# TWO-PERSON ZERO-SUM GAMES FOR NETWORK INTERDICTION

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A single evader attempts to traverse a path between two nodes in a network while a single interdictor attempts to detect the evader by setting up an inspection point along one of the network arcs. For each arc there is a known probability of detection if the evader traverses the arc that the interdictor is inspecting. The evader must determine a probabilistic "path-selection" strategy which minimizes the probability of detection while the interdictor must determine a probabilistic "arc-inspection" strategy which maximizes the probability of detection. The interdictor represents, in a simplified form, U.S. and allied forces attempting to interdict drugs and precursor chemicals as they are moved through river, road, and air routes in Latin America and the Caribbean. We show that the basic scenario is a two-person zero-sum game that might require the enumeration of an exponential number of paths, but then show that optimal strategies can be found using network flow techniques of polynomial complexity. To enhance realism, we also solve problems with unknown origins and destinations, multiple interdictors or evaders, and other generalizations.

This paper investigates game-theoretic models for the following scenario and a number of generalizations: Each day, an "evader" selects and attempts to traverse a path through a network from node $s$ to node $t$, without being detected by an "interdictor." Each day, the interdictor selects an arc $k$ in the network and sets up an inspection site there. If the evader traverses arc $k$, he is detected with probability $p_k$; otherwise, he goes undetected. Both the evader and the interdictor know the detection probability for each arc in the network. The problem for the interdictor is to find a probabilistic "arc-inspection strategy" which maximizes the average probability of detecting the evader called the *interdiction probability*, while the problem for the evader is to find a "path-selection strategy" which minimizes the interdiction probability. This problem fits into the form of a two-person zero-sum game.

The basic problem was investigated in a military context by Wollmer (1964) who described, without proof, the optimal strategy for the interdictor; he did not describe the optimal strategy for the evader. Interest in network interdiction has been revived because of the United States' drug interdiction efforts in the 1980s and 1990s, primarily those aimed at cocaine coming from Latin America. Law enforcement agencies, such as the U.S. Customs Service and the Drug Enforcement Agency, naturally took the lead in drug interdiction, but the Department of Defense (DoD) became significantly involved starting with the Defense Authorization Act of 1989 (U.S. Public Law 100-456, 1989). This act allows the use of DoD assets in the detection and tracking of drug smugglers although not directly in their arrest.

Much research on drug interdiction, as opposed to network interdiction, has focused on estimating changes in the probabilities of detecting drug smugglers when interdiction assets are manipulated in various ways, e.g., Mitchell and Bell (1980), and Godshaw, Pancoast and Koppel (1987). More recently, Caulkins, Crawford and Reuter (1993) used an adaptive simulation to investigate the effect of changing interdiction efforts on drug smuggling from South America into the United States. The simulation is "adaptive" in that drug smugglers learn about and react to past interdictions. Also, the objective of the drug smugglers is to maximize profit, not just to minimize probability of detection.

The adaptive nature of the simulation created by Caulkins, Crawford and Reuter represents an important aspect of drug interdiction, namely, that drug smugglers must be considered to be intelligent adversaries who know or can learn about an interdictor's strategy. Our game-theoretic models take this explicitly into account. The evader actually represents drug smugglers who are dispatched periodically on specific routes by an intelligent employer or "narco." The narco knows the optimal strategy of the interdictor or at least learns about the interdictor's strategy by noting the fraction of his smugglers who are caught using a particular route. The interdictor is an intelligent law-enforcement force that knows it must move its inspection site in a randomized fashion to avoid being predictable. In our multiple interdictor

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models, the interdictors can represent assets such as aircraft which are dispatched by a single intelligent "coordinator" such as JTF-4 (Joint Task Force Four) which schedules U.S. drug surveillance aircraft in the Caribbean and Central America (Dettbarn 1993).

The models mentioned above considered certain geographic aspects of drug smuggling but only in terms of generic routes between supply and demand points. Steinrauf (1991) identifies the need for detailed modeling of the movement of drugs along a network of roads and rivers in parts of South America. Steinrauf (1991) and Wood (1993) describe a number of deterministic network interdiction models (integer programming models) which, given limited assets, minimize the maximum flow of a commodity through a network by selectively destroying arcs—these models might be useful within a simulation for some of the scenarios we are considering, but are probably more relevant to short-term, war-fighting situations. The game-theoretic models we develop here are appropriate for modeling drugs being transported through networks and could be applied to models using generic "routes."

A number of generalizations to the basic model are developed which address practical aspects of drug interdiction, such as multiple interdictors and unknown origin and destination nodes. This paper, however, is largely theoretical in nature; it describes only the general structure of real-world optimal strategies. We seek to gain insight into what optimal or near-optimal strategies should look like, hoping that this will lead to better deployment of interdiction assets whether or not a formal model is used.

In the next section, we define network concepts and notation which will be used throughout the paper. In Section 2, we describe the basic network interdiction problem as an unwieldy, two-person zero-sum matrix game and then show how it can be solved efficiently using wieldy network flow techniques. The basic structures of the evader's and interdictor's optimal strategies are then derived. Section 3 makes simple extensions of the basic model, for instance, to cases where the origin and destination nodes are not known with certainty. Section 4 develops more complicated extensions of the basic model to include multiple interdictors. Section 5 summarizes our results and points out areas for future research.

## 1. DEFINITIONS AND NOTATION

Let $G = (N, A)$ denote a directed network with node set $N$ and arc set $A$. We will usually refer to an arc by its number $k$ although it can be represented by the ordered pair $(i(k), j(k))$ or simply $(i, j)$, where $i$ is the *tail* of the arc and $j$ is the *head* of the arc. The set of arcs directed out of a node $i$ is the *forward star* of $i$, denoted $FS(i)$, while the set of arcs directed into node $i$ is the *reverse star* of $i$, denoted $RS(i)$.

