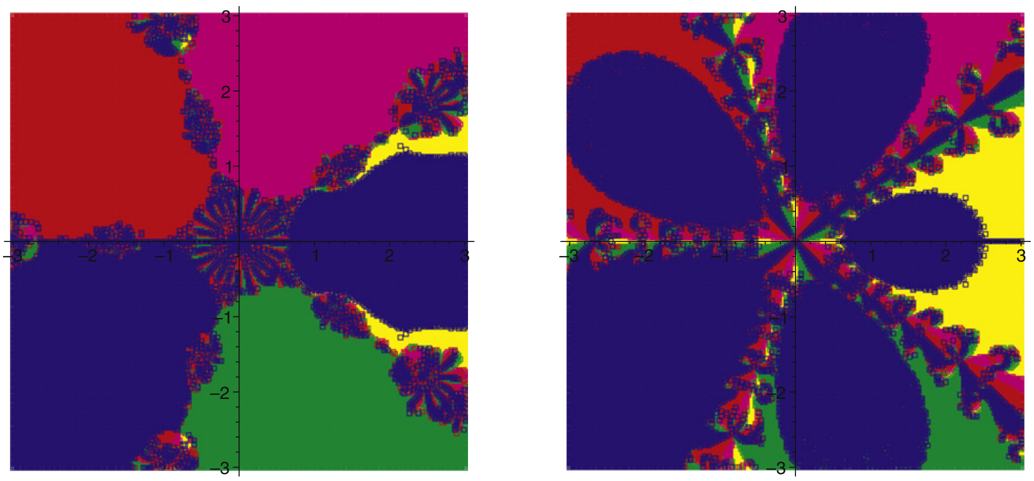

Fig. 13. King's fourth order method with $\beta = -1/2$ (left) and Kung and Traub's fourth order method (right) for the five roots of unity.

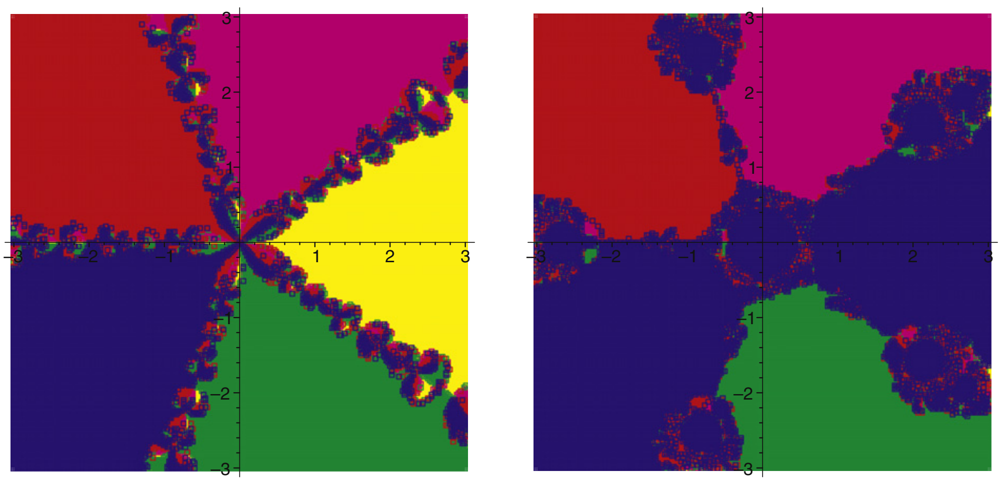

Fig. 14. Murakami's fifth order method (left) and Neta's sixth order method with $\beta = -\frac{1}{2}$ (right) for the five roots of unity.

