NUMERICAL METHODS IN FLUID DYNAMICS I

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C05

ANALYSIS OF FINITE ELEMENT METHODS FOR THE SOLUTION OF THE VORTICITY-DIVERGENCE FORM OF THE SHALLOW WATER EQUATIONS

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Abstract

Various finite element approximations to the vorticity-divergence form of the shallow water equations are analyzed. The vorticity-divergence equation schemes give superior solutions to those which are based on the primitive equations. The best results come from the finite element schemes on isosceles triangles and rectangles.

1 INTRODUCTION

The hydrostatic primitive equation numerical models which are used for weather-prediction, permit inertial gravity waves, Rossby waves and advective effects. The influence of a numerical method on each of these types of motion is analyzed by separating the linearized prediction equations into vertical modes with an equivalent depth analysis(Gill/1- 1982). Thus one obtains the linearized shallow water equations with the appropriate equivalent depth.

Four finite difference grids were analyzed by Winninghoff(/2- 1968) and Arakawa and Lamb (/3-1977). These grids are labeled A-D (see Fig.1). Their analysis showed that the geostrophic adjustment for the unstaggered grids A, D is poor. Schoenstadt(/4-1980) studied geostrophic adjustment for finite elements (piecewise linear basis functions). He concluded that the unstaggered finite element (grid A) gives poor adjustment for small scale motions, but grids B, C are excellent. Williams(/5-1981) examined geostrophic adjustment in the vorticity-divergence form of the shallow-water equations with finite elements and finite differences. In this formulation, the unstaggered schemes give good geostrophic adjustment. Finite elements are currently used in atmospheric prediction models, see Staniforth and Mitchell(/6-1977, /7-1978), Staniforth and Daley(/8-1979) and Cullen and Hall(/9-1979).

Here we compare the treatment of Rossby waves by finite element approximations of the vorticity-divergence formulation of the shallow water equations. We also include in our comparison the fourth order method due to Staniforth and Mitchell(/6-1977).

2 VORTICITY-DIVERGENCE FORMULATION

The linearized shallow water equations are

$$\begin{aligned} u_t - fv + gh_x &= 0 \\ v_t + fu + gh_y &= 0 \\ h_t + gH(u_x + v_y) &= 0 \end{aligned} \tag{1}$$

where u , v are the perturbations in velocity, h is the perturbation in height, H is the equivalent depth and f is the Coriolis parameter,

$$f = f_0 + \beta y. \tag{2}$$

The vorticity-divergence formulation with the quasi-geostrophic approximation is (see, for example Haltiner and Williams(/10- 1980))

$$\begin{aligned} \zeta_t \; + \; f_0 D \; + \beta_0 g h_x / f_0 \qquad &= 0 \\ - \; f_0 \zeta \qquad + \qquad g(h_{xx} + h_{yy}) \; &= 0 \\ h_t \; + \; DH \qquad &= 0\,. \end{aligned} \tag{3}$$

The Rossby wave frequency ω is given by (see, Neta and Williams(/11-1989))

$$\omega = -\frac{\mu \beta_0}{\mu^2 + k^2 + \lambda^{-2}} \tag{4}$$

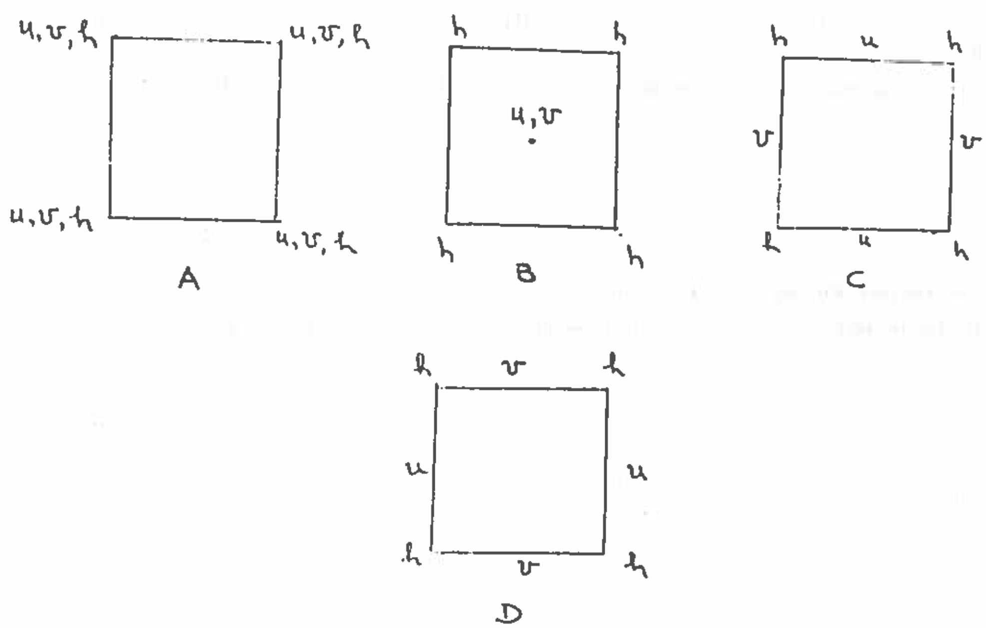

Fig.1: Various elements

where

$$\lambda = \frac{1}{f_0}\sqrt{gH} \tag{5}$$

is the Rossby radius of deformation. The group velocities are

$$\frac{\partial\omega}{\partial\mu} = \beta_0 \frac{\mu^2 - k^2 - \lambda^{-2}}{(\mu^2 + k^2 + \lambda^{-2})^2}$$

$$\frac{\partial\omega}{\partial k} = \beta_0 \frac{2\mu k}{(\mu^2 + k^2 + \lambda^{-2})^2}$$

