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A Higher Order Method for Multiple Zeros of Nonlinear Functions

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A method of order three for finding multiple zeros of nonlinear functions is developed. The method requires two evaluations of the function and one evaluation of the derivative per step.

KEY WORDS: Non-linear equations, order of convergence, multiple root, iteration

C.R. CATEGORIES: 5.1, 5.15

1. INTRODUCTION

Newton's method for computing a zero ξ of multiplicity m of a nonlinear equation $f(x)=0$ is of order one. There is a well-known modification (see e.g. Rall [4] or Schröder [5, p. 324]) of the method to obtain a second order. This uses a knowledge of the multiplicity.

Here we develop a method of order three. An iteration consists of one Newton substep followed by a substep of "modified" Newton (i.e., using the derivative of f at the first substep instead of the current one). The research for this article was motivated by the research of the second author in [2], where a sixth-order method was derived for simple roots, and required three evaluations of the given function and only one evaluation

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of the derivative per step. Indeed, the research in this paper represents initial endeavors in extending the analysis for higher order methods in [2–3] to include the case of multiple roots of either nonlinear functions or of Jacobian determinants of nonlinear systems.

Let us recall the definition of order (see e.g. [2]).

DEFINITION Let $x_1, x_2, \ldots, x_i$, be a sequence converging to ξ. Let $\varepsilon_i = x_i - \xi$. If there exist a real number p and a nonzero constant C such that

$$\frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} \to C \tag{1}$$

then p is called the order of the sequence.

2. DEVELOPMENT OF THE THIRD-ORDER METHOD

Let $m > 1$ be the multiplicity of the root ξ to be computed, and suppose f has a least $m+2$ derivatives in a neighborhood of ξ. The ith derivative of our function f is denoted by $f^{(i)}$.

Let

$$\left.\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= w_n - \frac{f(w_n)f(x_n) + Af(w_n)}{f'(x_n)f(x_n) + Bf(w_n)} \end{aligned}\right\} \tag{2}$$

where A, B are arbitrary constants. This is a family of methods which uses Newton's method in the first substep and a Newton-like in the second one. In each step we have to evaluate the function $f(x)$ at the two points x_n, w_n and to evaluate the derivative at one point x_n.

In order to find the order of the method we used MACSYMA (Project MAC's SYmbolic MAnipulation system, which is a large computer programming system written in LISP and used for performing symbolic as well as numerical mathematical manipulation [1]).

The error expression at w_n is given by

$$\begin{aligned} \varepsilon(w_n) = w_n - \xi &= \frac{m-1}{m}\varepsilon_n + \frac{F_2}{m^2(m+1)}\varepsilon_n^2 \\ &\quad - \left(\frac{F_2^2}{m^3(m+1)} - \frac{2F_3}{m^2(m+1)(m+2)}\right)\varepsilon_n^3 + \ldots, \end{aligned} \tag{3}$$

where

$$F_i = \frac{f^{(i+m-1)}(\xi)}{f^{(m)}(\xi)}, \tag{4}$$

and

$$\varepsilon_n = \varepsilon(x_n) = x_n - \xi. \tag{5}$$

For later use we also give the expression for

$$f(w_n)[f(x_n) + Af(w_n)] = \frac{\mu^m(1+A\mu^m)}{m!^2}\,\varepsilon_n^{2m}\left\{1 + \varepsilon_n\left[\frac{F_2(2m^2-2m+1)}{m(m^2-1)} + \frac{F_2 A\mu^m}{m(m^2-1)(1+A\mu^m)}\right] + \dots\right\}, \tag{6}$$

and

$$f'(x_n)[f(x_n) + Bf(w_n)] = \frac{B\mu^m+1}{m!(m-1)!}\,\varepsilon_n^{2m-1}\left\{1 + \varepsilon_n\left[\frac{F_2(2m^2-m-1)}{m(m^2-1)(1+B\mu^m)} + \frac{F_2 B\mu^m(2m^2-m)}{m(m^2-1)(1+B\mu^m)}\right] + \dots\right\}, \tag{7}$$

where

$$\mu = \frac{m-1}{m}. \tag{8}$$

The error expression at the point x_{n+1} is given by

$$\varepsilon_{n+1} = \left[\mu - \frac{\mu^m(1+A\mu^m)}{m(1+B\mu^m)}\right]\varepsilon_n + \frac{F_2}{m^2(m+1)}\varepsilon_n^2$$

$$+\frac{\mu^m(1+A\mu^m)}{m(1+B\mu^m)}\left[\frac{(m-2)F_2}{m(m^2-1)} - \frac{\mu A F_2}{m(m^2-1)(1+A\mu^m)} + \frac{\mu B F_2}{m(m^2-1)(1+B\mu^m)}\right]\varepsilon_n^2 + \dots$$

The expression for the coefficient for ε_n^3 is too long (over 2 pages) to copy from the computer output.