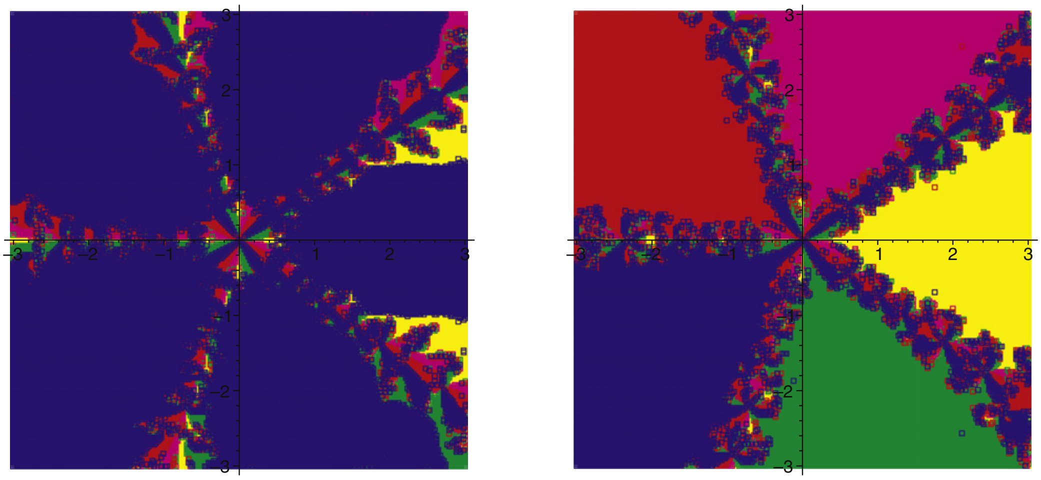

Fig. 15. Neta's sixth order method based on Kung–Traub scheme (left) and Neta and Johnson's eighth order (right) for the five roots of unity.

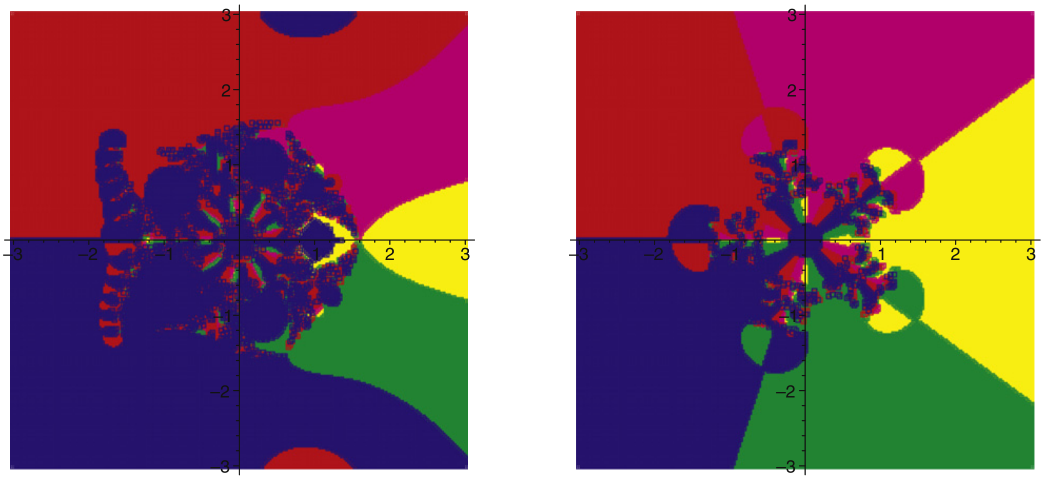

Fig. 16. Neta and Petkovic's optimal eighth order method (left) and Neta's 16th order method with $\beta = 2$ (right) for the five roots of unity.

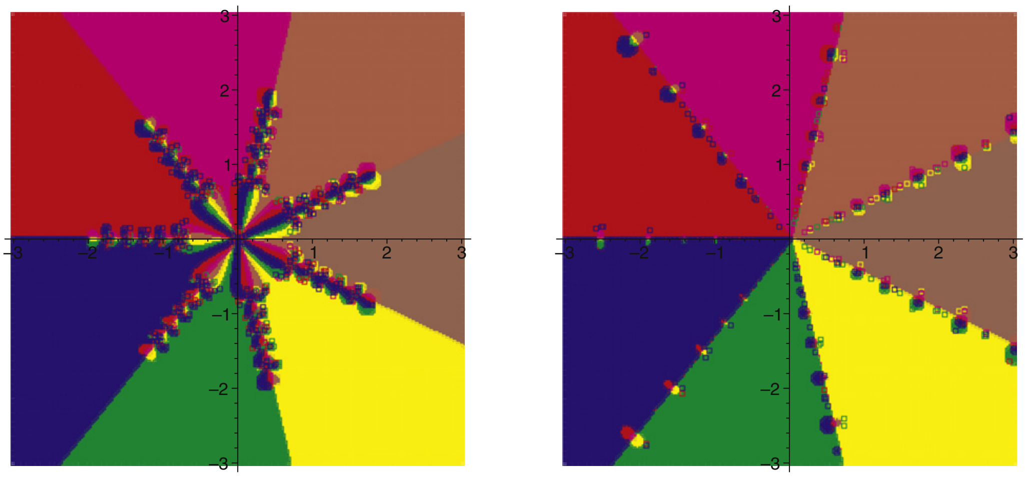

Fig. 17. Newton's (left) and Halley's method (right) for the seven roots of unity.

In the next two examples we have taken a polynomial yielding the roots of unity. The first is a quintic and the second is a polynomial of degree 7.

$$x^5 - 1. \tag{19}$$

As can be seen in Figs. 12–16, the optimal eighth order and the sixth order methods are not doing very well. The best methods are, as before, Newton's (Fig. 12), Halley's (Fig. 12), Murakami's (Fig. 14) and Neta–Johnson's (Fig. 15) schems. The 16th order Neta's method is also competitive.

