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The basins of attraction of Murakami's fifth order family of methods

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ABSTRACT

In this paper we analyze Murakami's family of fifth order methods for the solution of nonlinear equations. We show how to find the best performer by using a measure of closeness of the extraneous fixed points to the imaginary axis. We demonstrate the performance of these members as compared to the two members originally suggested by Murakami. We found several members for which the extraneous fixed points are on the imaginary axis, only one of these has 6 such points (compared to 8 for the other members). We show that this member is the best performer.

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1. Introduction

There is a vast literature on the solution of nonlinear equations, see for example Ostrowski [19], Traub [23], Neta [16] and Petković et al. [20]. In this paper we consider a fifth-order family of methods and show how to choose the best parameters. We will compare the performance of the two originally suggested members to two new ones by using the idea of basin of attraction and analyzing the extraneous fixed points.

Murakami [15] has developed a fifth order family of methods

$$x_{n+1} = x_n - a_1 u_n - a_2 w_2(x_n) - a_3 w_3(x_n) - \psi(x_n), \tag{1}$$

where

$$\begin{aligned} u_n &= \frac{f(x_n)}{f'(x_n)}, \\ w_2(x_n) &= \frac{f(x_n)}{f'(x_n - u_n)}, \end{aligned} \tag{2}$$

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$$w_3(x_n) = \frac{f(x_n)}{f'(x_n + \beta u_n + \gamma w_2(x_n))},$$

$$\psi(x_n) = \frac{f(x_n)}{b_1 f'(x_n) + b_2 f'(x_n - u_n)}.$$

This family is of order five when we take

$$\begin{aligned} a_1 &= \frac{1}{6}\left(1 + \frac{4\gamma+1}{\theta}\right), \qquad a_2 = \frac{1}{\theta-1}\left(\frac{1}{6}\theta - \frac{2}{3}\gamma - \frac{1}{3}\right), \qquad a_3 = \frac{2}{3}, \\ b_1 &= -\frac{6\theta(\theta-1)^2}{4\gamma+1}, \qquad b_2 = \frac{6\theta^2(\theta-1)}{4\gamma+1}, \qquad \beta = -\gamma - \frac{1}{2}, \end{aligned} \tag{3}$$

and

$$\theta = \frac{16\gamma+5}{4(4\gamma+1)}. \tag{4}$$

Murakami suggested the following two possibilities:

$$\begin{aligned} &\gamma = 0, \qquad a_1 = 0.3, \qquad a_2 = -0.5, \qquad a_3 = \frac{2}{3}, \\ &b_1 = -\frac{15}{32}, \qquad b_2 = \frac{75}{32}, \qquad \beta = -\frac{1}{2} \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\gamma = -0.5, \qquad a_1 = -\frac{1}{18}, \qquad a_2 = -\frac{1}{2}, \qquad a_3 = \frac{2}{3}, \\ &b_1 = \frac{9}{32}, \qquad b_2 = \frac{27}{32}, \qquad \beta = 0. \end{aligned} \tag{6}$$

The idea there probably to choose one of the parameters to be zero, i.e. either $\gamma = 0$ or $\beta = 0$. As it turns out these parameters are not far from the best.

In this paper, we find the best possible value of the parameter γ. We will use two criteria we have developed in previous work [7] based on the location of the extraneous fixed points. In the next section, we discuss the extraneous fixed points. In section 3 we will discuss the two criteria and give the best parameter based on these criteria. In section 4 we describe the basins of attraction for the best members of the family for 7 different examples. We close with conclusions.

2. Extraneous fixed points

For the Murakami family $z_{n+1} = M_f(z_n)$, where

$$\begin{aligned} M_f(z) &= z - a_1 u_f(z) - a_2 w_{2,f}(z) - a_3 w_{3,f}(z) - \psi_f(z), \\ u_f(z) &= \frac{f(z)}{f'(z)}, \\ w_{2,f}(z) &= \frac{f(z)}{f'(z - u_f(z))}, \\ w_{3,f}(z) &= \frac{f(z)}{f'(z + \beta u_f(z) + \gamma w_{2,f}(z))}, \\ \psi_f(z) &= \frac{f(z)}{b_1 f'(z) + b_2 f'(z - u_f(z))}, \end{aligned} \tag{7}$$

we explore its conjugacy on quadratic polynomials. We begin with a preliminary result.

Lemma 1. *Let $f(z)$ be an analytic function on the Riemann sphere, and let $T(z) = \alpha z + \beta$, $\alpha \neq 0$, be an affine map. If $g(z) = (f \circ T)(z)$, then we have*

$$\begin{aligned}
&u_f(T(z)) = \alpha u_g(z),\\
&w_{2,f}(T(z)) = \alpha w_{2,g}(z),\\
&w_{3,f}(T(z)) = \alpha w_{3,g}(z),\\
&\psi_f(T(z)) = \alpha \psi_g(z).
\end{aligned} \tag{8}$$

