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NAVAL POSTGRADUATE SCHOOL

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Random Meetings at Sea

by

Alan R. Washburn

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NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA 93943-5001

Daniel T. Oliver
President

Leonard A. Ferrari
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Provost

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This report was prepared by:

ALAN R. WASHBURN
Distinguished Professor of
Operations Research

Reviewed by:

R. KEVIN WOOD
Associate Chairman for Research
Department of Operations Research

ROBERT F. DELL
Chairman
Department of Operations Research

Released by:

KARL VAN BIBBER
Vice President and
Dean of Research

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13. ABSTRACT (maximum 200 words)

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ABSTRACT

As ownship moves through a sea of targets moving at constant velocity in random directions, it will occasionally meet some of them when the intervening distance becomes sufficiently small. The very fact of meeting is significant in estimating the properties of a met target. We derive joint distributions for the properties of met targets, and show how to sample those properties in Monte Carlo simulations. We also introduce "instant" target motion analysis in the form of various conditional distributions of target properties, given observations made at the time of meeting.

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1. INTRODUCTION

Equation Section 1We consider situations where a certain vehicle, called "ownship", moves at constant velocity through an unbounded, two-dimensional medium full of other vehicles that are themselves moving at constant velocities in independent random directions. It sometimes happens that ownship meets one of these other vehicles (a "target"), by which we mean that the distance between them decreases to some critical value. The details of such meetings can be important to both parties—a collision may be imminent, or the target may be about to commit a hostile act that requires immediate action by ownship, with the action depending on the details. The detection mechanism strongly influences what can be said about those details. If detection is due to an active sensor like radar, then an accurate target location may be immediately available, but there are also passive sensors that can sometimes announce the fact of detection without providing much detail, especially detail about the distance to the target.

This subject is particularly important for ships, and so, even though there is nothing particularly nautical about the rest of this paper, we will refer to the vehicles as ships from here on. We will also use nautical terms such as "range" (distance to the target), "course", and "bearing", but all angles are measured counterclockwise in radians from East, the conventional mathematical system.

Even though all ships are assumed to be moving uniformly at random *a priori*, that statement is not true of targets that are met. It is often far from being true, but nonetheless it will be tempting for an analyst to suppose that the course or bearing of a met target is (say) uniformly distributed over some plausible range. In this paper, we will derive nonuniform distributions for such quantities that follow directly from the *a priori* assumptions and Bayes' theorem. Monte Carlo simulations (e.g., Forman, 2005) sometimes have ownship meeting a succession of targets as it moves, with each met target being assigned properties such as range, bearing, speed, and course. The aforementioned distributions can be used to set these properties realistically. Section 6.3 includes some sampling functions.

The careful distinction between targets in general and targets that have been met goes back to at least World War Two. Let b be the (relative) bearing from ownship to a met target at the moment of meeting, by which we mean the angle from ownship's velocity vector to the line-of-sight to the target (Figure 1). Meetings are more likely to occur with small bearings (bow sightings) than large ones (stern sightings); indeed, stern sightings are impossible if ownship has a higher speed than the target. It follows that moving sensors desiring to detect targets as early as possible should spend more of their time looking near the bow than in other directions. The density function of b, given detection of a target with known speed, was derived in World War Two (Koopman, 1956) in the process of dealing quantitatively with the issue of where lookouts should concentrate their effort. This paper generalizes that work.

Although we anticipate that the main application of our results will be in setting the parameters of newly met targets, there are also potential applications to Bayesian inference. The most notable of these is the determination of the target's range from bearing measurements, sometimes known as target motion analysis (TMA). It is well known (Wagner, Mylander, & Sanders, 1999, Theorem 11.4) that range cannot be exactly inferred from bearing measurements, no matter how numerous and accurate, unless ownship maneuvers (that is, unless ownship departs from the constant course and speed assumptions that we make throughout this paper). However, that fact does not prevent the application of Bayes theorem—an activity that we dub "instant" TMA to distinguish it from single-leg TMA where ownship does not turn, but makes a sequence of measurements separated in time (e.g., Ganesh & Baylog, 2004).

If ownship detects a target using passive sensors, it is likely that b and perhaps its time derivative $z \equiv db/dt$ (the bearing rate) will be known. Those known quantities can then be used in instant TMA. We will not consider errors in measuring either of those quantities. Doing so would be essential if we would otherwise obtain exact estimates of operational quantities, but that is not the case when one is merely applying Bayes' theorem to data measured or discovered at one point in time. Intuition is that the target must be nearby if the bearing rate is high. This turns out to be correct, but, as will be seen in Section 7, considerable uncertainty about the target's range will remain even when b and z are known exactly. Instant TMA typically provides only a slight refinement in knowledge of operational quantities such as the target's range.

Problems where the speed and detection range of the met target are either known or irrelevant are taken up in Section 4. Such problems include Koopman's derivation of the bearing distribution referred to above. For other questions, an application of Bayes' theorem will require a joint *a priori* distribution for the target's speed and detection range. A useful class of such distributions, based on the passive radar/sonar equation, is introduced in Section 5.

The reader may wish to have available the Excel™ workbook *Meetings.xls*, which contains Visual Basic code for some of the aforementioned distribution functions, and also illustrates their application. That workbook can be found among the downloads at http://faculty.nps.edu/awashburn/.

2. BASIC DEFINITIONS AND RELATIONSHIPS

Figure 1 shows two ships, each moving in a straight line with constant velocity. Without loss of generality, we assume ownship is proceeding to the right. The target is at bearing b and range r, proceeding at speed u. The target's course is c, relative to ownship's course, or ψ, relative to the line-of-sight (LOS) from ownship to the target ($\psi = c - b$).

Figure 2 shows the same situation in a coordinate system where ownship remains stationary. The velocity of the target relative to ownship is the vector difference between target velocity and ownship velocity, shown in both polar and Cartesian forms. The target's relative speed is w and its relative course is ϕ, or alternatively, the target's velocity can be resolved into two components: one (x) within the LOS and directed toward ownship, and the other (y) normal to the LOS and directed counterclockwise. The formulas relevant to our analysis are:

$$\begin{aligned} x &\equiv v\cos(b) - u\cos(c-b) \\ y &\equiv v\sin(b) + u\sin(c-b) \\ w &= \sqrt{u^2 + v^2 - 2uv\cos(c)} \end{aligned} \tag{1.1}$$

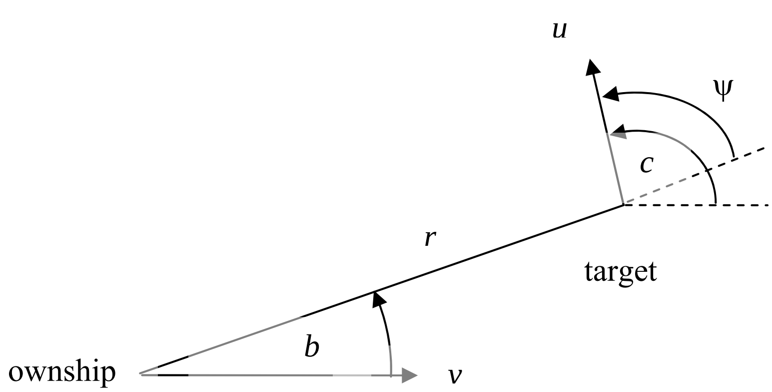

Figure 1: Ownship and target moving at constant velocity in two dimensions.