A *path* in network $G$, starting at node $i_0$ and ending at node $i_m$, is a sequence of nodes and arcs of the form $i_0$, $(i_0, i_1)$, $i_1$, $(i_1, i_2)$, $\ldots$, $i_{m-1}$, $(i_{m-1}, i_m)$, $i_m$. When the start and end nodes of the path are of particular importance, the path can be referred to as an *s-t path*, where $s = i_0$ and $t = i_m \neq i_0$. The path is a *simple path* if all nodes are unique. A path is a *cycle* if $i_0 = i_m$ and it is a *simple cycle* if all nodes are unique except that $i_0 = i_m$. The set $L$ will denote the set of all simple $s$-$t$ paths, and in some cases, a path $l \in L$ will be denoted by its arc set $A(l)$. We define $D$ to be the *arc-path incidence matrix with respect to L* by

$$d_{kl} = \begin{cases} 1 & \text{if path } l \in L \text{ includes arc } k \\ 0 & \text{otherwise.} \end{cases}$$

The $l$th column of $D$, denoted $\mathbf{d}(l)$, is the *arc-path incidence vector* for path $l$.

It will also be useful to view $G$ as a "flow network" with a single commodity flowing through its arcs. Associated with each node $i$ is a value $b_i$ which represents an exogenous source of the commodity if $b_i > 0$, or an exogenous demand if $b_i < 0$. Letting $y_k$ be the flow of the commodity on arc $k$, the standard flow balance equations for $G$ are

$$\sum_{k \in FS(i)} y_k - \sum_{k \in RS(i)} y_k = b_i \quad \text{for all } i \in N$$

$$y_k \geq 0 \quad \text{for all } k \in A$$

which we represent more succinctly as

$$\mathbf{F}\mathbf{y} = \mathbf{b}$$
$$\mathbf{y} \geq \mathbf{0}.$$

Associated with each arc $k$ is a *probability of detection* $p_k$, which is the probability that an evader traversing arc $k$ will be detected if the interdictor inspects that arc. We assume that $p_k > 0$ for all arcs $k$ to avoid tedious details when $p_k = 0$ is allowed. If $\mathbf{p}$ is the $n$-vector of detection probabilities, then $P \equiv \text{diag}(\mathbf{p})$.

Given two distinct nodes in $G$, $s$ and $t$, an *s-t cut* $C$ is a partition of $N$ into two subsets $N_s$ and $N_t$ such that $s \in N_s$ and $t \in N_t$. With respect to that cut, an arc is a *forward arc* if it is directed from a node in $N_s$ to a node in $N_t$ and it is a *backward arc* if it is directed from a node in $N_t$ to a node in $N_s$. When each arc $k$ has a *capacity* $u_k > 0$, which is an upper bound on arc-flow $y_k$, the capacity of a cut $C$ is $\sum_{k \in A_C} u_k$, where $A_C$ is the set of forward arcs of $C$.

## 2. THE BASIC MODEL

Our interest is in solving a two-person zero-sum game $Q$ where the evader's (**P2**'s) pure strategies select simple $s$-$t$ paths $l$ to traverse, and the interdictor's (**P1**'s) pure strategies select single arcs to inspect. Define the vector $\mathbf{z}$ such that $z_k = 1$ if **P1** inspects arc $k$ and $z_k = \mathbf{0}$ otherwise. Then, the payoff function for $Q$ is

$$V(\mathbf{z}, l) = \sum_{k \in A(l)} p_k z_k$$

which is the probability that **P2** is detected by **P1**. The expected value of $V(\mathbf{z}, l)$, denoted $\phi$, is the interdiction probability. For **P1**, the objective of the game is to maximize $\phi$ by determining a randomized, i.e., a mixed strategy for inspecting an arc. For **P2**, the objective is to minimize $\phi$ by developing a mixed strategy for selecting a path $l$. Let $x_k$ be the probability that **P1** inspects arc $k$ and let $\hat{y}_l$ be the probability that **P2** selects path $l$ so that the vectors $\mathbf{x}$ and $\hat{\mathbf{y}}$ represent **P1**'s and **P2**'s mixed strategies, respectively. Since interdiction occurs only if **P1**'s arc $k$ is on **P2**'s path $l$, and even then only with probability $p_k$, we have

$$\phi = E(V(\mathbf{z}, l)) = \sum_{k \in A} \sum_{l \in L} x_k p_k d_{kl} \hat{y}_l = \mathbf{x} PD\hat{\mathbf{y}}.$$

This problem fits into the form of a matrix game $Q$ with matrix $PD$ and can be stated as the "maxmin" problem.

### Maxmin0

$$\begin{aligned} &\max_{\mathbf{x}} \min_{\hat{\mathbf{y}}} \mathbf{x} PD\hat{\mathbf{y}} \\ &\text{subject to} \quad \mathbf{x1} = 1 \\ &\qquad \mathbf{x} \geq \mathbf{0} \\ &\qquad \mathbf{1}\hat{\mathbf{y}} = 1 \\ &\qquad \hat{\mathbf{y}} \geq \mathbf{0}. \end{aligned}$$

Being a finite matrix game, $Q$ could be solved by solving the following linear program. (For example, see Owen 1982, pp. 34–41, or simply fix $\hat{\mathbf{y}}$ in **Maxmin0** and take the dual with respect to $\mathbf{x}$.)

### Problem LP1

$$\begin{aligned} &v^* = \min_{\hat{\mathbf{y}}, v} v \\ &\text{subject to} \quad PD\hat{\mathbf{y}} - \mathbf{1}v \leq \mathbf{0} \\ &\qquad \mathbf{1}\hat{\mathbf{y}} = 1 \\ &\qquad \hat{\mathbf{y}} \geq \mathbf{0}, \end{aligned}$$

where $\mathbf{1}v$ denotes a column vector of 1s, all of whose elements are multiplied by $v$. The $k$th constraint $(PD\hat{\mathbf{y}})_k - v \leq \mathbf{0}$ states that the interdiction probability must not exceed $v$ no matter which arc is chosen by **P1**. The optimal dual variables for those constraints yield **P1**'s optimal strategy.

It will be useful to rewrite **LP1** in terms of the *arc-traversal probabilities* $\mathbf{y} \equiv D\hat{\mathbf{y}}$ as follows:

### Problem LP1a

$$\begin{aligned} &v^* = \min_{\mathbf{y}, v} v \\ &\text{subject to} \quad P\mathbf{y} - \mathbf{1}v \leq \mathbf{0} \\ &\qquad \mathbf{y} \in \mathbf{Y}^1, \end{aligned}$$

where $Y^1 = \{\mathbf{y} | \mathbf{y} = D\hat{\mathbf{y}}, \mathbf{1}\hat{\mathbf{y}} = 1, \hat{\mathbf{y}} \geq \mathbf{0}\}$. The problem with solving **LP1**, or **LP1a**, is that the number of variables in $\hat{\mathbf{y}}$, i.e., the number of simple $s$-$t$ paths in the network $G$, may grow exponentially in $|A|$. We overcome this difficulty by replacing $Y^1$ in **LP1a** with an easily stated relaxation $Y^2$, and show that the resulting LP can be used to solve **LP1** efficiently. We say that a candidate vector of arc-traversal probabilities $\mathbf{y}$, derived by any means, is *playable* if $D\mathbf{y} = \hat{\mathbf{y}}$ for some probability distribution $\hat{\mathbf{y}}$ over the set of simple $s$-$t$ paths $L$.