Proof. We have $g'(z) = \alpha f'(T(z))$, so that

$$g^{(n)}(z) = \alpha^n f^{(n)}(T(z)),\ n \geq 1. \tag{9}$$

From this we obtain

$$u_f(T(z)) = \frac{f(T(z))}{f'(T(z))} = \alpha u_g(z). \tag{10}$$

We then have by the comparison of Taylor series expansions on f and g,

$$\begin{aligned}
f'(T(z) - u_f(T(z))) &= f'(T(z) - \alpha u_g(z))\\
&= f'(T(z)) - f''(T(z))\alpha u_g(z) + \cdots\\
&= \frac{1}{\alpha} g'(z) - \frac{1}{\alpha^2} g''(z)\alpha u_g(z) + \cdots\\
&= \frac{1}{\alpha} g'(z - u_g(z)),
\end{aligned} \tag{11}$$

and we obtain

$$w_{2,f}(T(z)) = \frac{f(T(z))}{f'(T(z) - u_f(T(z)))} = \alpha w_{2,g}(z). \tag{12}$$

Similarly as in (11), it can be shown from (9), (10), (12) that

$$f'(T(z) + \beta u_f(T(z)) + \gamma w_{2,f}(T(z))) = \frac{1}{\alpha} g'(z + \beta u_g(z) + \gamma w_{2,g}(z)), \tag{13}$$

from which,

$$w_{3,f}(T(z)) = \alpha w_{3,g}(z). \tag{14}$$

It follows from (9) and (11) that

$$\psi_f(T(z)) = \alpha \psi_g(z). \tag{15}$$

We can now obtain the scaling theorem for the Murakami family.

Theorem 2. *Let $f(z)$ be an analytic function on the Riemann sphere, and let $T(z) = \alpha z + \beta$, $\alpha \neq 0$, be an affine map. If $g(z) = (f \circ T)(z)$, then we have $T \circ M_g \circ T^{-1} = M_f(z)$, that is, M_f and M_g are topologically conjugated via T.*

Proof. We will prove that $(T \circ M_g)(z) = (M_f \circ T)(z)$ for all z. By Lemma 1 we have

$$\begin{aligned}
(T \circ M_g)(z) &= \alpha M_g(z) + \beta\\
&= \alpha z - [a_1 \alpha u_g(z) + a_2 \alpha w_{2,g}(z) + a_3 \alpha w_{3,g}(z) + \alpha \psi_g(z)] + \beta\\
&= T(z) - [a_1 u_f(T(z)) + a_2 w_{2,f}(T(z)) + a_3 w_{3,f}(T(z)) + \psi_f(T(z))]\\
&= (M_f \circ T)(z).
\end{aligned} \tag{16}$$

Every quadratic polynomial $p(z)$ with distinct roots reduces, via a linear map, to a polynomial belonging to the one parameter family $q(z) = z^2 - \mu$, and so by the above scaling theorem, the study of the dynamics of the Murakami family for any complex quadratic polynomial with distinct roots reduces to its dynamical study for $f(z) = z^2 - \mu$. The scaling theorems and the conjugacy classes of the Murakami family for degree three and four polynomials can be discussed as in Amat et al. [2].

In solving a nonlinear equation iteratively we are looking for fixed points which are zeros of the given nonlinear function. Many multipoint iterative methods have fixed points that are not zeros of the function of interest. Thus, it is imperative to investigate the number of extraneous fixed points, their location and their properties.

Theorem 3. *The extraneous fixed points for the Murakami family for $f(z) = z^2 - \mu$ can be found by solving*

$$\frac{N_\gamma(z)}{D_\gamma(z)} = 0, \tag{17}$$

where

$$\begin{aligned} N_\gamma(z) &= (128\gamma^3 + 352\gamma^2 + 522\gamma + 87)z^8 - 2\mu(256\gamma^3 + 224\gamma^2 - 528\gamma - 129)z^6 \\ &\quad + 12\mu^2(64\gamma^3 + 83\gamma + 24)z^4 - 2\mu^3(256\gamma^3 + 32\gamma^2 - 216\gamma - 63)z^2 \\ &\quad + \mu^4(2\gamma + 1)(3 + 8\gamma)^2, \\ D_\gamma(z) &= 6((3 + 16\gamma)z^2 + 16\mu(5 + \gamma))((3 + 2\gamma)z^4 + 4\mu(1 - \gamma)z^2 + \mu^2(1 + 2\gamma))(z^2 + \mu). \end{aligned}$$

Proof. In order to obtain the extraneous fixed points for the method (1), we have to rewrite it in the following form:

$$x_{n+1} = x_n - u_n H_f(x_n), \tag{18}$$

where

$$\begin{aligned} H_f(x_n) \;=\; & a_1 + a_2 \frac{f'(x_n)}{f'(x_n - u_n)} + a_3 \frac{f'(x_n)}{f'(x_n + \beta u_n + \gamma w_2(x_n))} \\ & + \frac{f'(x_n)}{b_1 f'(x_n) + b_2 f'(x_n - u_n)} \end{aligned} \tag{19}$$

Upon using $f(z) = z^2 - \mu$ in (19), we have (17).

In the sequel, the case $\mu = 1$ will be considered, so the best possible value of the parameter of the Murakami family will be found for $f(z) = z^2 - 1$.

For $\gamma = 0$ and $\mu = 1$, the equation (17) becomes

$$\frac{1}{2}\frac{29z^6 + 57z^4 + 39z^2 + 3}{(z^2 + 1)(3z^2 + 5)(3z^2 + 1)} = 0. \tag{20}$$

The roots are as follows;

$$\xi = \pm 0.296058904382937i, \xi = \pm 0.271524554061638 \pm 1.00630947859782i.$$

All the fixed points are repulsive.