The range, bearing, speed, and course of the target are all assumed to be random variables, and uppercase versions of r, b, u, and c will be used to represent them. The general idea is that the two-dimensional medium is initially assumed to be homogeneously populated with targets located according to some constant Poisson density. Each of these targets is then assigned parameters (R, U, C), independently of the others. Specifically, R and U are assigned according to some density $g(r,u)$ that will be left indefinite for the moment, and C is independently assigned to be uniform over a full circle. Each target moves according to its assigned speed and course, and ownship moves to the right at speed v. Meetings occasionally happen when the distance between ownship and some target decreases to the detection range assigned to that target (we ignore any initial detections that happen when R is larger than the initial distance to the target). Each of these meetings is characterized by parameters (R, B, U, and C), and we wish to describe various distributions and conditional distributions involving those four random variables. The joint distribution of the four is the most fundamental, and the subject of the next section.

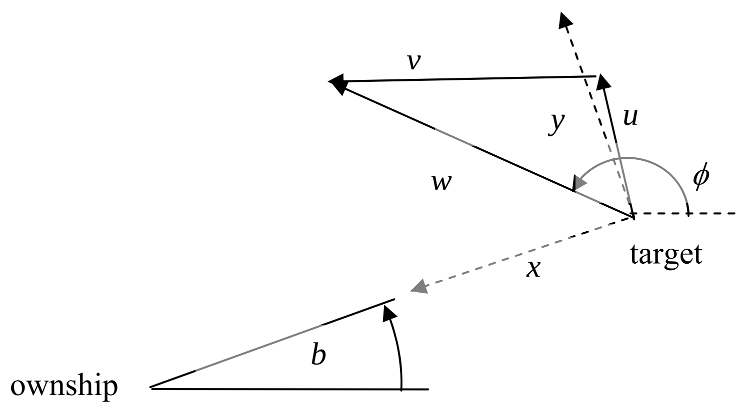

Figure 2: Illustrating target velocity relative to ownship, resolved into components *x* directly toward ownship and *y* across the LOS. Meetings can happen only when *x* is positive.

The assumption that C is *a priori* uniform over a full circle is significant, and could be false if, for example, the targets were all merchant ships moving within a traffic lane. The uniformity assumption is probably more realistic for military ships than for merchant ships.

For brevity, we will, in most cases, use the notation $f(r,b,u,c)$ for the joint density, in place of the more precise notation $f_{R,B,U,C}(r,b,u,c)$, trusting the name of the dummy variable to cue the name of the random variable under discussion, and likewise for discovery rates such as $\lambda(r,b,u,c)$. Similarly, $f(x|y)$ is the density of random variable X whenever the random variable Y is given to be y.

3. THE JOINT DENSITY OF (R, B, U, C) UPON MEETING

First consider some class of targets, all of which have speed u and detection range r, and assume that these targets are initially located in a two-dimensional Poisson field relative to ownship, with density h targets per unit area. All targets initially closer than r to the origin (ownship's location) are eliminated from consideration, since they are detected immediately. Each of the remaining targets is independently assigned a constant course C that is a uniform random variable in the interval $[-\pi, \pi]$. Should one of these targets be subsequently met, it will also have a bearing B, as well as a range r from ownship. Following Koopman's 1956 argument, the rate at which ownship meets such targets with specific course c and bearing b is

$$\lambda(r,b,u,c) = 2rhx^{+}(db)(dc/2\pi);\, b,c \in [-\pi,\pi]\,, \tag{1.2}$$

where x is the approach velocity given by (1.1) and the + sign means "positive part"; that is, x^{+} is x if $x \geq 0$ or 0 if $x < 0$. It should be intuitively clear that meetings are impossible if $x < 0$, since such targets are moving away from ownship, rather than toward it. The product of $2r$ with x^{+} is an area coverage rate.

By integrating (1.2) over all values of b for which x is positive, we find that

$$\lambda(r,u,c) = 2rhw(dc/2\pi) = 2rh\sqrt{u^2+v^2-2uv\cos(c)}\,(dc/2\pi);\, c \in [-\pi,\pi], \tag{1.3}$$

where w is the relative speed from (1.1). If we now integrate over all values of c, we obtain the rate of finding targets of this class:

$$\lambda(r,u) = 2rh\frac{1}{2\pi}\int_0^{2\pi}\sqrt{u^2+v^2-2uv\cos(c)}\,dc \equiv 2rhS(u,v)\,. \tag{1.4}$$

The equivalent speed function $S(u,v)$ is a complete elliptic integral of the second kind. It represents the expected relative speed between two targets, with speeds u and v moving in uniformly random directions. The function $\lambda(r,u)$ is the rate parameter for a Poisson process of target discovery. If $u = 0$, $S(u,v)$ reduces to v, and (1.4) reduces to the familiar idea that the rate of discovering targets is the rate of examining area ($2rv$) multiplied by the density of targets per unit area (h).

Now suppose that each target is initially assigned a detection range and speed (R,U) according to the density function $g(r,u)$, in which case the discovery rate for targets with specific (dr,db,du,dc) is $\lambda(r,b,u,c)g(r,u)(dr)(du)$. By integrating over b and c as before, the discovery rate for targets with specific range and speed (dr, du) is $\lambda(r,u)g(r,u)(dr)(du)$. Let

$$R_g(u) = \int_0^{\infty} rg(r,u)dr\,. \tag{1.5}$$

Then the total rate of discovering targets is

$$\int_0^\infty\int_0^\infty \lambda(r,u)g(r,u)(dr)(du) = 2h\int_0^\infty R_g(u)S(u,v)(du) \equiv 2h\Lambda\ . \tag{1.6}$$

The quantity 2Λ has dimensions of area per unit time, and can be thought of as an overall rate of examining area for targets.

The joint density function of (R,B,U,C) for a discovered target is now given by a ratio of rates:

$$f(r,b,u,c) = \frac{\lambda(r,b,u,c)g(r,u)}{2h\Lambda} = \frac{rg(r,u)(v\cos(b) - u\cos(c-b))^+}{\Lambda}\ . \tag{1.7}$$

Note that h cancels when taking this ratio, so the spatial density of targets is immaterial. The joint density (1.7) can be used to derive various other quantities. In particular, the joint density of (R,U) for a met target is

$$f(r,u) = \lambda(r,u)g(r,u)/(2h\Lambda) = rg(r,u)S(u,v)/\Lambda\ . \tag{1.8}$$

Note that $f(r,u)$ is not $g(r,u)$. Due to the factor r in (1.8), met targets tend to have longer detection ranges than do targets in general. Fast targets also tend to be over-represented because $S(u, v)$ increases with u. This is in contrast to the observed fact that fast fish are *underrepresented* in towed sampling nets, presumably because fish use their speed to avoid the net when they see it (Barkley, 1964), rather than continuing to move at constant velocity as we assume here.