**Lemma 1.** *Necessary conditions for* $\mathbf{y}$ *to be playable are* $F\mathbf{y} = \mathbf{b}$ *and* $\mathbf{y} \geq 0$, *where* $b_s = 1$, $b_t = -1$ *and* $b_i = 0$ *for all* $i \in N - s - t$.

**Proof.** Since every path in $L$ leaves node $s$ once, enters node $t$ once, and leaves every other node as often as it enters, the probability of leaving $s$ must be 1, the probability of entering $t$ must be 1, and the probability of entering any other node must be the same as the probability of leaving that node. It must, therefore, be true of any arc-traversal probabilities $\mathbf{y}$ that $\mathbf{Fy} = \mathbf{b}$ and $\mathbf{y} \geq \mathbf{0}$.

Lemma 1 implies that $Y^1 \subseteq Y^2$, where $Y^2 = \{\mathbf{y} | F\mathbf{y} = \mathbf{b}, \mathbf{y} \geq \mathbf{0}\}$. The relationship can be strict because $\mathbf{y} \in Y^2$ may contain some flow around a cycle that is infeasible in $Y^1$, i.e., the conditions of Lemma 1 are not sufficient for playability. So, a relaxed version of **LP1** is as follows.

### Problem LP2

$$\begin{aligned} &v^{**} = \min_{\mathbf{y}, v} v \\ &\text{subject to} \quad P\mathbf{y} - \mathbf{1}v \leq \mathbf{0} \\ &\qquad \mathbf{y} \in Y^2. \end{aligned}$$

To use **LP2** to solve **LP1**, we must first solve **LP2**. Divide the $k$th constraint of $P\mathbf{y} - \mathbf{1}v \leq \mathbf{0}$ by $p_k$, and define $u_k = p_k^{-1}$. Then, **LP2** becomes:

### Problem LP2a

$$\begin{aligned} &v^{**} = \min_{\mathbf{y}, v} v \\ &\text{subject to} \quad F\mathbf{y} = \mathbf{b} \\ &\qquad \mathbf{y} \leq \mathbf{u}v \\ &\qquad \mathbf{y} \geq \mathbf{0}, \end{aligned}$$

where $\mathbf{u}v$ is the column vector $\mathbf{u}$, each of whose elements is multiplied by $v$. This problem has a simple interpretation: Each arc $k$ has a capacity of $u_k v$ and the problem is to reduce these capacities by decreasing $v$ to the optimal value $v^{**}$, where the problem of getting one unit of flow from $s$ to $t$ is just feasible. From the max-flow min-cut theorem (Ford and Fulkerson 1962) the problem is feasible as long as the capacity of the minimum capacity cut is at least 1. Since $v$ is just a linear scale factor, the minimum capacity cut is minimum for any $v > 0$, for example, $v = 1$. Therefore, the problem can be solved as follows: Create the max flow problem from $s$ to $t$ using arc capacities $\mathbf{u}$. Solve this problem to find the maximum flow value $f$ and obtain the maximum flow vector $\mathbf{y}^{++}$ and the forward arcs $A_C$ of some minimum capacity cut $C$. All of this is easy to do using standard maximum flow techniques, e.g., see Ahuja, Magnanti and Orlin 1993, pp. 177–185 and pp. 213–218. Then, scale $\mathbf{y}^{++}$ to obtain the optimal solution to **LP2**, $\mathbf{y}^{**} = \mathbf{y}^{++} f^{-1} = \mathbf{y}^{++}(1/$

$\sum_{k \in A_C} u_k$) and $v^{**} = f^{-1} = 1/\sum_{k \in A_C} u_k$. It is now easy to prove the following theorem.

**Theorem 1.** *The optimal solution values of* **LP1** *and* **LP2** *are equal.*

**Proof.** There always exists a maximum flow vector $\mathbf{y}^+$ which solves **LP2a** and can be expressed in terms of only simple paths, i.e., there always exists $\hat{\mathbf{y}}^+ \geq \mathbf{0}$ such that $\mathbf{1}\hat{\mathbf{y}}^+ = f$ and $\mathbf{y}^+ = D\hat{\mathbf{y}}^+$ is a maximum flow of value $f$ (Lawler 1976, pp. 119–120). Thus, we can replace $\mathbf{y}^{++}$ in the discussion above with $\mathbf{y}^+$, let $\mathbf{y}^* = \mathbf{y}^+ f^{-1}$, and let $\hat{\mathbf{y}}^* = \hat{\mathbf{y}}^+ f^{-1}$. Then, $\hat{\mathbf{y}}^*$ is a feasible solution for **LP1** with solution value $v^*$, and $\mathbf{y}^*$ solves **LP2** with optimal solution value $v^*$. Since **LP2** is a relaxation of **LP1** the optimal solutions of **LP1** and **LP2** must be equal.

So, the solution value of **LP2** yields the value of $Q$ although optimal strategies for **P1** and **P2** are not yet apparent.