For $\gamma = -0.5$ and $\mu = 1$, the equation (17) becomes

$$\frac{N_\gamma(z)}{(z^2 + 3)(z^2 + 1)(z^2 + 0.6)} = 0, \tag{21}$$

where

$$\begin{aligned} N_\gamma(z) &= 1.7(z^2 + 0.467734312641029z + 1.58640402523038) \\ &\quad (z^2 + 0.163614395584542)(z^2 - 0.467734312641029z + 1.58640402523038). \end{aligned}$$

The roots are as follows;

$$\xi = \pm 0.4044927633i, \xi = \pm 0.2338671563 \pm 1.237622793i.$$

All the fixed points are repulsive.

In the next section we find that, based on the criterion used, the parameter γ should be either $\gamma = -0.62$ or $\gamma = -0.125$.

For $\gamma = -0.62$ and $\mu = 1$, the equation (17) becomes

$$\frac{N_\gamma(z)}{D_\gamma(z)} = 0, \tag{22}$$

where

$$\begin{aligned} N_\gamma(z) &= 1.804130321(z^2 + 0.698167010067880z + 1.89250439483774) \\ &\quad (z^2 + 0.0593972226948352)(z^2 + 0.0328734688882183) \\ &\quad (z^2 - 0.698167010067880z + 1.89250439483774), \\ D_\gamma(z) &= (z^2 + 0.7109826589)(z + 0.191498766272523)(z - 0.191498766272523) \\ &\quad (z^2 + 3.71848995930208)(z^2 + 1). \end{aligned}$$

The roots are as follows;

$$\xi = \pm 0.3490835050 \pm 1.330655891i, \xi = \pm 0.2437154544i, \xi = \pm 0.1813104213i.$$

All the fixed points are repulsive.

For $\gamma = -0.125$ and $\mu = 1$, the equation (17) becomes

$$\frac{N_\gamma(z)}{D_\gamma(z)} = 0, \tag{23}$$

where

$$N_\gamma(z) = 1.636363636(z^2 + 0.0463327506379597)(z^2 + 1)(z^2 + 2.39811169380649),$$
$$D_\gamma(z) = (z^2 + 1.44801847547959)(z^2 + 3)(z^2 + 0.188345160884045).$$

The roots are as follows;

$$\xi = \pm 1.548583770i, \xi = \pm i, \xi = \pm 0.2152504370i.$$

All the fixed points are repulsive.

Remark. Notice that this is the only case out of the 4 above where the extraneous fixed points are all on the imaginary axis. There are other values of γ in the range $(-0.1891, -0.125)$ for which we have purely imaginary extraneous fixed points. In these cases the number of points is 8 versus 6 in the case we chose to present here.

In the next section we discuss two criteria to choose the parameters in the Murakami family. These criteria were developed in [8].

3. Best possible parameters

The parameters can be chosen to position the extraneous fixed points on the imaginary axis or, at least, close to that axis, (see, for example, Chun and Neta [7] and [10]).

We have searched the parameter space (γ) and found that the extraneous fixed points are on the imaginary axis for certain values of the parameter γ. We have considered one measure of closeness to the imaginary axis, denoted by d, and another measure of averaged stability of the extraneous fixed points, denoted by A. These measures are defined below. We have experimented with those members from the parameter space.

Let $E = \{z_1, z_2, ..., z_{n_\gamma}\}$ be the set of the extraneous fixed points corresponding to the value of the parameter γ. We define

$$d(\gamma) = \max_{z_i \in E} |Re(z_i)|. \tag{24}$$

We look for the parameters γ which attain the minimum of $d(\gamma)$. The minimum of $d(\gamma) = 0$ occurs at $\gamma = -0.125$. We will call the corresponding method **Murakamid**. We will not consider other values of γ for which $d(\gamma) = 0$, since they have more extraneous fixed points.

Another method to choose the parameter is by considering the stability of $z \in E$ defined by

$$dq(z) = \frac{dq}{dz}(z), \tag{25}$$

where q is the iteration function of the Murakami family of methods. We define a function called the averaged stability value of the set E by

$$A(\gamma) = \frac{\sum_{z_i \in E} |dq(z_i)|}{n_\gamma}, \tag{26}$$

where n_γ is the number of elements of the set E.

The smaller A becomes, the less chaotic the basin of attraction tends to. The minimum of $A(\gamma)$ occurs at $\gamma = -0.62$. We will call the corresponding method **MurakamiA**.

It was shown before that methods for which the extraneous fixed points are on or close to the imaginary axis perform better than others. $d(\gamma)$ measures the distance of the extraneous fixed points from the imaginary axis and thus predicts best performance. The measures $d(\gamma)$ and $A(\gamma)$ provide ways of choosing the best performers for any parameter-dependent families of methods. In our previous work, we found that methods suggested by these measures perform better than others.

The values of the parameter γ presented in [15] are $\gamma = 0$ and $\gamma = -0.5$ that we call them **Murakami1** and **Murakami2**, respectively.