The density of U upon meeting can be obtaining by integrating (1.8) over all r:

$$f(u) = \frac{R_g(u)S(u,v)}{\Lambda}\ . \tag{1.9}$$

In fact, the joint density of any subset of (r,b,u,c) can be obtained by integrating the joint density $f(r,b,u,c)$ over the variables not in the subset. Equation (1.7) is the joint density, and all others can be derived from it.

If there are actually several classes of targets indexed by i, with class i having spatial density h_i and initial density $g_i(r,u)$ for (R,U), then the rate of discovering targets of class i is $2h_i\Lambda_i$, where

$$\Lambda_i \equiv \int_0^\infty S(u,v)(\int_0^\infty rg_i(r,u)du)dr\ , \tag{1.10}$$

and the total rate of discovering targets is $D \equiv \sum_i 2h_i\Lambda_i$. If the discovery process for each class of targets is an independent Poisson process with rate $2h_i\Lambda_i$, then the discovery of all targets is a Poisson process with overall rate D. Equivalently, one can imagine an overall Poisson discovery process with rate D where the target class is determined upon discovery to be i with probability $2h_i\Lambda_i/D$, with the class determining the joint distribution of (R,B,U,C). Here we have used results about the classification and superposition (summing) of independent Poisson processes (Heyman & Sobel, 1990; Ross, 2000). We note these multiclass possibilities only in passing, and will consider only a single class from here on. Section 4 considers quantities that do not involve the as yet unspecified prior density $g(r,u)$. Section 5 deals with inferences that require specific assumptions about $g(r,u)$.

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4. DISTRIBUTIONS INDEPENDENT OF *g*(*r*,*u*)

4.1 JOINT DENSITY OF (*B*,*C*), GIVEN *U*: *f*(*b*,*c*|*u*)

In random meetings, the target's bearing B and course C, relative to the line-of-sight, are neither independent nor uniform. From (1.2) and (1.4),

$$f(b,c\,|\,r,u) = f(r,b,u,c)/f(r,u) = \frac{x^+}{4\pi S(u,v)} = \frac{(v\cos(b) - u\cos(c-b))^+}{4\pi S(u,v)}. \quad (1.11)$$

Since the right-hand side of (1.11) is independent of r, the conditional density is the same thing as $f(b,c|u)$. Equivalently, since $\psi = c-b$,

$$f(b,\psi\,|\,u) = \frac{x^+}{4\pi S(u,v)} = \frac{(v\cos(b) - u\cos(\psi))^+}{4\pi S(u,v)}. \quad (1.12)$$

4.2 DENSITY OF *B*, GIVEN *U*: *f*(*b*|*u*)

To find $f(b|u)$, we need to integrate the density (1.12) over all ψ for which x^+ is positive. If $v\cos(b) < -u$, then $x^+ = 0$ for all ψ, and consequently $f(b|u) = 0$. If $v\cos(b) > u$, then $x^+ = x$ for all ψ, and, since the integral of $\cos(\psi)$ over a full circle is 0, $f(b|u) = v\cos(b)/(2S(u,v))$. There remains the case where $|v\cos(\theta)| \le u$, in which case there is an angle ψ_0 such that $u\cos(\psi_0) = v\cos(b)$ and $0 \le \psi_0 \le \pi$. Since $x^+ > 0$ if and only if $\psi_0 < \psi < 2\pi - \psi_0$, by symmetry

$$f(b\,|\,u) = \frac{2\int_{\psi_0}^{\pi} (v\cos(b) - u\cos(\psi))d\psi}{4\pi S(u,v)} = \frac{(v\cos(b)(\pi - \psi_0) + u\sin(\psi_0))}{2\pi S(u,v)}.$$

Since $\pi - \psi_0 = \mathrm{ArcCos}(-v\cos(b)/u)$, and since $u\sin(\psi_0) = \sqrt{u^2 - v^2\cos^2(b)}$, the above can be summarized as

$$f(b\,|\,u) = \frac{1}{2\pi S(u,v)}\begin{cases} 0, \text{ if } v\cos(b) < -u \\ \pi v\cos(b), \text{ if } v\cos(b) > u \\ v\cos(b)ArcCos(-v\cos(b)/u) + \sqrt{u^2 - v^2\cos^2(b)}, \text{ if } |v\cos(b)| \le u. \end{cases} \quad (1.13)$$

This is the density first determined by Koopman (1956) in World War Two. Targets cannot appear outside the "limiting lines of approach" (the first condition), and are more likely to appear over the bow (near $b = 0$) than near any other bearing. Only the ratio of the two speeds is relevant. Figure 3 shows the distribution when $u/v = 0.5$, 1, and 2.

Since r is missing from the right-hand side of (1.11), formula (1.13) is also $f(b|r,u)$; the additional knowledge of the detection range is of no value as long as the target's speed is already known.

