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A New Family of Higher Order Methods for Solving Equations

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Neta's three step sixth order family of methods for solving nonlinear equations require 3 function and 1 derivative evaluation per iteration. Using exactly the same information another three step method can be obtained with convergence rate 10.81525 which is much better than the sixth order.

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INTRODUCTION

Neta [1] developed a sixth order family of three-step methods for solving the nonlinear equation $f(x)=0$. The method requires three evaluations of the function and one evaluation of the derivative per iteration. Given x_n, evaluate w_n by Newton's method

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

then evaluate z_n and x_{n+1} by a modified Newton

$$z_n = w_n - \frac{f(w_n)}{f'(x_n)} \frac{f(x_n)+Af(w_n)}{f(x_n)+(A-2)f(w_n)} \tag{2}$$

$$x_{n+1}=z_n-\frac{f(z_n)}{f'(x_n)}\frac{f(x_n)-f(w_n)}{f(x_n)-3f(w_n)}. \tag{3}$$

The error term is given by

$$\varepsilon_{n+1}=\tfrac{1}{144}[2F_3^2F_2-3(2A+1)F_2^3F_3]\varepsilon_n^6+\cdots, \tag{4}$$

where

$$F_i=\frac{f^{(i)}(\xi)}{f'(\xi)}, \tag{5}$$

$$\varepsilon_n=\varepsilon(x_n)=x_n-\xi. \tag{6}$$

The parameter A can be chosen equal to -1 so that the multiplier in the last two steps is the same, or so as to minimize the error constant.

In this note we construct a three step method using the same information and having order 10.81525.

CONSTRUCTION OF THE METHOD

Note that in the last step (3) one uses the values of f at 3 points x_n, w_n, z_n and the values of the derivative at x_n. Suppose we now construct the first two steps in a similar fashion, namely, use the value of f at 3 previously computed points and the value of f' at only one previous point. Let w_n be computed based on the values of f at x_n, z_{n-1}, w_{n-1} and the value of f' at x_n. Let z_n be computed based on the values of f at w_n, x_n, z_{n-1} and the value of f' at x_n. Clearly, the information used is the same as that of the sixth order method except that we need three starting values x_0, z_{-1}, w_{-1}.

We use inverse interpolation to compute w_n. Let

$$R(f(x))=a+b(f(x)-f(x_n))+c(f(x)-f(x_n))^2 + d(f(x)-f(x_n))^3 \tag{7}$$

be a polynomial of degree three satisfying

$$x_n = R(f(x_n)), \tag{8a}$$

$$\frac{1}{f'(x_n)} = R'(f(x_n)), \tag{8b}$$

$$w_{n-1} = R(f(w_{n-1})), \tag{8c}$$

$$z_{n-1} = R(f(z_{n-1})). \tag{8d}$$

It is easy to see from (8a) and (8b) that

$$a = x_n, \qquad b = \frac{1}{f'(x_n)}. \tag{9}$$

Thus, if we use the notations

$$\sigma = \sigma_n - x_n, \qquad F_\sigma = f(\sigma_n) - f(x_n), \qquad \phi_\sigma = \frac{\sigma}{F_\sigma^2} - \frac{1}{F_\sigma f'(x_n)}, \tag{10}$$

for $\sigma_n = w_{n-1}$, z_{n-1}, then (8c) and (8d) will yield

$$c + dF_w = \phi_w, \qquad c + dF_z = \phi_z. \tag{11}$$

Solving these equations we have

$$d = \frac{\phi_w - \phi_z}{F_w - F_z}, \qquad c = \frac{F_w \phi_z - \phi_w F_z}{F_w - F_z}. \tag{12}$$

Thus, after rearrangement,

$$w_n = R(0) = x_n - \frac{f(x_n)}{f'(x_n)} + (f(w_{n-1})\phi_z - f(z_{n-1})\phi_w) \times \frac{f^2(x_n)}{f(w_{n-1}) - f(z_{n-1})}, \tag{13}$$

where

$$\phi_w = \frac{w_{n-1} - x_n}{(f(w_{n-1}) - f(x_n))^2} - \frac{1}{(f(w_{n-1}) - f(x_n))f'(x_n)},$$
$$\phi_z = \frac{z_{n-1} - x_n}{(f(z_{n-1}) - f(x_n))^2} - \frac{1}{(f(z_{n-1}) - f(x_n))f'(x_n)}. \tag{14}$$

In a similar fashion we obtain z_n. The only difference is that w_{n-1} will be replaced by w_n

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} + (f(w_n)\phi_z - f(z_{n-1})\psi_w)\frac{f^2(x_n)}{f(w_n) - f(z_{n-1})}, \tag{15}$$

where

$$\psi_w = \frac{w_n - x_n}{(f(w_n) - f(x_n))^2} - \frac{1}{(f(w_n) - f(x_n))f'(x_n)}. \tag{16}$$

Similarly for x_{n+1}

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + (f(w_n)\psi_z - f(z_n)\psi_w)\frac{f^2(x_n)}{f(w_n) - f(z_n)}, \tag{17}$$

where

$$\psi_z = \frac{z_n - x_n}{(f(z_n) - f(x_n))^2} - \frac{1}{(f(z_n) - f(x_n))f'(x_n)}. \tag{18}$$

We now show that the method (13), (15), (17) is of order 10.81525. To this end, we shall use Herzberger's [2] matrix method according to which the order of a single step k-point method $x_n = G(x_{n-1}, x_{n-2}, \ldots, x_{n-k})$ is the spectral radius of the matrix M with elements $m_{1,i}$ = amount of information required at point x_{n-i}, $i = 1, 2, \ldots, k$, $m_{i,i-1} = 1$, $i = 2, 3, \ldots, k$, and $m_{i,j} = 0$ otherwise. The order of an s-step method $= G_1 \circ G_2 \circ \cdots \circ G_s$ is the spectral radius of the matrix $M_1 M_2 \ldots M_s$. In our case, $s = 3$, $k = 3$, and

$$M = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 6 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}. \quad (19)$$

The eigenvalues of M are the roots of the cubic polynomial

$$-\lambda^3 + 12\lambda^2 - 13\lambda + 2, \quad (20)$$

which are 1, 10.81525, +0.18475.

Thus, the spectral radius of M is 10.81525 which proves the order of the method.

References

[1] B. Neta, A sixth-order family of methods for nonlinear equations, *Intern. J. Computer Math., Sec. B*, 7 (1979), 157–161.

[2] J. Herzberger, Über Matrixdarstellungen für Iterationsverfahren bei nichtlinearen Gleichungen, *Computing* 12 (1974), 215–222.