We next describe how to efficiently construct a compact representation of an optimal solution to **LP1a**, and thus an optimal strategy for **P2**, starting with an optimal solution to **LP2**, $\mathbf{y}^{**}$. Ford and Fulkerson show that any feasible flow such as $\mathbf{y}^{**}$ can be decomposed into a set of (simple) path flows and (simple) cycle flows of cardinality at most $|A|$. This can be accomplished using a simple search algorithm on $G$ with respect to $y^{**}$. For instance, a depth-first search can be started at $s$ and continued along arcs with positive flow until a simple path from $s$ to $t$ is identified or a cycle is identified. If an $s$-$t$ path $l$ with path-arc incidence vector $\mathbf{d}(l)$ is found, the path is noted and given a path flow value $\hat{y}_l^*$ equal to the minimum flow over all arcs in that path. Then, the arc flows on that path are all reduced by $\hat{y}_l^*$. If a cycle is found, the cycle could be identified, but flow around a cycle does not contribute to a (scaled) maximum flow and it suffices for our purposes to reduce the flow on all arcs on the cycle by the minimum flow over all arcs in the cycle. (In fact, we can arrange the sequential search so that cycles are ignored entirely. Even using the search described, not all cycle flows may be found which is fine because cycle flows are irrelevant.) After identifying a path or cycle, the flow on at least one arc will go to 0 and thus, $|L'| + |W'| \leq |A|$, where $L'$ is the set of paths enumerated and $W'$ is the set of cycles enumerated before the total flow on all $s$-$t$ paths is reduced to 0. The arc flows $\mathbf{y}^*$ represented by the path flows, i.e., $\mathbf{y}^* = \sum_{l \in L'} \mathbf{d}(l)\hat{y}_l^*$ must move one unit of flow from $s$ to $t$ and $\sum_{l \in L'} \hat{y}_l^* = 1$. Additionally, $y_k^* = y_k^{**}$ for all $k \in A_C$ because there can be no flow around a cycle that includes an arc in a minimum capacity cut. Thus, it follows that $\mathbf{y}^*$ solves **LP1a** and the $\hat{y}_l^*$ are path-selection probabilities for the "small" (polynomial in $|A|$) set of paths $L'$ and the corresponding path-selection strategy is optimal for **P2**. For the rest of this section the vector $\mathbf{y}^*$ will denote a playable solution of **LP2** for **P2**, i.e., a solution to **LP1a**. Note that $\mathbf{y}^*$ and the optimal path-selection strategy $\hat{\mathbf{y}}^*$ can be obtained efficiently, that is, in polynomial time. This is true because a maximum flow can be obtained in polynomial time and the path-extraction process will require at most $O(|A|^2)$ work because at most $|A|$ paths plus cycles will be found using a depth-first search, and each search requires $O(|A|)$ work.

We have found one optimal strategy for **P2** which is sufficient for solving Game 1, but there can be many others. The Markovian strategy is one such; it might be of benefit to **P2** in practice because the path need not be decided upon before starting out on a smuggling run and would be less subject to compromise by an informant. We define the Markovian strategy for **P2** with respect to optimal arc-traversal probabilities $\mathbf{y}^*$ as follows: **P1** starts at node $i = s$ and randomly chooses the next arc to traverse $k' \in FS(i)$ using probabilities $\bar{y}_{k'} = y_{k'}^* / \sum_{k \in FS(i)} y_k^*$. He repeats this process at each node until he reaches $t$, which must occur with probability 1. The Markovian strategy can be shown to be optimal if it is based on a vector $\mathbf{y}^*$ that does not have positive flow on all arcs around a cycle. An optimal "acyclic" vector $\mathbf{y}^*$ can always be found by sequentially reducing flows around cycles in a procedure similar to path extraction, or by solving the following pure network LP:

$$\min_{\mathbf{y}} \mathbf{1y}$$

$$\text{subject to} \quad F\mathbf{y} = \mathbf{b}$$

$$\mathbf{y} \leq \mathbf{u}v^*$$

$$\mathbf{y} \geq \mathbf{0}.$$

We will next consider **P1**'s optimal arc-inspection probabilities.

**Theorem 2.** *An optimal strategy for* $\mathbf{x}^*$ *for* **P1** *in* $Q$ *is* $x_k^* = u_k f^{-1} = p_k^{-1} f^{-1}$ *for* $k \in A_C$ *and* $x_k^* = 0$ *for* $k \in A - A_C$, *where* $A_C$ *is any minimum capacity cut.*

**Proof.** Here $\mathbf{x}^*$ is a feasible strategy because $\mathbf{x}^* \geq \mathbf{0}$ and $\mathbf{x}^*\mathbf{1} = 1$. We must show that the expected return for **P1** using $\mathbf{x}^*$ is at least $v^*$ no matter which path is selected by **P2**, i.e.,

$$\sum_{k \in A(l)} p_k x_k^* \geq v^* \quad \text{for all } l \in L.$$

Every path $l$ must include at least one arc $k \in A_C$ because $C$ is a cut. Let $k_l$ be the first such arc. Then,

$$\sum_{k \in A(l)} p_k x_k^* \geq p_{k_l} x_{k_l}^* = p_{k_l}(p_{k_l}^{-1} f^{-1}) = f^{-1} = v^*.$$

Any standard "flow-augmenting path" algorithm for the maximum flow problem (e.g., Edmonds and Karp 1972), identifies both a maximum flow and a minimum capacity cut in polynomial time. Thus, optimal strategies for both **P1** and **P2** can be obtained in polynomial time.

## 3. SIMPLE EXTENSIONS

In this section, we wish to consider the single-evader/single-interdictor problem where some of the assumptions about the form of the problem and/or network are generalized for more realism. For instance, the evader may start and end at any of a number of nodes, or the network may be undirected.

### 3.1. Multiple Sources and Sinks

In practice, the actual source node and/or terminal node for the evader may be unknown. For instance, a source node may be a drug laboratory at any one of a number of possible sites, or sink nodes may be any of a number of border crossing points. Here we consider two variants on this theme.

Assume that the evader may begin, according to his own desires, at any node in a set of nodes $N^s$ and finish at any node in a disjoint set of nodes $N^t$. This problem can be solved as follows: First create $G'$ from $G$ by the usual technique of creating a super source $s'$ and directing artificial arcs from $s'$ to each node in $s'' \in N^s$ and creating a super sink $t'$ and directing artificial arcs from each node $t'' \in N^t$ to $t'$. Then, derive a playable $\mathbf{y}^*$ from a solution to **LP2a** defined on $G'$, where $u_k = \infty$ for artificial arcs (the max-flow problem on $G'$ must still have a finite solution), and then extract paths and path-selection probabilities from $\mathbf{y}^*$, as before, to obtain $\hat{\mathbf{y}}^*$. Flow on an artificial arc $(s', s'')$ corresponds to the probability that the evader will leave from node $s''$, and flow on $(t'', t')$ corresponds to the probability of ending at node $t''$.