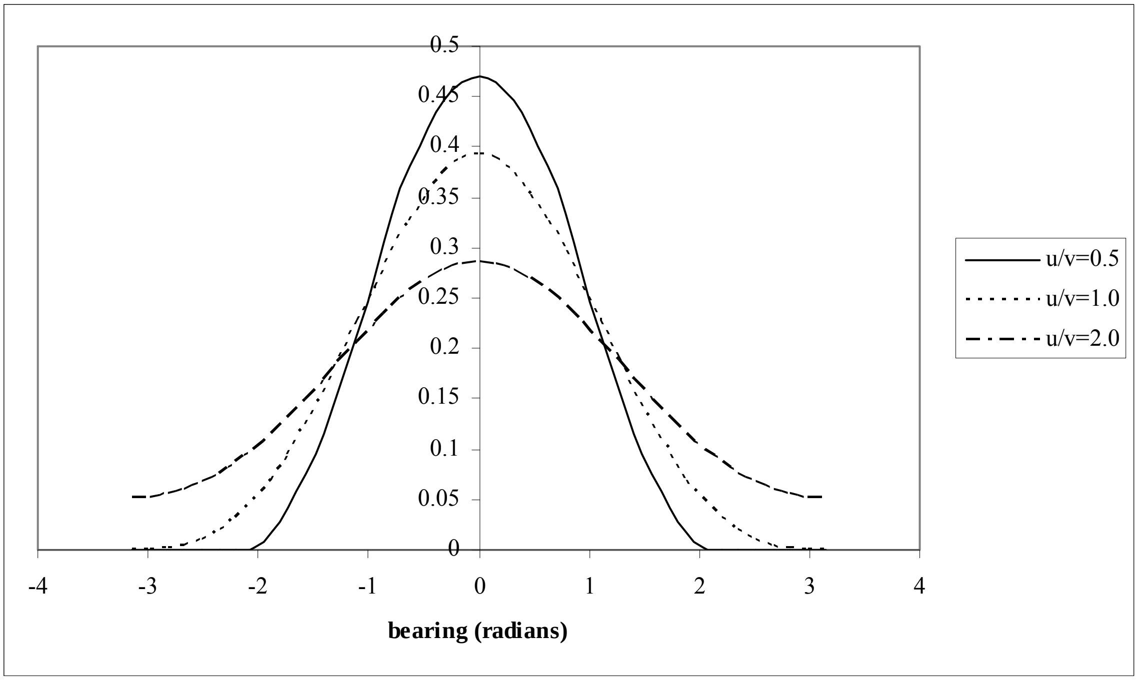

Figure 3: Density of the bearing at meeting (*B*) for three speed ratios.

4.3 DENSITY OF ψ, GIVEN *B* AND *U*: *f*(ψ|*b*,*u*)

Given that a target has been detected on bearing b, this is the density of the target's course relative to the line of sight, which is of interest in forecasting the target's motion. This density is simply the ratio of (1.12) to (1.13). Note that $S(u,v)$ cancels in taking the ratio, so this is the easiest to evaluate of any of the densities in this section. If the ratio is 0/0, the interpretation is that b is impossible.

4.4 JOINT DENSITY OF *B* AND *Y*, GIVEN *U*: *f*(*b*,*y*|*u*)

Our interest in Y, the target's speed across the LOS, stems from the random variable's close relationship to Z, the bearing rate, specifically $Y = RZ$. Since Z is passively observable, we can potentially condition on it, as well as on B, when applying Bayes' theorem.

Since Y is a function of both B and ψ according to (1.1), this section is an exercise in calculating the density of a scalar function of two random variables, complicated by the fact that the function involved is not monotonic. In general, if the angles for which $Y(b,\psi) = y$ are ψ_1, ψ_2, …, then $f(b,y\,|\,u) = \sum_i f(b,\psi_i\,|\,u)/|Y'(\psi_i)|$, where $Y'(\psi_i)$ denotes the derivative of Y with respect to ψ at the i^{th} angle. In the present application, the sum may include 0, 1, or 2 terms. From (1.1), the required derivatives are all $Y'(\psi_i) = u\cos(\psi_i)$.

There are multiple cases. Refer to Figure 4. If $v\cos(b) < -u$, then $f(b,y|u)$ is 0 for all y because b is impossible (x is negative regardless of the target's course). Otherwise, let $\delta = y - v\sin(b)$, so that the equation $Y(b,\psi) = y$ is equivalent to the equation $\delta = u\sin(\psi)$. If $|\delta| > u$, then $f(b,y|u) = 0$ because there are no solutions to the equation. Otherwise there are two solutions, namely $\psi_1 = \text{ArcSin}(\delta/u)$ and $\psi_2 = \pi - \psi_1$. For these two solutions we have $u\cos(\psi_1)=\sqrt{u^2-\delta^2}$ and $u\cos(\psi_2) = -u\cos(\psi_1)$, with each solution qualifying for inclusion in the sum if and only if the closure rate x at that solution is nonnegative. If $v\cos(b) > u\cos(\psi_1)$, then both qualify, and the sum of the two joint densities is $f(b,\psi_1|u) + f(b,\psi_2|u) = 2v\cos(b)/(4\pi S(u,v))$. If $u\cos(\psi_2) \le v\cos(b) \le u\cos(\psi_1)$, then only ψ_2 qualifies, with joint density $f(b,\psi_2 \mid u) = (v\cos(b) + \sqrt{u^2-\delta^2})/(4\pi S(u,v))$. If $v\cos(b) < u\cos(\psi_2)$, then neither solution qualifies. In all cases, the joint density must be divided by $|Y'(\psi_i)|$, which is $\sqrt{u^2-\delta^2}$, before being summed.

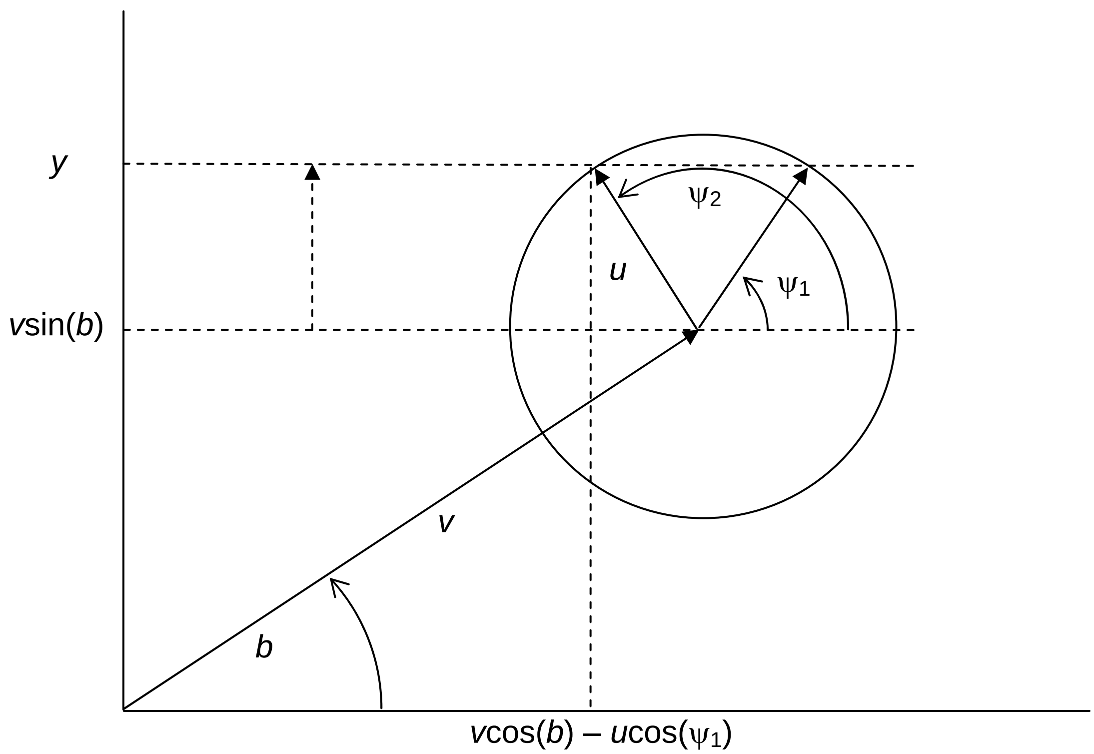

Figure 4: Velocity across the line of sight (y) on the vertical axis, and closing velocity along the line of sight (x) on the horizontal axis. The angles ψ_1 and ψ_2 both qualify in this diagram, but there are other possibilities.