In order to annihilate the coefficient of ε_n the parameters A, B should satisfy:

$$\mu = \frac{\mu^m(1+A\mu^m)}{m(1+B\mu^m)}. \tag{10}$$

Combining (9) and (10) one has

$$\varepsilon_{n+1} = \frac{m-1}{m^2(m+1)} F_2 \varepsilon_n^2 - \frac{F_2}{m^2(m+1)} \frac{m\mu^{1-m}-1}{m\mu^{1-m}(1+B\mu^m)} \varepsilon_n^2 + \dots \tag{11}$$

To annihilate the second order terms one needs

$$B = -\frac{\left(\frac{m}{m-1}\right)^m (m-2)(m-1)+1}{(m-1)^2}. \tag{12}$$

Substituting this value of B in (10) one obtains

$$A = \left(\frac{m}{m-1}\right)^{2m} - \left(\frac{m}{m-1}\right)^{m+1}. \tag{13}$$

Note that A, B depend on m. The knowledge of m in this case increased the order to three. For a discussion of how to estimate the multiplicity m, see for example Traub [6, pp. 129–130].

3. NUMERICAL EXAMPLES

In our first experiment we obtained the double root $\xi = 1$ of the function

$$f(x) = x^2 - 2x + 1 \tag{14}$$

using our method and the following modified Newton method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}. \tag{15}$$

Our method obtained the root in 1 iteration using a starting value $x_0 = 0$. The modified Newton method (15) required 2 interations.

The second experiment was to obtain a triple root at $\xi=1$ of the function

$$f(x)=x^5-8x^4+24x^3-34x^2+23x-6. \tag{16}$$

Again, we used $x_0=0$ as starting value. The convergence of our method was so fast that we were not able to measure the rate of convergence. The results were obtained in quadruple precision on ITEL AS-6 and are summarized in Table I.

TABLE I

Iteration number	x		Absolute Error	
	Newton	This paper	Newton	This paper
0	0.	0.	1.	1.
1	0.7826087	0.9294938	0.2173913	0.0705062
2	0.9816479	0.9999038	0.0183521	0.0000962
3	0.9998356	1.0000000	0.0001644	0.
4	1.0000000	–	0.	–

TABLE II

x_{old}	x_{new}	$\lvert x_{\text{old}}-1\rvert^3$	$\lvert x_{\text{new}}-1\rvert$
0.	0.9294938	1.	$0.705062.10^{-1}$
0.9294938	0.9999038	$0.350486.10^{-3}$	$0.962.10^{-4}$
0.9999038	1.0000000	$0.890277.10^{-12}$	0.

In Table II we compute $|x_{\text{old}}-1|^3$ and $|x_{\text{new}}-1|$ for the second problem. Note that our method produced an error which is smaller than the cube of the error at the previous step.

In our next experiment we were able to measure the rate of convergence. The function

$$f(x)=3x^4+8x^3-6x^2-24x+19 \tag{17}$$

has a double root at $\xi=1$. Starting with $x_0=0$ our method used 2/3 the number of iterations required by the modified Newton. The results are summarized in Table III.

TABLE III

Iteration number	x Newton	x This paper	rate Newton	rate This paper
0	0.	0.	—	—
1	1.583333	0.8904491	—	—
2	1.071987	0.9998828	—	—
3	1.001386	1.000000	1.7500	3.2624
4	1.000001	1.000000	1.9850	—
5	1.000000	—	2.0002	—
6	1.000000	—	—	—

In our last experiment we compare our method to Werner's [7] for approximating a double root at $\xi=0$ of

$$f(x)=x^2\exp(x). \tag{18}$$

The results are summarized in Table IV.

TABLE IV

Iteration number	x Werner	x This paper		rate Werner	rate This paper	
0	0.2	0.2	0.1	—	—	—
1	$0.2160.10^{-1}$	$0.1487.10^{-2}$	0.1141	2.383	4.045	—
2	$0.2379.10^{-3}$	$0.6844.10^{-9}$	$0.2904.10^{-3}$	2.176	3.241	3.752
3	$0.2830.10^{-7}$	—	$0.5103.10^{-11}$	2.083	—	3.193

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