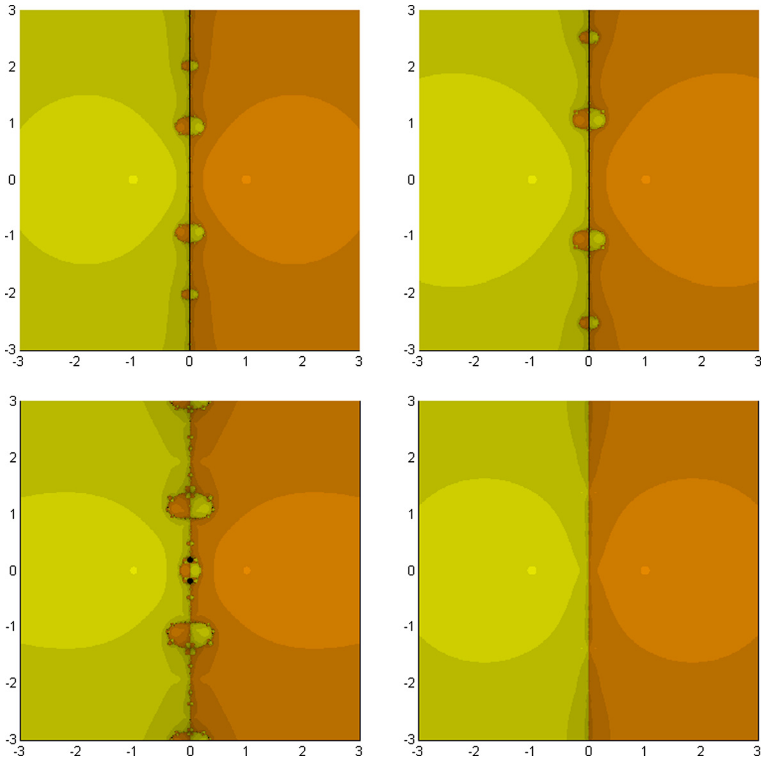

Fig. 1. The top row for **Murakami1** (left) and **Murakami2** (right), and second row for **MurakamiA** (left) and **Murakamid** (right) for the roots of the polynomial $z^2 - 1$.

4. Numerical experiments

The Basin of Attraction is a method to visually understand how a method behaves as a function of the various starting points. This idea was started by Stewart [22] and continued in the work of Amat et al. [1–3], Argyros and Magreñan [4], Chicharro et al. [5], Chun et al. [6,9,11], Cordero et al. [12], Geum et al. [13], Magreñan [14], Neta et al. [17,18], and Scott et al. [21].

We have used the 4 members of the Murakami family for 7 different polynomials. The choice of the parameters in the families used is based on the analysis in the previous section. All the examples have roots within a square of $[-3, 3]$ by $[-3, 3]$. We have taken 360,000 equally spaced points in the square as initial points for the methods and we have registered the total number of iterations required to converge to a root and also to which root it converged. We have also collected the CPU time (in seconds) required to run each method on all the points using Dell Optiplex 990 desktop computer. We then computed the average number of iterations required per point and the number of points requiring 40 iterations.

Example 1. In our first example, we have taken the polynomial

$$p_1(z) = z^2 - 1 \tag{27}$$

whose roots are $z = \pm 1$. In Fig. 1 we have presented the basins for the 4 members of the Murakami family of methods. In the top row, we have $\gamma = 0$ (left) and $\gamma = -0.5$ (right). On the bottom row we have the case $\gamma = -0.62$ based on the A criterion (left) and $\gamma = -0.125$ based on the d criterion (right). It is clear from the Fig. 1 that **Murakamid** is the best method (bottom right subplot). The parameter based on the d criterion for closeness has improved the basins (compared to the cases suggested by [15]). The parameter based on the A criterion made the situation slightly worse. In order to get a quantitative comparison, we have collected the average number of iterations per point in Table 1, the CPU time for each method and each example in Table 2 and the number of points requiring 40 iterations in Table 3. It is clear that **Murakamid** requires the least number (2.70), and the original ones require slightly more. **MurakamiA** is the worst, requiring 2.94 iterations per point on average. Note that the difference is very small. In terms of CPU time (see Table 2) now **Murakami2** is the fastest (150 seconds) followed by **Murakamid** (151 seconds). **MurakamiA** is the slowest with 162 seconds. In terms of the number of points requiring 40 iterations (see Table 3) we find that **MurakamiA** is the worst with 765 points requiring 40 iterations. All other methods have 601 points requiring 40 iterations.

Example 2. Our next example is a cubic polynomial having the three roots of unity,

Table 1
Average number of iterations per point for each example (1–7) and each method.

Method	Ex1	Ex2	Ex3	Ex4	Ex5	Ex6	Ex7	Average
Murakami1 ($\gamma = 0$)	2.81	3.43	3.52	2.98	4.64	4.21	5.93	3.93
Murakami2 ($\gamma = -0.5$)	2.72	3.39	3.38	2.92	4.62	4.24	6.06	3.90
MurakamiA ($\gamma = -0.62$)	2.94	3.58	3.47	3.09	4.84	4.48	6.35	4.11
Murakamid ($\gamma = -0.125$)	2.70	3.36	3.29	2.88	4.34	3.63	5.09	3.61

Table 2
CPU time (in seconds) required for each example (1–7) and each method.

Method	Ex1	Ex2	Ex3	Ex4	Ex5	Ex6	Ex7	Average
Murakami1	156.297	252.331	258.228	289.428	417.116	1297.023	648.995	474.203
Murakami2	150.338	248.837	240.382	278.508	420.501	1296.010	657.076	470.236
MurakamiA	162.584	259.336	244.734	295.357	438.565	1358.800	679.961	491.334
Murakamid	151.024	244.407	244.235	276.402	386.477	1111.538	572.367	426.636

Table 3
Number of points requiring 40 iterations for each example (1–7) and each method.