In summary, $f(b,y|u)$ is 0 if $|y - v\sin(b)| \ge u$. Otherwise, let $s \equiv \sqrt{u^2 - (y - v\sin(b))^2}$. Then

$$f(b,y\,|\,u)=\frac{1}{4\pi S(u,v)}\begin{cases}\dfrac{2v\cos(b)}{s}\ \text{if}\ v\cos(b)>s\\[2ex]\dfrac{s+v\cos(b)}{s}\ \text{if}\ -s\le v\cos(b)\le s.\\[2ex]0\ \text{if}\ v\cos(b)<-s\end{cases}\tag{1.14}$$

The conditional density $f(y|b,u)$ is just the ratio of (1.14) to (1.13). With the same exclusions and the same definition of s as in (1.14), it is

$$f(y\,|\,b,u)=\begin{cases}1/(\pi u)\ \text{if}\ v\cos(b)>u\\[1ex]\dfrac{v\cos(b)}{s(v\cos(b)ArcCos(-v\cos(b)/u)+\sqrt{u^2-v^2\cos^2(b)})}\ \text{if s<}v\cos(b)\le u\\[2ex]\dfrac{.5(s+v\cos(b))}{s(v\cos(b)ArcCos(-v\cos(b)/u)+\sqrt{u^2-v^2\cos^2(b)})}\ \text{if}\ -s\le v\cos(b)\le s.\\[2ex]0\ \text{if}\ v\cos(b)<-s\end{cases}\tag{1.15}$$

Note that the constant $S(u,v)$ cancels in taking this ratio. Figure 5 shows the conditional density function $f(y\,|\,b,u)$ for $b = 0$, $u = 2$, and $v \ge 2$. Large values of Y are more likely than small ones when $b = 0$; indeed, the density $f(y\,|\,b,u)$ is unbounded at the endpoints. A curiosity is that $f(y\,|\,b,u)$ is a uniform distribution on the interval $[v\text{–}u,\ v\text{+}u]$ when $b=\pm\pi/2$.

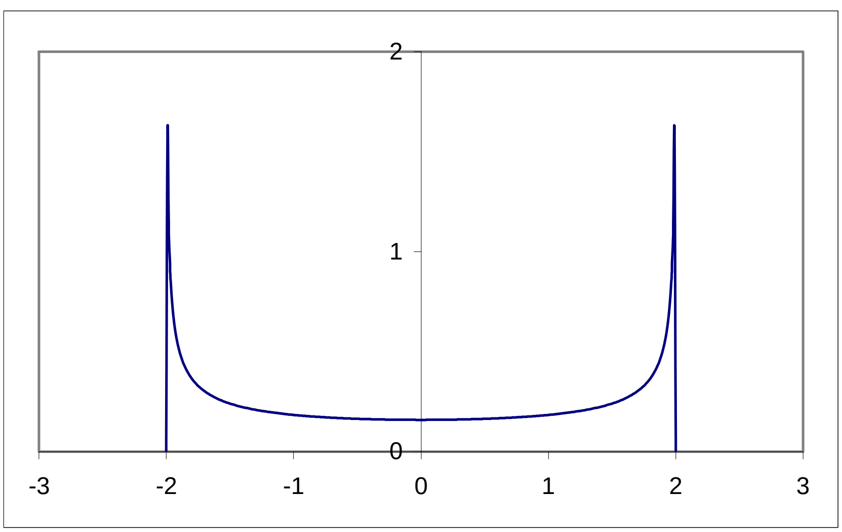

Figure 5. Conditional density of speed across the LOS, given detection dead ahead, for a target with speed 2, as long as ownship speed is at least 2.

4.5 JOINT DENSITY OF *B* AND *X*, GIVEN *U*: *f*(*b*,*x*|*u*)

Random variable X is the target's velocity towards ownship, or range rate, taken positive when the range is decreasing. Meetings can only happen when X is positive, and large values of X are potentially alarming to ownship because of the collision potential, depending on the bearing b. Range rate is potentially observable through a Doppler shift, even when the range itself cannot be directly sensed. The density that is derived in this section is the first step in conditioning on the observed value of X.

The procedure for determining $f(b,x|u)$ is similar to the procedure for determining $f(b,y|u)$. Since X is a function of both B and ψ according to (1.1), we again have a problem in determining the density of a scalar function of two random variables. If the angles for which $X(b,\psi) = x$ are ψ_1, ψ_2, …, then $f(b,x\,|\,u) = \sum_i f(b,\psi_i\,|\,u) / |\,X'(\psi_i)\,|$, where $X'(\psi_i)$ denotes the derivative of X with respect to ψ at the i^{th} angle. The required derivatives are all $X'(\psi_i) = -u\sin(\psi_i)$. Let $\delta = v\cos(b) - x$, so that $u\cos(\psi_i) = \delta$ according to (1.1). If $|\delta| > u$, there are no solutions to this equation. If $|\delta| < u$, there are two solutions ψ_1 and ψ_2, for each of which $|\,X'(\psi_i)\,| = \sqrt{u^2-\delta^2}$. Thus, using (1.12),

$$f(b,x\,|\,u) = \frac{1}{4\pi S(u,v)}\begin{cases} 0 \text{ if } |\delta| > u \text{ or } x \le 0 \\ \dfrac{2x}{\sqrt{u^2-\delta^2}} \text{ if } |\delta| < u \text{ and } x > 0. \end{cases} \tag{1.16}$$

This joint density function is unbounded when $|\delta|$ is nearly u, which corresponds to extreme situations where the target is moving directly toward or directly away from ownship. Figure 6 shows a typical transect.