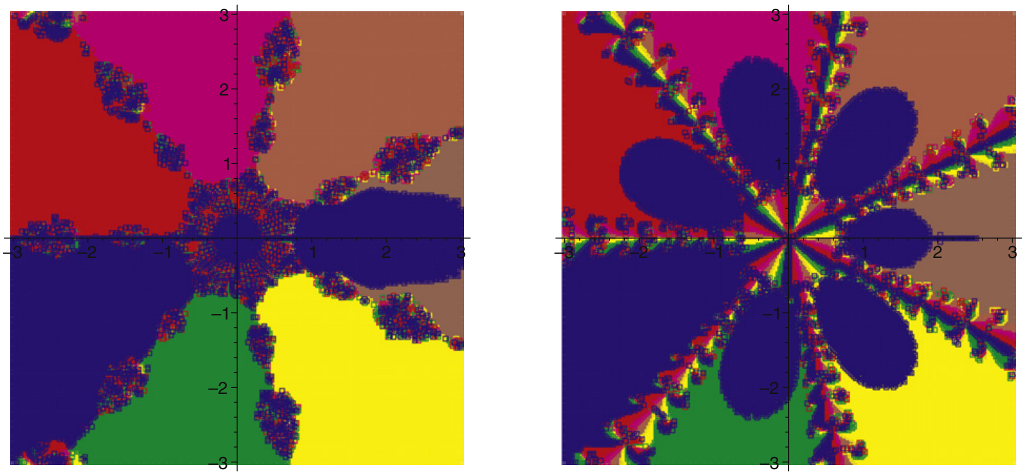

Fig. 18. King's fourth order method with $\beta = -1/2$ (left) and Kung and Traub's fourth order method (right) for the seven roots of unity.

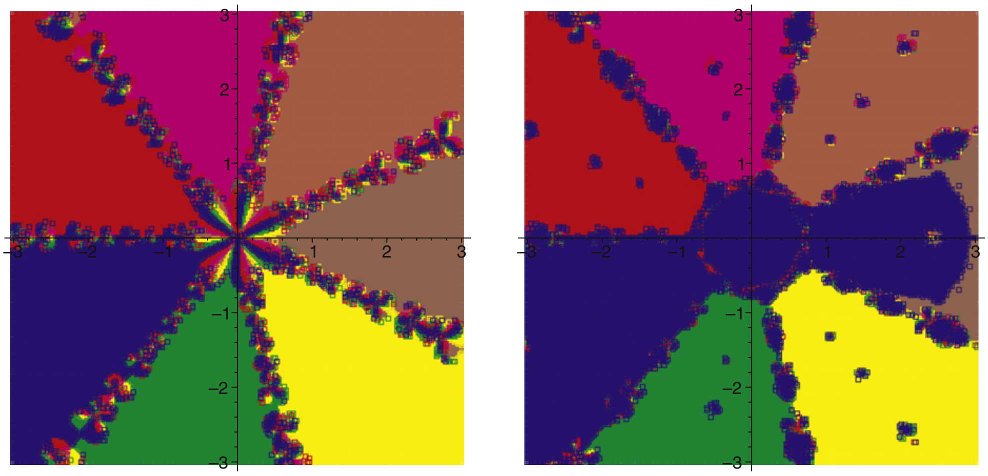

Fig. 19. Murakami's fifth order method (left) and Neta's sixth order method with $\beta = -\frac{1}{2}$ (right) for the seven roots of unity.