If, on the other hand, we know that a fraction $f_i^s$ of all illicit shipments begins at node $i \in N^s$ and that a fraction $f_i^t$ must end at node $i \in N^t$ for logistical reasons, then **P2**'s optimal strategy may be found by first solving **LP2** with $\mathbf{b}$ defined by

$$b_i = \begin{cases} f_i^s & \text{if } i \in N^s \\ -f_i^t & \text{if } i \in N^t \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

to yield solution $\mathbf{y}^{**}$. Then, that $\mathbf{y}^{**}$ is sent into a modified path-decomposition procedure that extracts paths from each node in $N^s$ to any node in $N^t$. At most $|A| + \max\{|N^s|, |N^t|\}$ paths for **P2** will be found in this way. **P1**'s optimal strategy can be determined from the optimal duals of the modified **LP2**. One might speculate that **P1**'s optimal strategy for this problem involves finding a min capacity cut (where the arc capacities are defined as before) with all the source nodes on one side of the cut and all sink nodes on the other side. However, this is not true and it may be advantageous for **P1** to leave certain paths uninspected if only "low capacity" sources and/or sinks are involved.

### 3.2. Undirected Networks and Node Interdiction

In most cases, the network being traversed by the evader is, at least partially, undirected. For instance, a drug smuggler can traverse a section of road in either direction if need be, although drug traffic on certain sections of river may only move downstream. If an arc $k = (i, j)$ in the network is undirected with probability of detection $p_k$, make the standard transformation by replacing it with two directed arcs $k' = (i, j)$ and $k'' = (j, i)$ with $p_{k'} = p_{k''} = p_k$. Then, solve **LP2** for the resulting directed network and derive an optimal acyclic $\mathbf{y}^*$. If $y_{k'}^* > 0$, then $y_{k''}^* = 0$, or vice versa, because $\mathbf{y}^*$ is acyclic. Consequently, **P2** need only traverse an undirected arc $k$ in one direction. On the other hand, by the proof of the max-flow min-cut theorem, if $y_{k'}^* > 0$ and $k'$ is a forward arc on the minimum capacity cut, then $y_{k''}^* = 0$ because $k''$ is a backward arc on that cut. It then follows by duality that $x_{k'}^* > 0$ and $x_{k''}^* = 0$, and thus, **P1** need only look in one direction for **P2**'s approach along such an arc. If the probabilities of detection on an arc are dependent on the direction of travel, upstream or downstream, for instance, make the same substitution as above, but let $p_{k'}$ and $p_{k''}$ be the appropriate, nonequal probabilities. The solution then proceeds as outlined.

In addition to inspecting arcs, **P1** may set up inspection points at nodes in the network, for instance, at road intersections. Assume that we begin with a directed network because otherwise we can apply the conversion described above, and assume that the probability of detection at node $i$ is $p_i$. Using another standard transformation, first replace node $i$ with two nodes $i'$ and $i''$ and direct all arcs that were entering $i$ into $i'$ and direct all arcs that were leaving $i$ out of $i''$. Then, add an arc $k''' = (i', i'')$ with detection probability $p_{k'''} = p_i$, solve the resulting problem as before, and translate the solution back into terms of the original network.

### 3.3. Multiple Evaders

Suppose that the narco is going to send out $r > 1$ smugglers on a single day and assume that multiple smugglers traversing the same arc, even at the same time, are each subject to the arc's probability of detection. If the narco is only concerned with the average number of smugglers detected, he can choose the smugglers' routes independently, just as if they were being sent out on separate days. Thus, the single-evader/single-interdictor model can be used without modification.

However, the narco may have a concave "utility function" which causes him, for instance, to prefer having at least one shipment, of two attempted, successfully delivered with probability 1, rather than having two shipments successfully delivered with probability ½. Assuming that all cuts have a cardinality of at least 2, in this case the shipments should be routed in a dependent fashion, using paths that may have no arcs in common. Although some analysis of this situation is possible without enumerating

$r$-tuples of arc-independent paths, this complication appears to be significant, and we leave the topic for future research.

## 4. MULTIPLE INTERDICTORS AND A SINGLE EVADER

In this section, we discuss interdiction/evasion games where **P1** is not an interdictor, but has $m > 1$ interdictors with which to inspect arcs. As before, we think of **P2** as a single evader. These problems can be solved by enumerating paths $l$ for **P2**, and enumerating "arc-group-inspection" strategies for **P1** which consist of the different ways of assigning $z_k \geq 0$ indistinguishable interdictors to each arc $k \in A$, so that $\sum_k z_k = m$. We will show that difficulties of enumeration can be avoided or much simplified, as in the one interdictor case, under certain conditions.

A variety of payoff functions may be of interest when $m > 1$. For instance, we may be interested in evaluating the average number of detections, or we may be interested in the probability of at least one detection assuming that the $m$ interdictors detect independently of each other, or under some sort of dependence. Additionally, the acceptable strategies for **P1** may be varied. In particular, we are interested in the case where the number of interdictors per arc is limited to one. Justifications for considering some of these variations and notational conventions are given next.

We consider the following payoff functions:

$$AD(\mathbf{z}, l) = \sum_{k \in A(l)} p_k z_k$$

$$IND(\mathbf{z}, l) = 1 - \prod_{k \in A(l)} (1 - p_k)^{z_k}$$

$$MAX(\mathbf{z}, l) = \max_{k \in A(l)} p_k I(z_k),$$

where $I(z_k)$ is an indicator function which is 1 if $z_k \geq 1$ and is 0 otherwise. $AD(\mathbf{z}, l)$ is the average number of detections. The other two payoff functions are best interpreted as the probability of at least one detection, but under different assumptions about statistical independence. The assumption that all interdiction attempts are independent leads to $IND(\mathbf{z}, l)$. Payoff function $MAX(\mathbf{z}, l)$ corresponds to a particular kind of dependence such that $p_k = P(S \geq s_k)$, $S$ being the random "size" of the evader, and $s_k$ being the smallest "size" detectable on arc $k$. In this case, the chances of detection are determined entirely by the interdicted arc in $A(l)$ for which $s_k$ is smallest, or, equivalently, for which $p_k$ is largest. It is easy to show that $MAX(\mathbf{z}, l) \leq IND(\mathbf{z}, l) \leq AD(\mathbf{z}, l)$ with all three functions being equal to $V(\mathbf{z}, l)$ of Section 2 when $m = 1$.