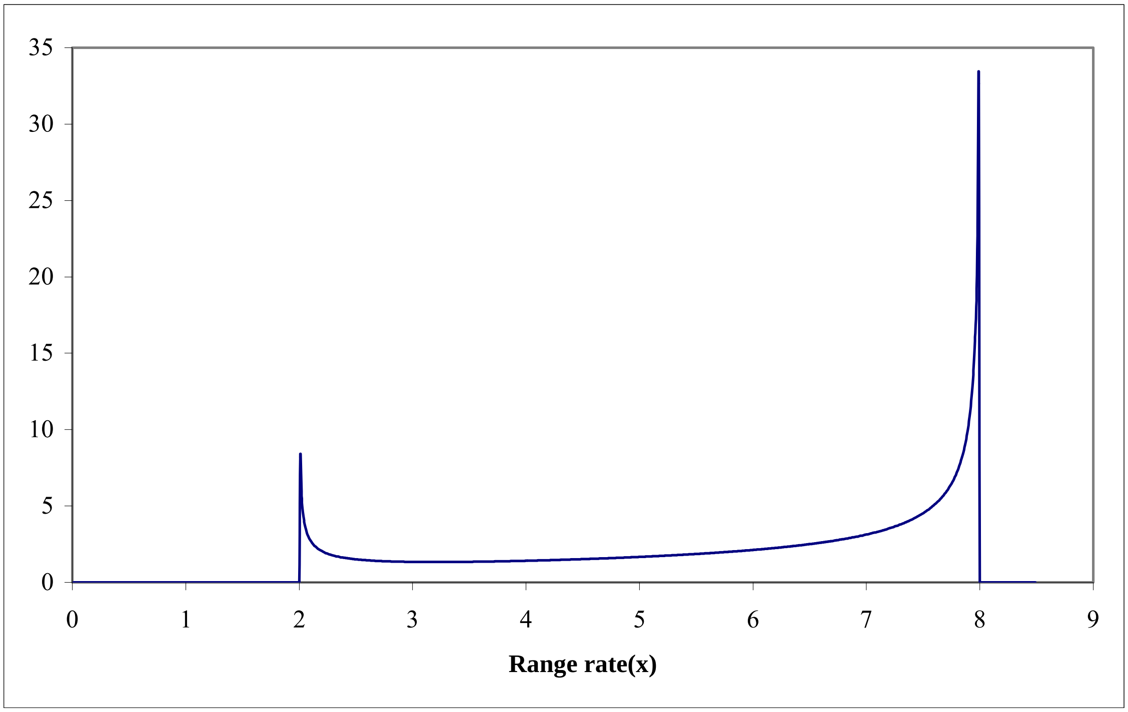

Figure 6: A graph of *f*(*b*,*x*|*u*) versus *x* (range rate) for *v* = 5, *u* = 3, and *b* = 0. Targets discovered dead ahead (*b* = 0) have a strong tendency to have a range rate almost as large as *v* + *u*, with a somewhat smaller tendency for *x* to be just above *v* – *u*.

Strongly peaked densities, such as those shown in Figure 6, can be problematic for techniques that evaluate densities only on a grid. In some cases, the cumulative form of the density may be more useful. It happens that (1.16) can be integrated analytically on x to obtain $F(b,x\,|\,u) \equiv P(X \le x, B \in db\,|\,U = u)/db$, a partially cumulative joint distribution. With $t \equiv v\cos(b)$, this distribution is given by

$$F(b,x\,|\,u) = \begin{cases} 0 \text{ if } x < 0 \text{ or } x < t - u, \\ f(b\,|\,u) \text{ if } x \ge t + u, \text{ or otherwise,} \\ \dfrac{t(\pi - Arc\cos((x-t)/u)) - \sqrt{u^2 - (x-t)^2}}{2\pi S(u,v)} \text{ if } t \ge u \\ \dfrac{\sqrt{u^2 - t^2} - \sqrt{u^2 - (x-t)^2} + t(Arc\cos(-t/u) - Arc\cos((x-t)/u))}{2\pi S(u,v)} \text{ if } t < u. \end{cases} \qquad (1.17)$$

4.6 THE PROBABILITY OF LEADING THE TARGET

When two ships meet, assuming that neither ship subsequently changes either course or speed, it is possible that the target will eventually pass in front of ownship, at which time the bearing will be 0. In that case ownship is said to be “leading” the target. If the initial bearing B is positive (negative), this will happen if and only if Y is negative (positive); that is, B and Y must have different signs (all angles are restricted to the interval $[-\pi,\pi]$ in this paper). Since the event E of crossing over one's bow is an important one for mariners, it is of interest to find its probability. We have

$$P(E)=2\int_{B>0,Y<0} f(b,\psi\,|\,u)dbd\psi=\frac{1}{2\pi S(u,v)}\int_{B>0,Y<0}(v\cos(b)-u\cos(\psi))^{+}dbd\psi\,. \quad (1.18)$$

The initial factor of 2 in (1.18) is by symmetry—the possibility that $B<0$ is not accounted for, but clearly the chances that the target will pass in front of ownship are the same regardless of the sign of B.

There appear to be only three cases where the integral in (1.18) can be accomplished analytically. The first two are almost obvious. If $u = 0$ and $v > 0$, $P(E)$ can easily be shown to be 0, or if $v = 0$ and $u > 0$, $P(E)$ is 0.5. Both of these results should make intuitive sense; the first because a stationary target will never cross ownship's bow, and the second because the direction of motion for a stationary ownship should be irrelevant. The third case is when the two speeds are equal. One-fourth of such meetings will be leading, as we next show. Without loss of generality, we assume both speeds are 1.

When $u = v = 1$, $P(E)$ is $M/(2\pi S(1,1))$, where M is the right-hand integral in (1.18):

$$M=\int_0^{\pi} db\int_{Q(b)}(\cos(b)-\cos(\psi))d\psi. \quad (1.19)$$

Here $Q(b)$ is the set of angles ψ for which both (1) $\cos(b) > \cos(\psi)$ and (2) $\sin(b) + \sin(\psi) < 0$. The first requirement is satisfied when $|\psi| > b$, but the second cannot be satisfied if $\psi > b$ because the sum of two positive terms cannot be negative, so we must have $\psi < -b$. If $b \le \pi/2$, the second is satisfied if and only if $\psi\in(b-\pi,-b)$, and $Q(b)$ is therefore that interval. If $b \ge \pi/2$, the second is satisfied if and only if $\psi\in(-b,b-\pi)$, but being in that interval is inconsistent with the requirement that $\psi<-b$, so $Q(b)$ is empty. We therefore have $M=\int_0^{\pi/2}db\int_{b-\pi}^{-b}(\cos(b)-\cos(\psi))d\psi$. It is now just a matter of evaluating standard integrals to conclude that $M = 2$. Since $S(1,1)$ is $4/\pi$, we conclude that $P(E) = 0.25$.

Mariners sometimes refer to a "lag" situation where the target's and ownship's velocity vectors point to opposite sides of the LOS. Such situations never result in the target crossing ownship's bow, so such events are not included in E. Even if the target's velocity vector points to the same side of the LOS as ownship's, it is still possible that the target will not cross ownship's bow. Such situations are sometimes described as "overlead". We mention this only to emphasize that our current usage does not include "overlead" in the event E, which remains the event that the target will (assuming that neither ship changes speed or course) cross ownship's bow. In other words, the opposite of E is not "lag", but rather "lag or overlead".

We are unable to evaluate $P(E)$ analytically, except in the three cases described above, but it is not difficult to evaluate numerically. As expected, $P(E)$ decreases from 0.5 to 0 as ownship becomes faster and faster relative to the target, with speed equality being about "half way to infinity".