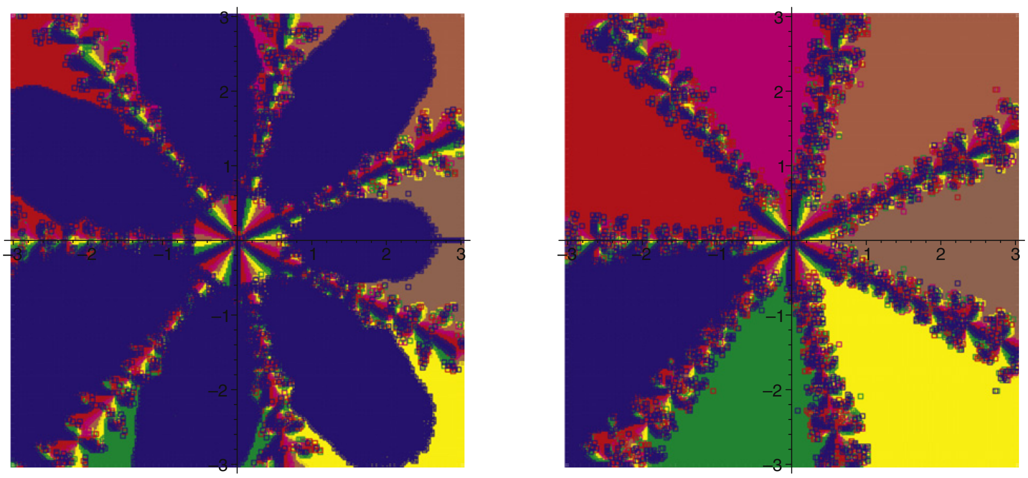

Fig. 20. Neta's sixth order method based on Kung–Traub scheme (left) and Neta and Johnson's eighth order (right) for the seven roots of unity.

$$x^7 - 1. \quad (20)$$

As can be seen in Figs. 17–21, the best methods are, as before, Newton's (Fig. 17), Halley's (Fig. 17), Murakami's (Fig. 19) and Neta–Johnson's (Fig. 20) schems. The 16th order Neta's (Fig. 21) method is also competitive.

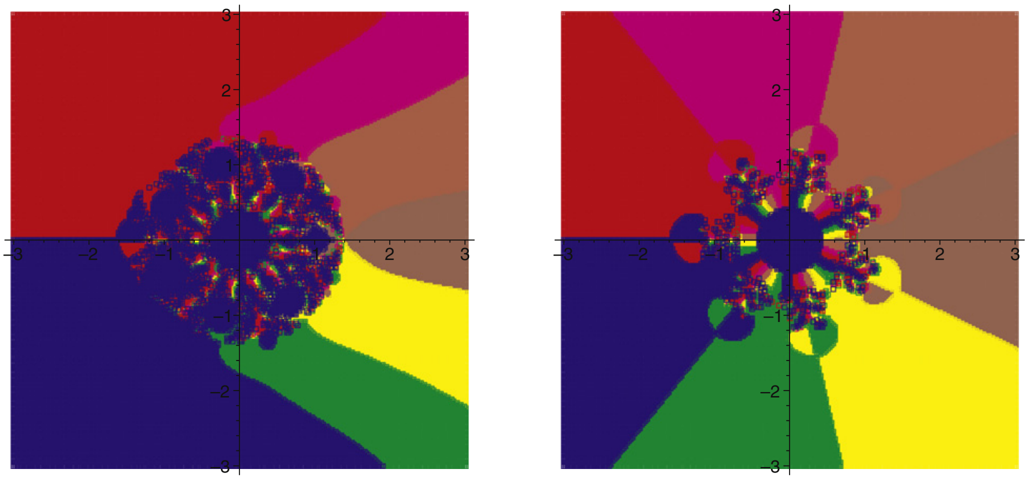

Fig. 21. Neta and Petkovic's optimal eighth order method (left) and Neta's 16th order method with $\beta = 2$ (right) for the seven roots of unity.

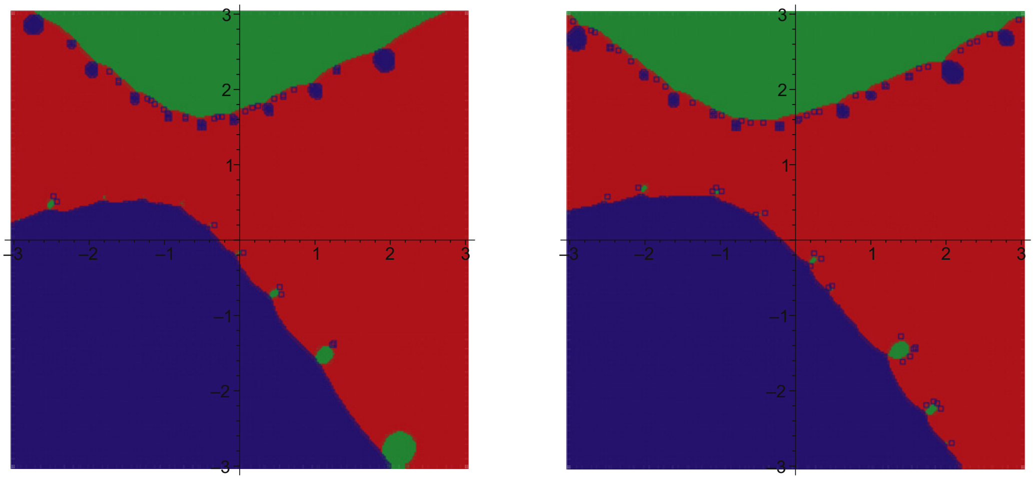

Fig. 22. Newton's (left) and Halley's method (right) for the complex cubic polynomial whose roots are: $-.5 + 2i$, $-.5 + i$, -1.