There may be good reasons for restricting the number of interdictors per arc to at most one. Sensors might interfere with each other, or in the military analog of this problem, fratricide could be an issue. In fact, JTF-4 schedules at most one aircraft for drug surveillance on a section of air corridor at any time because detection probabilities would not increase much with additional aircraft (Dettbarn 1993).

The above considerations force consideration of a variety of games. Let $Q_{m,POF}$ denote the game with $m$ interdictors and payoff function $POF$, and let $Q^1_{m,POF}$ denote the same game with the additional restriction that $z_k \leq 1$ on all arcs $k$. The game $Q$ in Section 3 is either $Q_{1,V}$ or $Q^1_{1,V}$ in this generalized notation. Note that $Q_{m,MAX}$ and $Q^1_{m,MAX}$ are essentially the same game because the interdictor gains nothing in $Q_{m,MAX}$ by setting $z_k > 1$.

In the previous section, we essentially argued that the game $Q_{1,V}$ can be solved by solving the following maxmin problem.

**Problem Maxmin1($m$)**

$$\max_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x}P\mathbf{y}$$

subject to $\mathbf{x1} = m$

$\mathbf{x} \geq \mathbf{0}$

$\mathbf{y} \in Y^2$,

where $m = 1$, via **LP2**. Now, if we interpret $x_k$ as the average number of interdictors placed on arc $k$, the solution to **Maxmin1**($m$) solves $Q_{m,AD}$ and can by implemented by sending each evader to arc $k$ independently with probability $x^*_k/m$ and using procedure 1 to extract a set of paths and path-selection probabilities for **P2**. This solution can be obtained by solving **LP2**($m$) which is **LP2** with the objective function $v$ replaced by $mv$. **LP2**($m$) obviously has the same playable primal solution $\mathbf{y}^*$ as **LP2**, but $\mathbf{x}^*$ is $m$ times the value obtained in **LP2**. Therefore, $Q_{m,AD}$ can be solved by simply using the max-flow technique described in the previous section.

On the other hand, $Q_{m,MAX}$ and $Q_{m,IND}$ are potentially much more complicated for $m > 1$ than for $m = 1$. The difficulty in these two cases is that $E(POF) = \mathbf{x}P\mathbf{y}$ only if the probability of more than one interdictor on an arc is 0 for all arcs, a restriction that may not be desirable from **P1**'s standpoint. Pure strategies for **P1** that do not use multiple interdictors per arc are equivalent to subsets of $A$ of size $m$. If there is some way for **P1** to randomly inspect one of those subsets in such a way that the probability of arc $k$ being included is $x_k$ for all $k$, then $\mathbf{x}$ will be said to be *1-playable*. A vector $\mathbf{x}$ satisfying the constraints from **Maxmin1**($m$), i.e., $\mathbf{x1} = m$ and $\mathbf{x} \geq \mathbf{0}$, is called an *m-distribution*, and will be 1-playable if and only if one additional condition is met.

**Lemma 2.** *If* $\mathbf{x}$ *is an* $m$*-distribution on* $A$, *then* $\mathbf{x}$ *is 1-playable if and only if* $\mathbf{x} \leq \mathbf{1}$.

**Proof.** The only-if part is obvious, so assume that $\mathbf{x} \leq \mathbf{1}$. Let $R$ be the set of all subsets of $A$ of size $m$, and let $D'$ be the $|A| \times \binom{|A|}{m}$ matrix such that

$$d'_{ks} = \begin{cases} 1 & \text{element } k \text{ is in subset } s \\ 0 & \text{otherwise.} \end{cases}$$

Then, we must show that there exists a probability distribution $\hat{\mathbf{x}}$ on $R$ such that $D'\hat{\mathbf{x}} = \mathbf{x}$. It suffices to show that there always exists a solution to

$$D'\hat{\mathbf{x}} = \mathbf{x}$$
$$\mathbf{1}\hat{\mathbf{x}} = 1$$
$$\mathbf{0} \leqslant \hat{\mathbf{x}} \leqslant \mathbf{1}.$$

Summing the constraints of $D'\hat{\mathbf{x}} = \mathbf{x}$ and dividing by $m$ gives $\mathbf{1}\hat{\mathbf{x}} = 1$, so the latter constraint is redundant. Given that $\mathbf{1}\hat{\mathbf{x}} = 1$ and $\hat{\mathbf{x}} \geqslant 0$, the constraints $\hat{\mathbf{x}} \leqslant \mathbf{1}$ are also redundant. Thus, we need only show that there is a solution to

$$D'\hat{\mathbf{x}} = \mathbf{x}$$
$$\hat{\mathbf{x}} \geqslant \mathbf{0}.$$

By Farkas' lemma (e.g., Bazaraa, Jarvis and Sherali 1990, pp. 219–220) that system has a solution if and only if

$$\mathbf{w}D' \leqslant \mathbf{0}$$
$$\mathbf{wx} > \mathbf{0}$$

has no solution, or given any $\mathbf{w}$ satisfying $\mathbf{w}D' \leqslant \mathbf{0}$, then $\mathbf{wx} \leqslant 0$. Let $\mathbf{w}$ satisfy $\mathbf{w}D' \leqslant \mathbf{0}$ and assume, without loss of generality, that $w_1 \geqslant w_2 \geqslant \ldots \geqslant w_m$. Then, $\sum_{k=1}^{m} w_k \geqslant \mathbf{wx}$ because $\mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{1}$ and $\mathbf{1x} = m$. But one of the constraints of $\mathbf{w}D' \leqslant \mathbf{0}$ is $\sum_{k=1}^{m} w_k \leqslant 0$, so $\mathbf{wx} \leqslant 0$.

Given that $\mathbf{x}^*$ is 1-playable, solving for $\hat{\mathbf{x}}^*$ can be much simpler than implied by Lemma 2. The system of equations to be solved can be simplified by deleting any row $k$ such that $x_k^* = 0$ and deleting any column such that $d'_{ks} = 1$. Thus, $D'\hat{\mathbf{x}}^* = \mathbf{x}^*$ could effectively be reduced to a system involving $|A_C|$ rows and $\binom{|A_C|}{m}$ variables if, for instance, $x_k^* > 0$ only for arcs in some cut $C$. Furthermore, the solution could be obtained by a column generation scheme using an $|A_C| \times |A_C|$ basis matrix, and as few as $|A_C|$ pure strategies might have to be generated.