Method	Ex1	Ex2	Ex3	Ex4	Ex5	Ex6	Ex7	Average
Murakami1	601	1	0	601	19	0	1027	321.29
Murakami2	601	1	0	601	20	0	886	301.29
MurakamiA	765	1	0	601	41	1	948	336.71
Murakamid	601	2086	4	609	5483	28	6158	2138.43

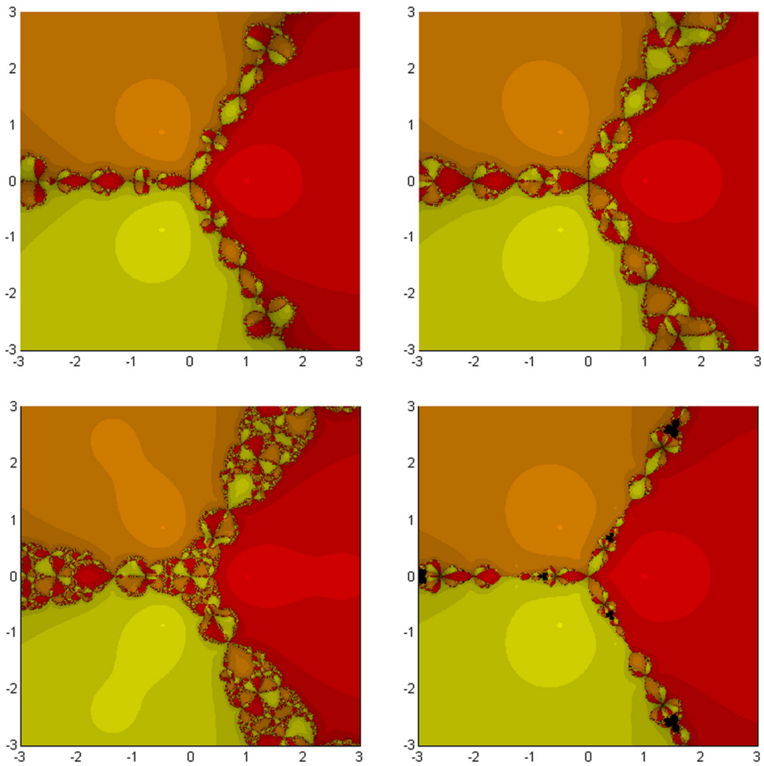

Fig. 2. The top row for **Murakami1** (left) and **Murakami2** (right), and second row for **MurakamiA** (left) and **Murakamid** (right) for the roots of the polynomial $z^3 - 1$.

$$p_2(z) = z^3 - 1. \tag{28}$$

The basins of attraction are plotted in Fig. 2. Again **MurakamiA** is most chaotic and the others are about the same. The average number of iterations per point is lowest for **Murakamid** (3.36) and highest for **MurakamiA** (3.58). The CPU time

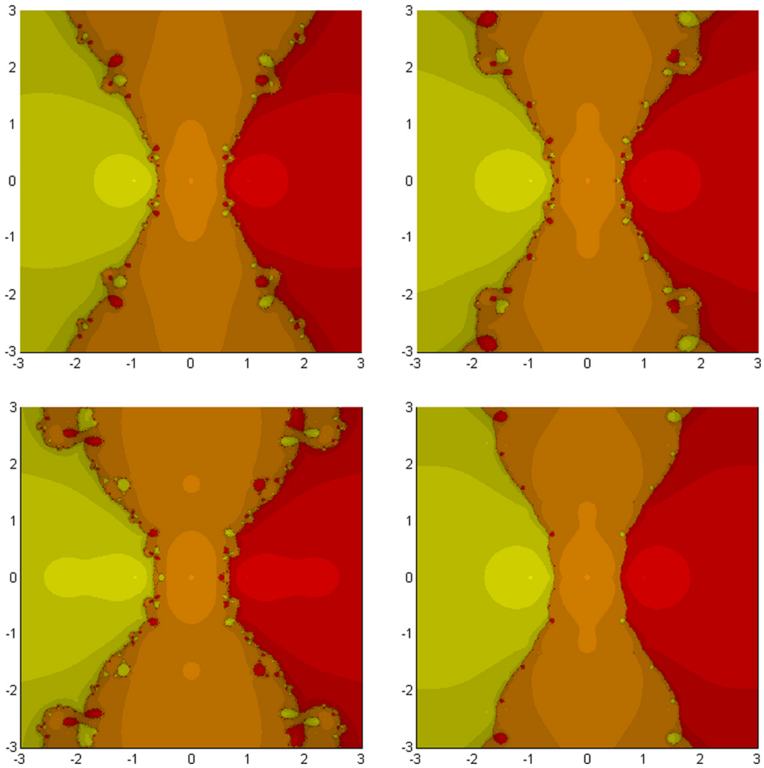

Fig. 3. The top row for **Murakami1** (left) and **Murakami2** (right), and second row for **MurakamiA** (left) and **Murakamid** (right) for the roots of the polynomial $z^3 - z$.

is about the same for all (249–259 seconds) with **MurakamiA** being the slowest and **Murakamid** being the fastest. The methods with the fewest number of points requiring 40 iterations are: **Murakami1**, **Murakami2** and **MurakamiA** (all with one point). The worst is **Murakamid** with 2086 points.

Example 3. Our next example is the cubic polynomial,

$$p_3(z) = z^3 - z. \tag{29}$$

The basins of attraction are given in Fig. 3. It is clear that **Murakamid** is best, but **Murakami1** and **Murakami2** are not far behind. In terms of the average number of iterations per points, **Murakamid** is best (3.29) and **Murakami1** is worst (3.52). The fastest is **Murakami2** (240 seconds) and the slowest is **Murakami1** (258 seconds). All members have no black points (see Table 3) except **Murakamid** with 4 black points.