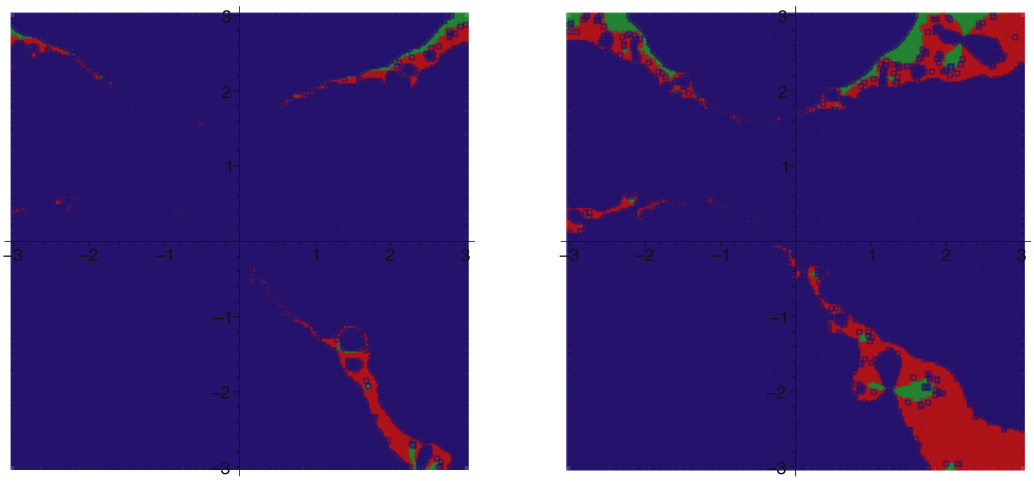

Fig. 23. King's fourth order method with $\beta = -1/2$ (left) and Kung and Traub's fourth order method (right) for the complex cubic polynomial whose roots are: $-.5 + 2i$, $-.5 + i$, -1.

The last two examples are using complex polynomials with simple real and complex roots. In both cases the same 4 methods shine. See Figs. 22–26 for the first example with complex coefficients.

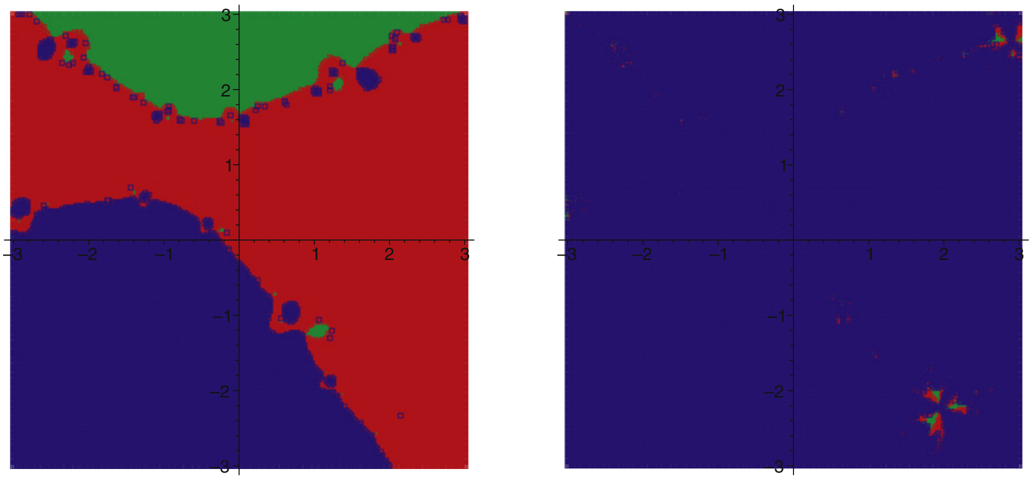

Fig. 24. Murakami's fifth order method (left) and Neta's sixth order method with $\beta = -\frac{1}{2}$ (right) for the complex cubic polynomial whose roots are: $-.5 + 2i$, $-.5 + i$, -1.