Henceforth, $\mathbf{y}^*$ will denote a playable set of arc-traversal probabilities for the LP or maxmin problem at hand. The next theorem states that a solution $(\mathbf{x}^*, \mathbf{y}^*, v^*)$ to **Maxmin1**$(m)$ corresponds directly to a solution of a variety of generalized, multiple-interdictor games.

**Theorem 3.** *If, in a dual and primal solution* $(\mathbf{x}^*, \mathbf{y}^*, v^*)$ *to* **LP2**$(m)$, $\mathbf{x}^* \leqslant \mathbf{1}$ *and* $x_k^* > 0$ *only for arcs* $k \in A_C$ *for some cut* $C$, *then* $\hat{\mathbf{y}}^*$ *and* $\hat{\mathbf{x}}^*$ *are optimal for any game* $Q_{m,POF}$ *as long as*

$$\mathrm{MAX}(\mathbf{z}, l) \leqslant \mathrm{POF}(\mathbf{z}, l) \leqslant \mathrm{AD}(\mathbf{z}, l).$$

**Proof.** Since $\mathbf{y}^*$ is optimal to $Q$ (remember that $\mathbf{y}^*$ solves **Maxmin**$(m)$ and **LP2**$(m)$ for any positive $m$), which has optimal value $m^{-1}v^*$, it follows that $p_k y_k^* \leqslant m^{-1}v^*$ for all $k \in A$. Also,

$$E(AD(\mathbf{z}, l)) = \sum_{k \in A} x_k^* p_k y_k^* \leqslant m^{-1}v^* \sum_{k \in A} x_k^*$$
$$= m^{-1}v^* m = v^*,$$

so **P1** cannot obtain more than $v^*$ if **P2** uses $\mathbf{y}^*$ in the game $Q_{m,AD}$. Since $POF(\mathbf{z}, l) \leqslant AD(\mathbf{z}, l)$, it follows that use of $\mathbf{y}^*$ by **P2** will also guarantee a payoff of at most $v^*$ in $Q_{m,POF}$. Since it is also true that $MAX(\mathbf{z}, l) \leqslant AD(\mathbf{z}, l)$, the theorem will be proved if it can be shown that (a) $\mathbf{x}^*$ is 1-playable, and (b) by playing $\mathbf{x}^*$, **P1** guarantees a payoff of at least $v^*$ in $Q_{1,MAX}$. Part a follows from Lemma 2. To prove part b, first observe that for all $k \in A_C$, there is a path in $L$ that includes arc $k$ and no other arcs in $A_C$. Since $m^{-1}\mathbf{x}^*$ guarantees $m^{-1}v^*$ for **P1** in the game $Q_{1,MAX}$, it follows that $p_k m^{-1} x_k^* \geqslant m^{-1}v^*$, i.e., $p_k x_k^* \geqslant v^*$, for all $k \in A_C$. Every path $l \in L$ must include at least one arc of $A_C$, so let $k_l$ be the first such arc. Then, $MAX(\mathbf{z}, l) \geqslant p_{k_l} I(z_{k_l})$, and since $E(I(z_{k_l})) = x_{k_l}^*$, $E(MAX(\mathbf{z}, l)) \geqslant p_{k_l} x_{k_l}^* \geqslant v^*$.

Unfortunately, Theorem 3 does not always apply. Consider the game $Q_{2,IND}$ on the network of Figure 1a, where $p_1 = p_2 = p_3 = \epsilon$ for some small, positive $\epsilon$, $p_4 = p_5 = 0.5$ and $p_6 = p$. The optimal dual solution to **LP2(2)** for $p \geqslant \epsilon$ is given in the first column of Table I. For $0.25 \leqslant p \leqslant 1$, $\mathbf{x}^*$ is a 1-playable $m$-distribution, and the corresponding solution is optimal. For $\epsilon \leqslant p < 0.25$, $\mathbf{x}^*$ is not an $m$-distribution, it cannot be 1-played, and does not correspond to a solution. However, if $\mathbf{x}^*$ is interpreted as the average number of interdictors per arc, this leads to an approximate solution which can be implemented simply: Place one interdictor on arc 6 with probability 1, and place the other interdictor on arcs 4, 5, and 6 with probabilities $x_4^*$, $x_5^*$ and $x_6^* - 1$, respectively. The second column of the table shows an exact solution for the interdictor for $p \leqslant 0.25$ and $p - p^2 > \epsilon$, where $x_6^* - 1$ can be interpreted as the probability that both interdictors are assigned to arc 6. The exact solution can be implemented as suggested for the approximate solution. The deviation between the optimal and approximate values for $x_k^*$ is modest for this example. For instance, for $p = 0.1$, the approximate solution has $x_4^* = x_5^* = 0.2857$ and $x_6^* = 1.4286$, while the exact solution has $x_4^* = x_5^* = 0.2794$ and $x_6^* = 1.44412$. We mention below the simple method that was used to obtain the exact solution. However, we first consider $Q_{m,POF}^1$ which is of interest because practical considerations may limit the number of interdictors per arc to at most one.

$Q_{m,AD}^1$ is easy to solve. Assuming that $m \leqslant |A|$, just solve **Maxmin2**$(m)$, which we define as **Maxmin1**$(m)$

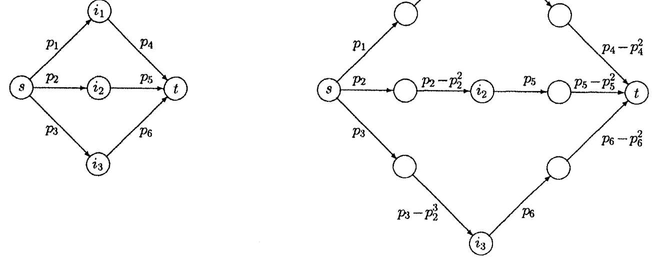


**Figure 1.** Illustrations for example.

**Table I**
Solutions to Example of Figure 1a With $p_1 = p_2 = \epsilon$, $p_4 = p_5 = 0.5$ and $p_6 = p > \epsilon$

| Probabilities | Dual Solution From **LP2(2)** Exact: $0.25 \leq p \leq 1$ Approximate: $\epsilon \leq p < 0.25$ | Exact Solution for $p \leq 25$ and $p - p^2 \geq \epsilon$ |
|---|---|---|
| $x_1^*, x_2^*, x_3^*$ | 0 | 0 |
| $x_4^*, x_5^*$ | $\frac{4p}{1+4p}$ | $\frac{4p-2p^2}{1+4(p-p^2)}$ |
| $x_6^*$ | $\frac{2}{1+4p}$ | $\frac{2-4p^2}{1+4(p-p^2)}$ |
| $x_6^* - 1$ | $\frac{1-4p}{1+4p}$ | $\frac{1-4p}{1+4(p-p^2)}$ |

with the added constraints $\mathbf{x}I \leq \mathbf{1}$. **Maxmin2**$(m)$ can be solved via the related LP (derive this by fixing $\mathbf{y}$ in **Maxmin2**$(m)$ and by then taking the dual of this LP with respect to $\mathbf{x}$):