5. INFERENCES ABOUT *R* IN THE CASE OF PASSIVE SONAR

Up to this point, we have not been specific about the *a priori* distribution of (R,U). Other than the idea that proximity implies detection, we have assumed nothing about the sensors involved. That must necessarily change in this section, which deals with estimating the range at detection (R). When detection is made passively, R is notoriously variable, in addition to being unobservable. Ownship generally becomes aware of a target's presence when some sound emitted by the target is heard by ownship, at which time the bearing B is measured. However, the received signal at first detection is typically near the threshold of audibility, so the strength of the received signal is of little use in determining R. Our interest here is in making inferences about R based on quantities that can be observed at the time of detection, particularly B and its rate of change Z. We therefore must make *a priori* assumptions about R that are tailored to the sensor involved.

The problem of making inferences about R is especially difficult when detection is made by passive sonar, since the acoustic transmission properties of the ocean are hard to predict. Seemingly one could reason that the signal excess, as predicted by the sonar equation (Urick, 1975), must be about 0 at the time of initial detection, and calculate the unique range for which that is true. The problem is that the sonar equation itself is only a rough approximation to reality—it can easily be off by 9 decibels in predicting the signal-to-noise ratio (Wagner et al., 1999, Section 509), which corresponds to a factor of about 8 in signal power. Adding to the difficulty is that R depends strongly on the signal level emitted by the target, which in turn depends strongly on the speed U of the target, which is also typically unknown. We cannot proceed under the assumption that either R or U is a known quantity, or that the two are independent of each other. We will develop a joint prior distribution for (R, U) by first assuming a distribution for U, then using the sonar equation to develop a distribution for R, given U, and finally multiplying the two together.

5.1 A PRIOR DISTRIBUTION *g*(*r*,*u*) FOR PASSIVE SONAR

We take the distribution of U to be lognormal with mode U_0; that is, the natural logarithm of (U/U_0) is assumed to be normal with mean 0 and standard deviation σ_U. U_0 can be thought of as a guess at the target's speed, with σ_U quantifying the vagueness of the guess. The lognormal distribution is concentrated on the positive real line, the natural domain of speed, and has both scaling (U_0) and shaping (σ_U) parameters.

We will use the sonar equation to develop the density of R, given U. The signal-to-noise ratio depends on ownship speed v, in addition to U, since ownship speed determines the level of noise with which the signal must compete. Let V_0 be the speed at which self-noise is equal to background noise, and assume that self noise is proportional to the α power of ownship speed. Total noise N is then proportional to $v^\alpha + V_0{}^\alpha$. The signal S is assumed proportional to U^β for some parameter β, and also proportional to the inverse square of R (spherical spreading). The signal-to-noise ratio is then

$$\frac{S}{N}=\frac{U^{\beta}(R/R_0)^{-2}}{v^{\alpha}+V_0^{\alpha}}K\ , \tag{1.20}$$

where K is approximately 1 and R_0 includes the aforementioned proportionality constants. To be precise, $\ln(K)$ is assumed to be normal with mean 0 and standard deviation $2\sigma_R$. If the sonar equation has a normal error with standard deviation 6 decibels, then σ_R is 0.69. If $\alpha = \beta$, R_0 can be thought of as the most likely detection range when U and v are both equal and large. We expect $\alpha = \beta = 3$, since the power required to drive a ship typically increases with the third power of its speed.

There is an important caveat to the assumption that K is approximately 1 at detection. Passive sonars are typically not omnidirectional, with the location of the sonar on ownship being such that directions forward ($B = 0$) are covered, but not directions aft ($B = \pi$). If the initial bearing is at or near the sonar's cutoff, but drawing forward (B and Z have opposite signs, see Figure 3), then K might be substantially greater than 1, and the target might be closer than would otherwise be expected. For such sonars, the Bayesian mathematics developed here applies only when the initial discovery is strictly within (not on the edge of) the angular coverage region of the sonar.

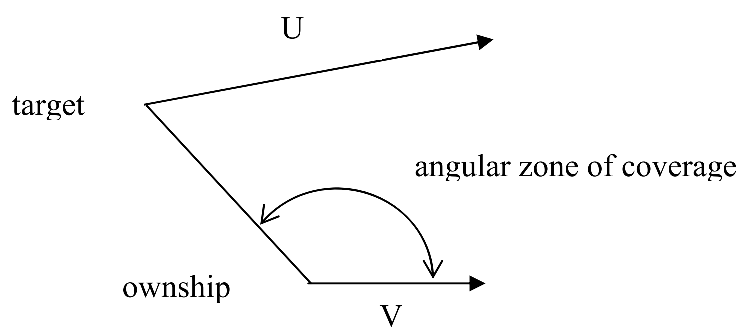

Figure 7: Illustrating a situation where the target is discovered when entering the zone of the sonar's coverage. The bearing *B* is positive and decreasing.

Since $S/N = 1$ at first detection, (1.20) implicitly defines the density of R, given $U = u$. Specifically, $\ln(R/R_0)$ is normal with mean $0.5\ln(\frac{u^{\beta}}{v^{\alpha}+V_0^{\alpha}})$ and standard deviation σ_R. Parameters that need to be known for determining the joint distribution of R and U are thus U_0, V_0, R_0, α, β, σ_U, and σ_R. Given those parameters, the joint density function is

$$g(r,u)=\frac{\exp(-0.5\{\frac{\ln(u/U_0)^2}{\sigma_U{}^2}+\frac{[\ln(r/R_0)-0.5\ln(u^{\beta}/(v^{\alpha}+V_0{}^{\alpha}))]^2}{\sigma_R{}^2}\})}{2\pi ru\sigma_U\sigma_R}. \tag{1.21}$$

This density function is illustrated on sheet "Prior" of *Meetings.xls*, with opportunities to explore the impact of different parameter choices.

There are other unimodal distributions with the appropriate scaling and shape parameters, but (1.21) uses standard functions and has some convenient analytic properties. One advantage of the lognormal assumption is that the function $R_g(u)$ defined in Section 1.2 can be expressed analytically (see Johnson & Kotz, 1970), as can certain other moments. Some useful formulas are recorded below:

$$E(U) = U_0 \exp(.5\sigma_U{}^2) \tag{1.22}$$

$$E(R) = R_0 \sqrt{\frac{U_0{}^{\beta}}{v^{\alpha} + V_0{}^{\alpha}}} \exp(0.5\sigma_R{}^2 + 0.125(\beta\sigma_U)^2) \tag{1.23}$$

$$E(R^2) = R_0{}^2 \frac{U_0{}^{\beta}}{v^{\alpha} + V_0{}^{\alpha}} \exp(2\sigma_R{}^2 + 0.5(\beta\sigma_U)^2) \tag{1.24}$$

$$E(RU) = R_0 U_0 \sqrt{\frac{U_0{}^{\beta}}{v^{\alpha} + V_0{}^{\alpha}}} \exp(0.5\sigma_R{}^2 + 0.5((1+0.5\beta)\sigma_U)^2) \tag{1.25}$$

$$R_g(u) = \int_0^{\infty} r g(r,u) dr = \frac{R_0}{u\sigma_U} \sqrt{\frac{u^{\beta}}{(2\pi)(v^{\alpha} + V_0{}^{\alpha})}} \exp(0.5[\sigma_R{}^2 - (\frac{\ln(u/U_0)}{\sigma_U})^2]). \tag{1.26}$$

It may be tempting to use the experience of subject matter experts (SMEs) to estimate some of the parameters required for this sonar model, but a word of caution is appropriate. It is likely that SMEs will have more knowledge of targets that are met than of targets in general, but the formulas in this section are for targets in general. It would be a mistake, for example, to use (1.22) with E(U) = 5 kt because of an SME's statement that "targets I have met have an average speed of 5 kt". Such SME knowledge applies to distributions upon meeting, the subject of the next section, rather than the *a priori* distributions that are the subject of this section.