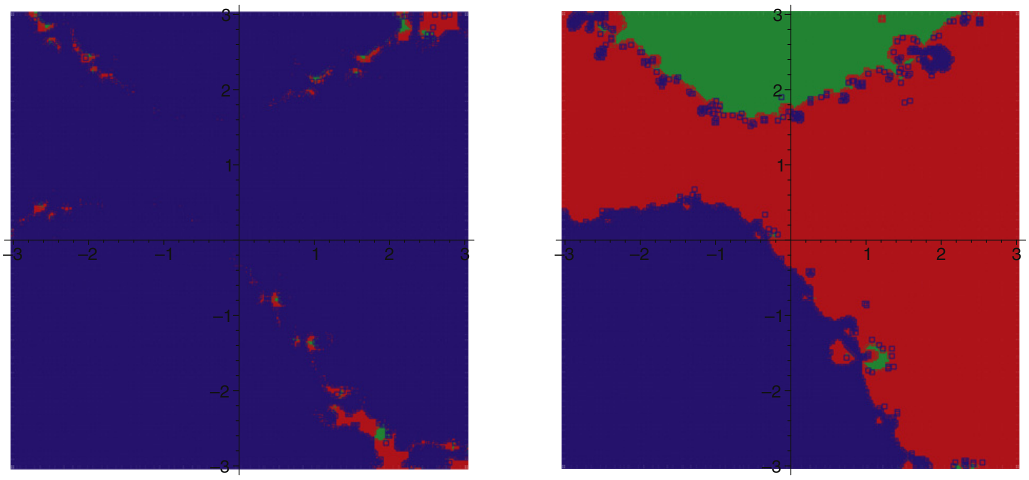

Fig. 25. Neta's sixth order method based on Kung–Traub scheme (left) and Neta and Johnson's eighth order (right) for the complex cubic polynomial whose roots are: $-.5 + 2i$, $-.5 + i$, -1.

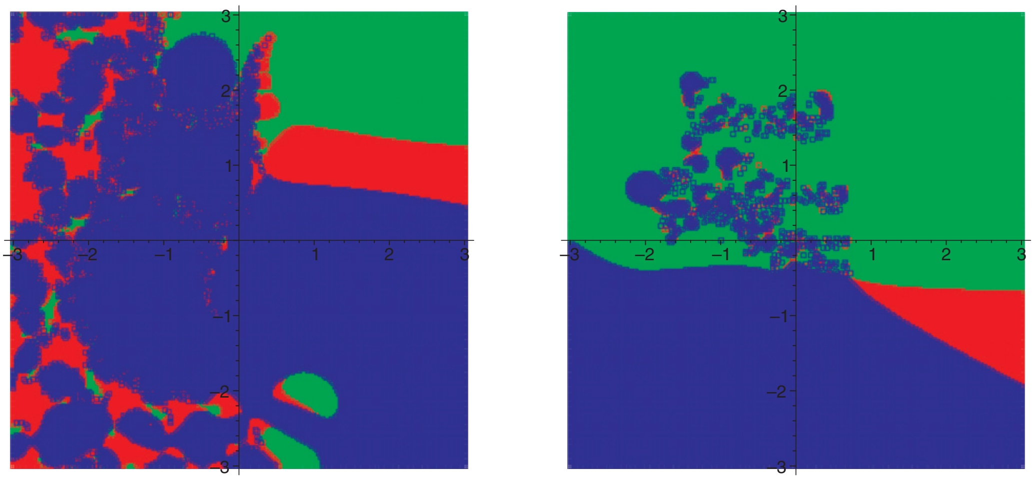

Fig. 26. Neta and Petkovic's optimal eighth order method (left) and Neta's 16th order method with $\beta = 2$ (right) for the complex cubic polynomial whose roots are: $-.5 + 2i$, $-.5 + i$, -1.

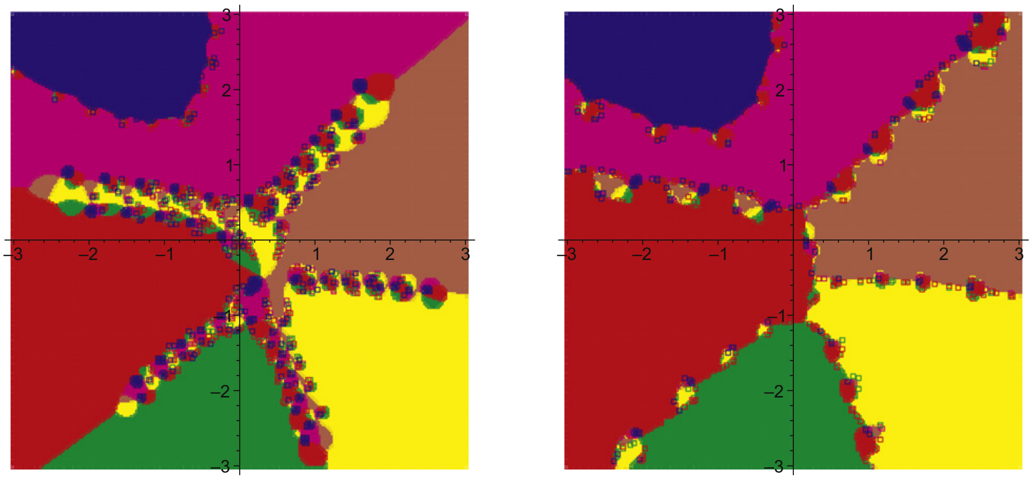

Fig. 27. Newton's (left) and Halley's method (right) for the complex sixth degree polynomial whose roots are: $-\frac{1}{2}-\frac{1}{2}i,\ -\frac{3}{2}i,\ -1+2i,\ 1-i,\ i,\ 1$.