**Problem LP3($m$)**

$$\min_{\mathbf{y},v,\mathbf{q}} mv + \mathbf{1q}$$

$$\text{subject to} \quad F\mathbf{y} = \mathbf{b}$$

$$\mathbf{1}v + I\mathbf{q} - P\mathbf{y} \geq \mathbf{0}$$

$$\mathbf{y} \geq \mathbf{0}$$

$$\mathbf{q} \geq \mathbf{0}.$$

**LP3**$(m)$ will not typically solve more difficult games like $Q^1_{m,IND}$, however. Consider the solution of **LP3(2)** on the example of Figure 1a with $p_1 = p_2 = p_3 = 0.01$, $p_4 = p_5 = 0.5$, and $p_6 = 0.1$, which has $x_4^* = x_5^* = 0.2115$, $x_6^* = 1$, and $x_3^* = 0.5769$. This solution modestly overestimates the true interdiction probability under independence because the probability of detecting an evader traversing the path $s$, $(s, i_3)$, $i_3$, $(i_3, t)$, $t$ with independent detections is $p_6 + x_3^*(p_6 + p_3 - p_6p_3)$, not the value implied by solving $Q^1_{2,AD}$ via **LP3(2)** which is $p_6 + x_3^*(p_6 + p_3)$. **LP3**$(m)$ could be used to obtain approximate solutions to $Q^1_{m,IND}$ and other games although the accuracy of such approximations would need to be investigated.

**LP3**$(m)$ is actually more likely to be useful for solving $Q_{m,IND}$ rather than $Q^1_{m,IND}$. For instance, we obtained the exact solution listed in column 2 of Table I for the example of $Q_{2,IND}$ using the following method: 1) Replace each arc $k$ with two arcs $k'$ and $k''$ in series and define $p_{k'} = p_k$ and $p_{k''} = p_k - p_k^2$ ($p_{k''}$ is the improvement in detection probability if a second interdictor is assigned to arc $k$) to create the problem of Figure 1b; 2) solve **LP3(2)** on the modified network to obtain optimal primal and dual solutions; 3) define $x_k^* = x_{k'}^* + x_{k''}^*$; 4) if the arcs $k$ such that $x_k^* > 0$ form a cut, the value of the game is optimal; 5) the optimal solution is created by implementing $x_k^*$ as the average number of interdictors assigned to arc $k$, and by extracting path-selection probabilities using the obvious correspondence between the modified and original problems. This method can be proven to be correct and can be generalized to $m$ interdictors and certain other payoff functions.

We note that while the solution techniques described here for multiple interdictor games are not always applicable, the techniques are failsafe and are much more efficient that brute-force enumeration.

## 5. CONCLUSIONS AND RECOMMENDATIONS

In the simplest model described, an evader moves through a network every day from nodes $s$ to $t$ to avoid an interdictor who inspects traffic along one arc of the network each day. Both the evader and the interdictor know the probability of detection $p_k$ on any arc $k$. The problem for the interdictor is to devise a probabilistic arc-inspection strategy which maximizes the expected probability of detecting the evader, i.e., the interdiction probability, while the evader must develop a probabilistic path-selection strategy which minimizes the interdiction probability. This results in a two-person zero-sum game which could be solved as a large matrix game, but we have shown that optimal strategies can be obtained by solving a simple maximum flow problem on the network where the capacity on any arc $k$ is $p_k^{-1}$. The minimum capacity cut $C$ is the key. The interdictor should find $C$ and then inspect arcs in $A_C$ at frequencies that are inversely proportional to the detection probabilities $p_k$, i.e., among this best set of arcs to inspect, the lower the detection probability on an arc, the more often that arc should be inspected. Similarly, the evader should select simple paths so that the probability he traverses any arc $k \in A_C$ in the worst (for him) cut $C$ is inversely proportional to $p_k$, i.e., the less likely he is to be detected on arc $k$, the more often he should traverse that arc—even though the interdictor will be spending more time inspecting that arc.

More realistic models that allow the evader to choose origin and destination nodes, or where the network is undirected, are simple modifications of the basic model. More complicated generalizations, which allow multiple interdictors, are sometimes solvable by the same network flow techniques; this will be more likely if the number of interdictors is small compared to the size of the minimum capacity cut and the $p_k$ values do not vary drastically. In particular, suppose that $\mathbf{x}^*$ is the optimal arc-inspection strategy in the single-interdictor game and we are concerned with the solution to an $m$-interdictor game. Then, if $m\mathbf{x}^* \leq \mathbf{1}$, $m\mathbf{x}^*$ are still optimal arc-inspection probabilities in the $m$-interdictor game and can be implemented by determining optimal "arc-subset-inspection" probabilities for subsets containing $m$ arcs from the forward arcs $A_C$ of the minimum capacity cut $C$. When an arc subset is chosen, exactly one interdictor will be assigned to each arc in the subset so that the overall probability that arc $k$ is inspected is inversely proportional to $p_k$, as in the single interdictor case; however, the frequency of inspection will be $m$ times higher than in the single interdictor case. The optimal strategy for the evader is identical to the strategy in the single interdictor case.

Some multiple interdictor games remain only partially solved, in the sense of "efficiently solved." If $\mathbf{x}^*$ solves the single-evader/single interdictor problem but there are $m$ interdictors and $mx_k^* > 1$ for some arc $k$, we have not provided a guaranteed, efficient solution method. Solving games that allow at most one interdictor per arc can also be hard. (One can always resort to the enumerated matrix games in an attempt to solve these problems.) Also, we have not considered any problems where **P1** has multiple types of interdiction assets, such as aircraft and ground inspection teams. We leave such problems as open research topics.

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