Example 4. The fourth example is the quartic polynomial,

$$p_4(z) = z^4 - 10z^2 + 9. \tag{30}$$

The basins of attraction are plotted in Fig. 4. Again **Murakamid** is better than the others. This is also confirmed by consulting Tables 1 and 2. All members have the same numbers of points requiring 40 iterations (601 points) except **Murakamid** with 609 points.

Example 5. The fifth example is a polynomial

$$p_5(z) = z^5 - 1. \tag{31}$$

The plots of the basins are given in Fig. 5. **Murakamid** and **Murakami1** are better than the others. The average number of iterations per point is 4.34 for **Murakamid** and higher (4.62–4.84) for the others (see Table 1). **Murakamid** is the fastest with 386 seconds (see Table 2) followed by **Murakami1** (417 seconds) and **Murakami2** (420 seconds). In terms of the number of points requiring 40 iterations, **Murakami1** and **Murakami2** have fewest number (19–20 points) and **Murakamid** the highest with 5483 points.

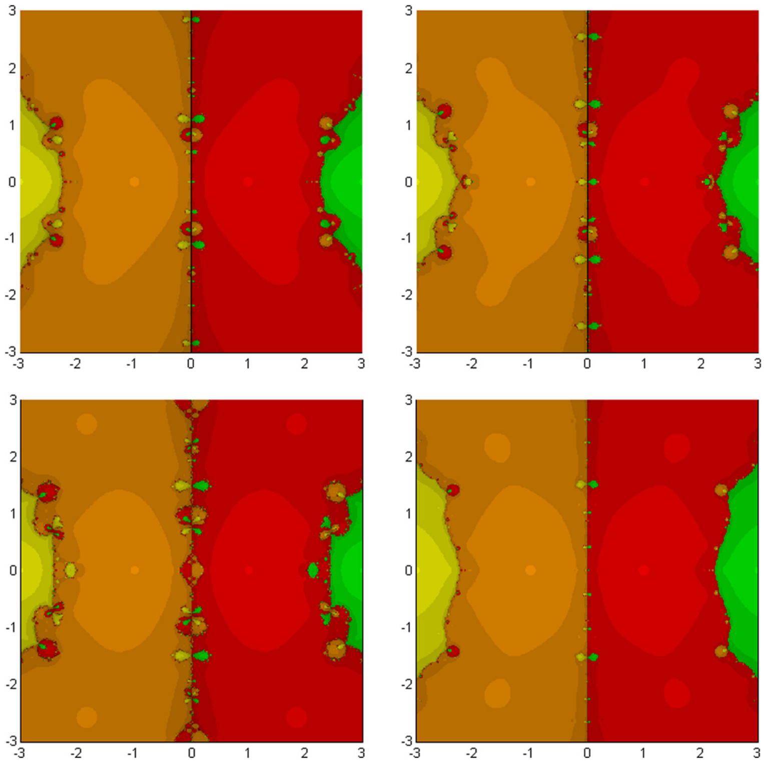

Fig. 4. The top row for **Murakami1** (left) and **Murakami2** (right), and second row for **MurakamiA** (left) and **Murakamid** (right) for the roots of the polynomial $z^4 - 10z^2 + 9$.

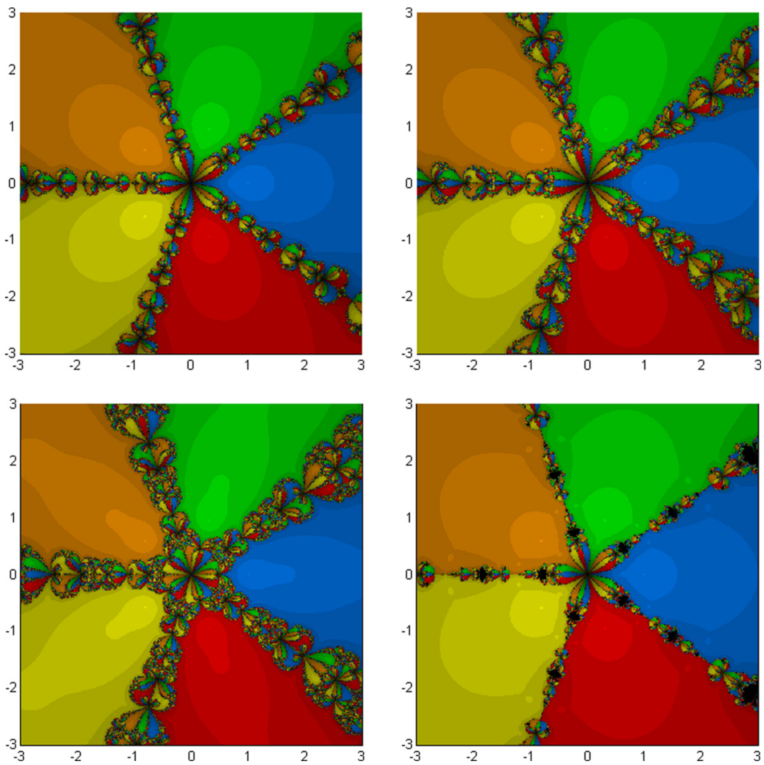

Fig. 5. The top row for **Murakami1** (left) and **Murakami2** (right), and second row for **MurakamiA** (left) and **Murakamid** (right) for the roots of the polynomial $z^5 - 1$.