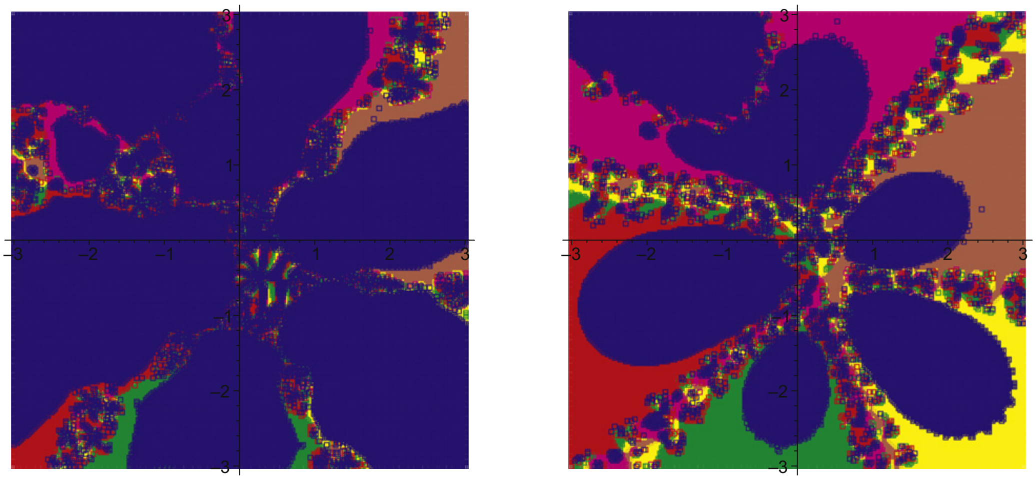

Fig. 28. King's fourth order method with $\beta = -1/2$ (left) and Kung and Traub's fourth order method (right) for the complex sixth degree polynomial whose roots are: $-\frac{1}{2}-\frac{1}{2}i,\ -\frac{3}{2}i,\ -1+2i,\ 1-i,\ i,\ 1$.

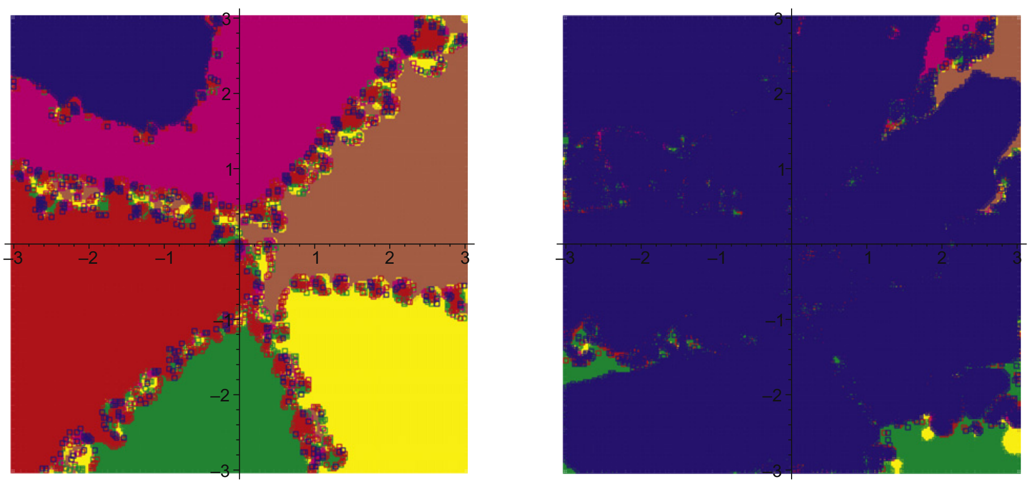

Fig. 29. Murakami's fifth order method (left) and Neta's sixth order method with $\beta = -\frac{1}{2}$ (right) for the complex sixth degree polynomial whose roots are: $-\frac{1}{2}-\frac{1}{2}i,\ -\frac{3}{2}i,\ -1+2i,\ 1-i,\ i,\ 1$.