3 NUMERICAL SCHEMES

It was shown by Neta and Williams(/11-1989) that, for most of the numerical schemes under investigation, the semi-discretized vorticity-divergence system is

$$\begin{array}{rcrcrl} \alpha\zeta_t & + & f_0\alpha D & + & \beta_0 i\theta gh/f_0 & = 0 \\ & - & f_0\alpha\zeta & - & g(\delta+\varepsilon)h & = 0 \\ \alpha h_t & + & H\alpha D & & & = 0 \end{array} \tag{6}$$

where α, θ, δ and ε depend on the scheme. The only exception is the fourth order method due to Staniforth and Mitchell(/7-1978). This scheme approximates first derivatives to fourth order. The boundary value problems resulting from this method are also approximated to fourth order.

In this case one can show that the system (6) should be replaced by

$$\begin{array}{rcrcrl} \alpha\zeta_t & + & f_0\alpha' D & + & \beta_0 i\theta gh/f_0 & = 0 \\ & - & f_0\alpha'\zeta & - & g(\delta+\varepsilon)h & = 0 \\ \alpha h_t & + & H\alpha' D & & & = 0\,. \end{array} \tag{7}$$

The parameters for all these methods are listed in Table 1.

The frequency can now be computed in terms of these parameters

$$\omega_F = -\frac{\beta_0\theta}{\delta + \varepsilon + \alpha\lambda^{-2}} \tag{8}$$

except for the method due to Staniforth and Mitchell for which

$$\omega_F = -\frac{\beta_0\theta\alpha'/\alpha}{\delta + \varepsilon - \alpha'\lambda^{-2}} \tag{9}$$

The group velocities can be obtained by deifferentiating the frequency with respect to μ and k.

4 COMPARISON

In our comparison we chose $\Delta x = \Delta y = d$. The frequencies and group velocities for the various schemes are given for $r^2 = d^2/4\lambda^2 = .1$, 1. and 10. The parameter r^2 measures the relative importance of the terms in the denominator of (7) when the wave scale is small. In Fig.2 we present the frequencies for $k = 0$ and $r^2 = .1$. The analytic formula for the frequency shows that $\omega \rightarrow 0$ as either $\mu \rightarrow 0$ or $\mu \rightarrow \infty$ and ω has a minimum at $\mu = \lambda^{-1}$. All schemes show the same behavior for small μ. However for $\mu > \pi/2d$ all underestimate the magnitude of the frequency. Staniforth and Mitchell's (SM) method is closest to the analytic but later ($\mu d/\pi > .85$) the isosceles finite element (FET) is better.

For $r^2 = 1$ (Fig.3), the FET scheme is the best for $\mu d/\pi > .75$. For $r^2 = 10$ (Fig.4), FET is best. These three cases show that SM, FET and FER (bilinear basis functions on rectangles) are best.

5 CONCLUSIONS

When the grid size is smaller than the Rossby radius of deformation, our results show all of the vorticity-divergence based methods are good, especially FET and SM. When the grid size is of the order of the Rossby radius, FET, FER and SM are best. When the grid size is larger, FER is best.

To combine these results with those of Neta and Williams(/11-1989) and Williams(/5-1981) for the geostrophic adjustment properties of schemes, one can recommend the use of finite element vorticity-divergence model.

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Table 1: Parameters in Eqs.(6) and (7)

oprat.	analy.	isosceles triangles	rectangles	Staniforth/Mitchell
α	1	$\frac{3+\cos X+2\cos\frac{X}{2}\cos Y}{6}$	$\frac{(2+\cos X)(2+\cos Y)}{9}$	$\frac{(2+\cos X)(2+\cos Y)}{9}$
θ	μ	$\frac{2(\sin X+\sin\frac{X}{2}\cos Y)}{3\Delta x}$	$\frac{\sin X(2+\cos Y)}{3\Delta x}$	$\frac{\sin X(2+\cos Y)}{3\Delta x}$
δ	μ^2	$\frac{\sin^2\frac{X}{2}}{(\frac{\Delta x}{2})^2}$	$\frac{\sin^2\frac{X}{2}}{(\frac{\Delta x}{2})^2}\frac{2+\cos Y}{3}$	$\frac{\sin^2\frac{X}{2}}{(\frac{\Delta x}{2})^2}\frac{5+\cos Y}{6}$
ε	k^2	$\frac{3+\cos X-4\cos\frac{X}{2}\cos Y}{2\Delta y^2}$	$\frac{\sin^2\frac{Y}{2}}{(\frac{\Delta y}{2})^2}\frac{2+\cos X}{3}$	$\frac{\sin^2\frac{Y}{2}}{(\frac{\Delta y}{2})^2}\frac{5+\cos X}{6}$
α'				$\frac{(5+\cos X)(5+\cos Y)}{36}$