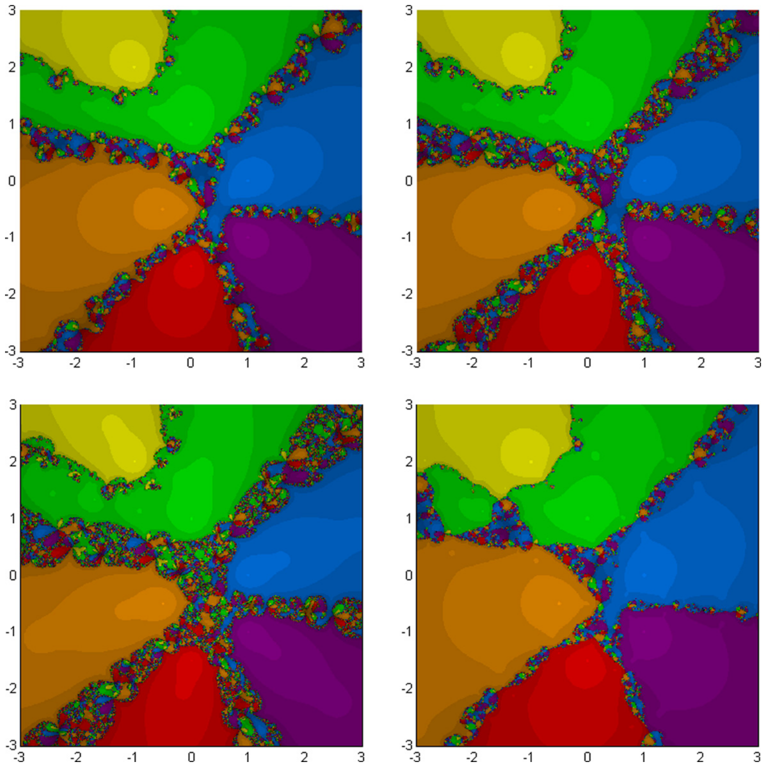

Fig. 6. The top row for **Murakami1** (left) and **Murakami2** (right), and second row for **MurakamiA** (left) and **Murakamid** (right) for the roots of the polynomial $z^6 - \frac{1}{2}z^5 + \frac{11}{4}(1+i)z^4 - \frac{1}{4}(19+3i)z^3 + \frac{1}{4}(11+5i)z^2 - \frac{1}{4}(11+i)z + \frac{3}{2} - 3i$.

Example 6. Our next example is a polynomial with complex coefficients

$$\begin{aligned} p_6(z) &= z^6 - \tfrac{1}{2}z^5 + \tfrac{11}{4}(1+i)z^4 - \tfrac{1}{4}(19+3i)z^3 + \tfrac{1}{4}(11+5i)z^2 \\ &- \tfrac{1}{4}(11+i)z + \tfrac{3}{2} - 3i. \end{aligned} \tag{32}$$

This example was the hardest for many iterative methods as we found out in our previous work. **Murakamid** is the best, as can be seen from Fig. 6. From Table 1 we find that **Murakamid** requires 3.63 iterations per point on average and the others require above 4.21. **Murakamid** is the fastest with 1111 seconds, followed by **Murakami2** (1296 seconds) and **Murakami1** (1297 seconds). The number of points requiring 40 iterations (see Table 3) is highest for **Murakamid** with 28 points. The rest have no black point or just one point.

Example 7. Our last example is a polynomial

$$p_7(z) = z^7 - 1. \tag{33}$$

The basins of attraction are plotted in Fig. 7. Based on the figure we conclude that **Murakamid** is best. Based on Table 1, we conclude that **Murakamid** uses least number of iterations per point (5.09) and the others use 5.93–6.35 iterations per point on average. The fastest method (see Table 2) is **Murakamid** (572 seconds) and the slowest is **MurakamiA** with 680 seconds. The number of points requiring 40 iterations is the smallest for **Murakami2** (886) and the highest (6158) for **Murakamid**.

5. Conclusions

In order to decide which member is best overall, we have averaged the numbers across the examples. We now find that **Murakamid** requires the least number of iterations per point (3.61) followed by **Murakami2** (3.90). **MurakamiA** requires the most (4.11). Same conclusion for the average CPU time across examples, i.e. **Murakamid** is the fastest (426 seconds) and **MurakamiA** is the slowest (491 seconds). In terms of the number of points requiring 40 iterations, the highest is 2138 for **Murakamid** and the lowest (301) for **Murakami2**. We can conclude that the criterion d is very useful in finding the best performer from a family of methods. Based on the averages in the 3 tables we conclude that **Murakamid** is best followed by **Murakami2** and **Murakami1**.