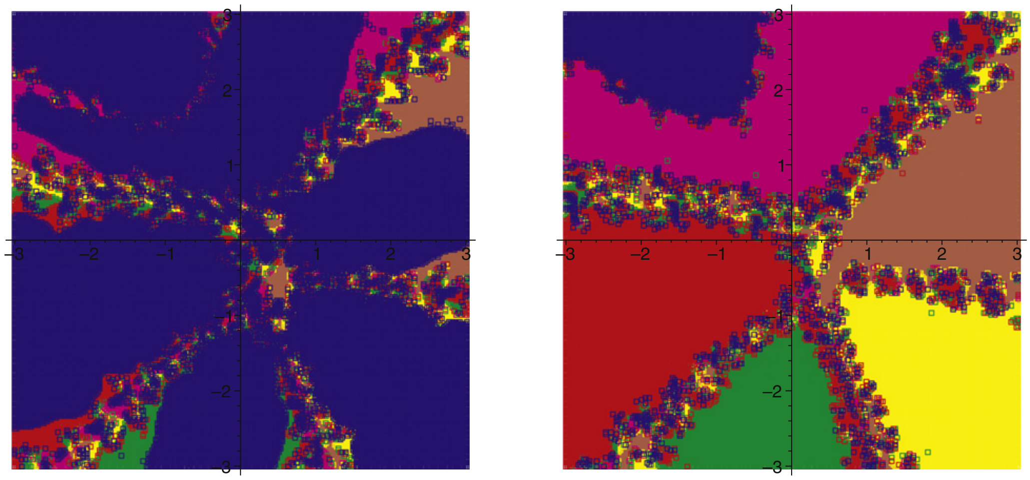

Fig. 30. Neta's sixth order method based on Kung–Traub scheme (left) and Neta and Johnson's eighth order (right) for the complex sixth degree polynomial whose roots are: $-\frac{1}{2}-\frac{1}{2}i,\ -\frac{3}{2}i,\ -1+2i,\ 1-i,\ i,\ 1$.

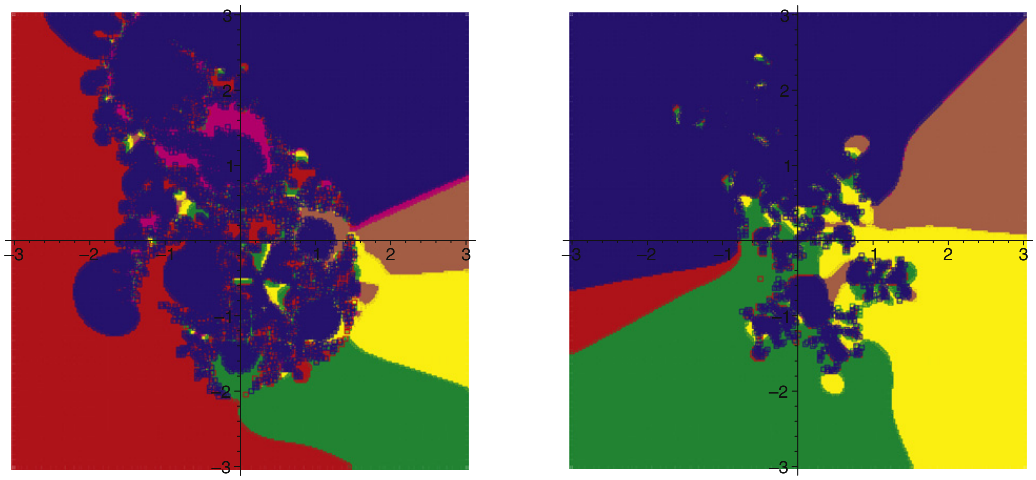

Fig. 31. Neta and Petkovic's optimal eighth order method (left) and Neta's 16th order method with $\beta = 2$ (right) for the complex sixth degree polynomial whose roots are: $-\frac{1}{2}-\frac{1}{2}i,\ -\frac{3}{2}i,\ -1+2i,\ 1-i,\ i,\ 1$.

$$x^3+2x^2-3ix^2-\frac{3}{4}x-\frac{9}{2}ix-\frac{7}{4}-\frac{3}{2}i, \tag{21}$$

$$x^6-\frac{1}{2}x^5+\frac{11}{4}(1+i)x^4-\left(\frac{3}{4}i+\frac{19}{4}\right)x^3-\left(\frac{5}{4}i+\frac{11}{4}\right)x^2-\left(\frac{1}{4}i+\frac{11}{4}\right)x+\frac{3}{2}-3i. \tag{22}$$

The results for the last example are given in Figs. 27–31.

3. Conclusions

The boundaries of basins of attraction of roots may have more complicated fractal structure as the order of methods increases. It seems that the basins of Kung–Traub method and the methods based on it have more chaotic and more fractal boundaries than those of King's method and the methods based on it. For most of higher order methods we considered, the presence of chaotic behaviors in their basins of attraction can be observed unlike Newton's and Halley's methods, this explains that the convergence behavior of a method does depend in a complicated and unpredictable way on the initial root, and why one needs more conditions on the initial root, especially for convergence of a higher-order method.

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