$X=\mu\Delta x, Y=k\Delta y$

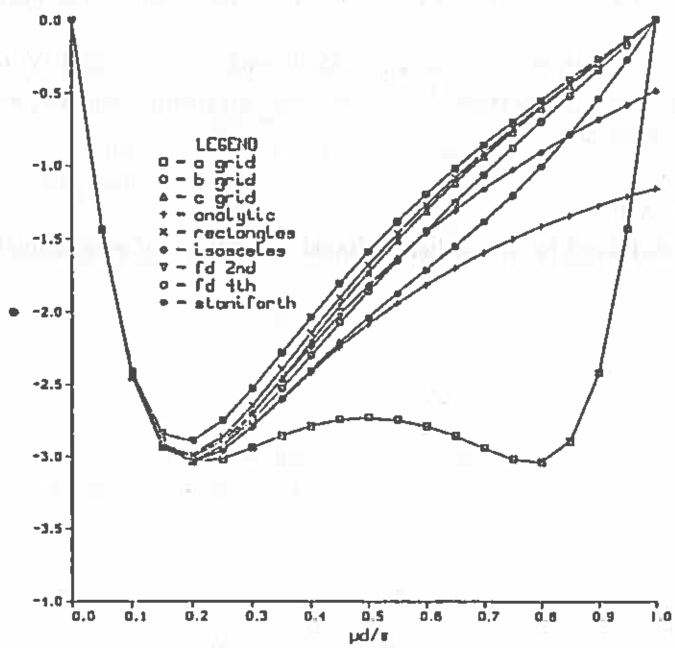

Fig.2: Comparison of frequencies ($r^2=.1$)

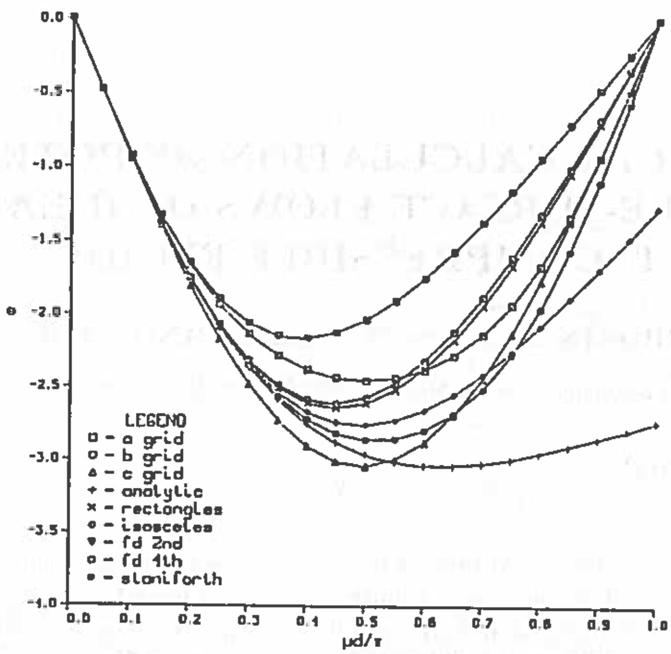

Fig.3: Comparison of frequencies ($r^2 = 1$)

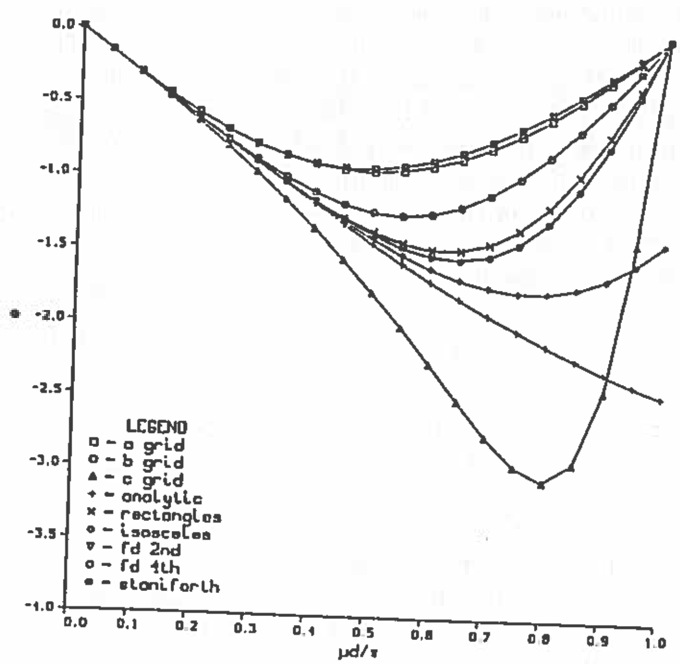

Fig.4: Comparison of frequencies ($r^2 = 10$)