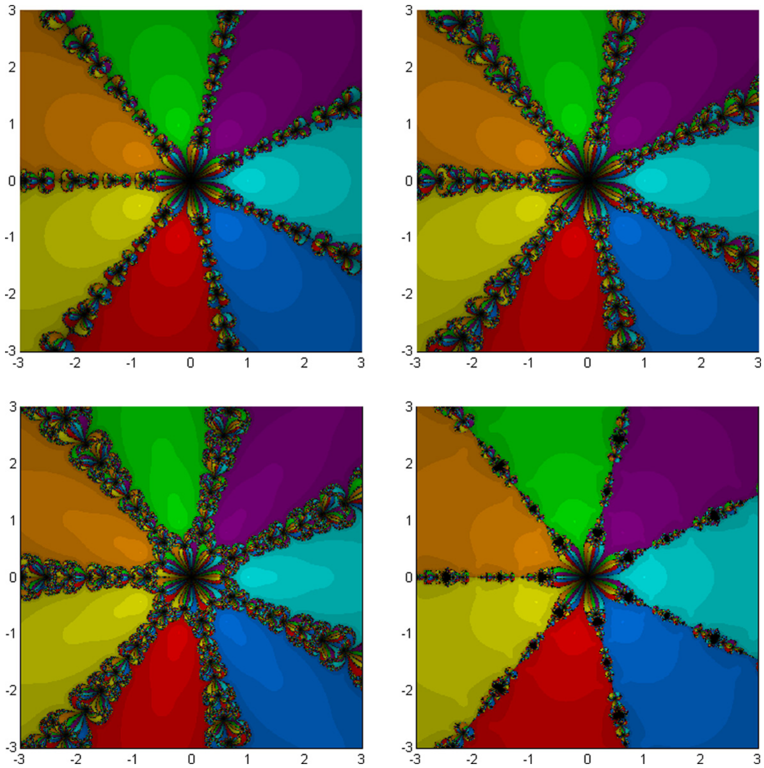

Fig. 7. The top row for **Murakami1** (left) and **Murakami2** (right), and second row for **MurakamiA** (left) and **Murakamid** (right) for the roots of the polynomial $z^7 - 1$.

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References

[1] S. Amat, S. Busquier, S. Plaza, Review of some iterative root-finding methods from a dynamical point of view, Scientia 10 (2004) 3–35.
[2] S. Amat, S. Busquier, S. Plaza, Dynamics of a family of third-order iterative methods that do not require using second derivatives, Appl. Math. Comput. 154 (2004) 735–746.
[3] S. Amat, S. Busquier, S. Plaza, Dynamics of the King and Jarratt iterations, Aequ. Math. 69 (2005) 212–223.
[4] I.K. Argyros, A.A. Magreñan, On the convergence of an optimal fourth-order family of methods and its dynamics, Appl. Math. Comput. 252 (2015) 336–346.
[5] F. Chicharro, A. Cordero, J.M. Gutiérrez, J.R. Torregrosa, Complex dynamics of derivative-free methods for nonlinear equations, Appl. Math. Comput. 219 (2013) 7023–7035.
[6] C. Chun, B. Neta, An analysis of a new family of eighth-order optimal methods, Appl. Math. Comput. 245 (2014) 86–107.
[7] C. Chun, B. Neta, An analysis of a family of Maheshwari-based optimal eighth order methods, Appl. Math. Comput. 253 (2015) 294–307.
[8] C. Chun, B. Neta, An analysis of a King-based family of optimal eighth-order methods, Am. J. Algorithms Comput. 2 (2015) 1–17.
[9] C. Chun, M.Y. Lee, B. Neta, J. Džunić, On optimal fourth-order iterative methods free from second derivative and their dynamics, Appl. Math. Comput. 218 (2012) 6427–6438.
[10] C. Chun, B. Neta, J. Kozdon, M. Scott, Choosing weight functions in iterative methods for simple roots, Appl. Math. Comput. 227 (2014) 788–800.
[11] C. Chun, B. Neta, S. Kim, On Jarratt's family of optimal fourth-order iterative methods and their dynamics, Fractals 22 (2014) 1450013, http://dx.doi.org/10.1142/S0218348X14500133.
[12] A. Cordero, J. García-Maimó, J.R. Torregrosa, M.P. Vassileva, P. Vindel, Chaos in King's iterative family, Appl. Math. Lett. 26 (2013) 842–848.
[13] Y.H. Geum, Y.I. Kim, B. Neta, On developing a higher-order family of double-Newton methods with a bivariate weighting function, Appl. Math. Comput. 254 (2015) 277–290.
[14] A.A. Magreñan, Different anomalies in a Jarratt family of iterative root-finding methods, Appl. Math. Comput. 233 (2014) 29–38.
[15] T. Murakami, Some fifth order multipoint iterative formulae for solving equations, J. Inf. Process. 1 (1978) 138–139.
[16] B. Neta, Numerical Methods for the Solution of Equations, Net-A-Sof, California, 1983.
[17] B. Neta, M. Scott, C. Chun, Basin of attractions for several methods to find simple roots of nonlinear equations, Appl. Math. Comput. 218 (2012) 10548–10556.
[18] B. Neta, C. Chun, M. Scott, Basins of attraction for optimal eighth order methods to find simple roots of nonlinear equations, Appl. Math. Comput. 227 (2014) 567–592.
[19] A.M. Ostrowski, Solution of Equations in Euclidean and Banach Space, Academic Press, New York, 1973.

[20] M.S. Petković, B. Neta, L.D. Petković, J. Džunić, Multipoint Methods for Solving Nonlinear Equations, Elsevier, Waltham, MA, 2013.
[21] M. Scott, B. Neta, C. Chun, Basin attractors for various methods, Appl. Math. Comput. 218 (2011) 2584–2599.
[22] B.D. Stewart, Attractor basins of various root-finding methods, M.S. thesis, Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA, June 2001.
[23] J.F. Traub, Iterative Methods for the Solution of Equations, Chelsea Publishing Company, New York, 1977.