5.2 DISTRIBUTIONS UPON MEETING

The joint density $f(r,u)$ upon meeting is given by (1.8), with (1.21) substituted for $g(r,u)$. We know of no analytic expression for the constant Λ from (1.8), so are forced to resort to numerical integration, but otherwise this density is an analytic function. Various moments of R and U can be derived by numerical integration, and possibly used for parameter estimation through SME judgments.

The joint density $f(r,u,b)$ is the product of (1.8) and (1.13). The equivalent speed $S(u,v)$ cancels when making that product, so $f(r,u,b)$ is again an analytic function except for Λ. If the bearing is observed to be b upon meeting, then this joint density easily enables the computation of the conditional density $f(r,u|b)$ and its marginals $f(r|b)$ and $f(u|b)$. In other words, everything that is to be known about the range and speed of the intercepted target, given only the bearing to the target, is at hand.

The joint density $f(r,u,b,y)$ is the product of (1.8) and (1.14). Let the bearing rate be Z, so that $Z=Y/R$. The joint density $f(r,u,b,z)$ is then $rf(r,u,b,y = rz)$, where this awkward notation means that rz is to be substituted for y. We can now condition on (b,z) to discover the effects of observations of bearing and bearing rate on the distribution of (R,U). This is the “instant TMA” referred to in the introduction.

The joint density $f(r,u,b,x)$ is the product of (1.8) and (1.16), enabling the computation of $f(r,u|b,x)$, the density of range and speed when both bearing and range rate are observed.

The distributions described above are all implemented and exercised in the workbook *Meetings.xls*. See Section 7.

6. VISUAL BASIC FOR APPLICATIONS (VBA) IMPLEMENTATION OF VARIOUS DISTRIBUTION FUNCTIONS

The VBA module included with *RandMeet.xls* contains code for all of the functions introduced above, plus some additional ones useful for simulation. In that module, densities defined upon meeting all start with the letter f, followed by the uppercase names of the random variables whose joint density is under consideration, possibly followed by the letter g (for "given"), followed by the uppercase names of the random variables whose values are given. Thus, fXgY(x,y) is the name of the function that would be called $f_{X|Y}(x|y)$, were subscripts available in VBA code. Some densities omit normalization by hard-to-compute constants such as Λ, in which case the name of the function ends in the letter k as a warning.

6.1 SECTION 2 FUNCTIONS

$S(u,v)$ (formula 1.4) is EquivSpd(u,v).

6.2 SECTION 4 FUNCTIONS

$f(b|u)$ (formula 1.13) is fBgUk(b,u,v). This function needs to be divided by 2S(u,v) to be a true density. The alternative function BearDen(b,u,v) is Koopman's original specification, and is a true density.

$f(\psi|b,u)$ (the ratio of formula 1.12 to 1.13) is fPgBU(u,v,b, p), a true density.

$f(y|b,u)$ (formula 1.15) is fYgBU(u,v,b,y), a true density.

$f(b,x|u)$ (formula 1.16) is fBXgUk(u,v,b,x). Divide by 2πS(u,v) to get a true density.

$F(b,x|u)$ (formula 1.17) is fBXgUc(u,v,b,x). The terminal "c" means "cumulative". Divide by 2πS(u,v) to get the true distribution.

$g(r,u)$ (formula 1.21) is prior(r, R0, u, U0, v, V0, SigU, SigR, *pow*). Parameters α and β are both assumed to be equal to input *pow*. The argument SigU is σ_U, SigR is σ_R, and other symbols are as used earlier.

E(U) (formula 1.22) is EU(R0, U0, v, V0, sig$U$, Sig$R$, pow), notation as in $g(r,u)$.

E(R) (formula 1.23) is ER(R0, U0, v, V0, sig$U$, Sig$R$, pow), notation as in $g(r,u)$.

E(R^2) (formula 1.24) is ER2(R0, U0, v, V0, sig$U$, Sig$R$, *pow*), notation as in $g(r,u)$.

E(*RU*) (formula 1.25) is ERU(*R*0, *U*0, *v*, *V*0, sig*U*, Sig*R*, *pow*), notation as in *g*(*r,u*).

$R_g(u)$ (formula 1.26) is Rg(*R*0, *u*, *U*0, *v*, *V*0, sig*U*, Sig*R*, *pow*), notation as in *g*(*r,u*).

Λ (formula 1.6 , using $R_g(u)$ and S(*u*,,*v*) from above) can be obtained using the function called integral(*v*, *R*0, Sig*R*, *U*0, sig*U*, *pow*, *V*0), with notation as in *g*(*r,u*). This is a numerical integral, so comparatively expensive to evaluate.

E(*R*|meeting) (not numbered) can be obtained from ERA(*v*, *R*0, SigR, *U*0, sigU, pow, *V*0), notation as in *g*(*r,u*). This numerical integral is the expected range of a met target when ownship moves at speed *v*, and might be of use in parameter estimation.

f(*r,u,b*) (not numbered) is fRUBk(*r*, *R*0, *u*, *U*0, *v*, *V*0, sig*U*, Sig*R*, *pow*, *b*). This needs to be multiplied by $4\pi\sigma_U\sigma_R\Lambda$ to be a true joint density.

f(*r,u,b,z*) (not numbered) is fRUBZk(*r*, *R*0, *u*, *U*0, *v*, *V*0, sig*U*, Sig*R*, *pow*, *b,z*).

6.3 SAMPLING AND OTHER FUNCTIONS

Various other functions are also provided in *Meetings.xls*, generally introduced by comments in the VBA code. Among these are some sampling functions that may be of use to those who have no ambition to do any Bayesian analysis, but would like to generate random meeting parameters in a Monte Carlo simulation. All use the rejection method (Ross, 2000). The functions are:

For *B* given *U*, use Bsample(*u*,*v*) or possibly Bsample2(*u*,*v*). The latter should be somewhat faster, but the code is more complicated.

For *C* given *U* and *B*, use Csample(*u*,*v*,*b*). If (*u*,*v*,*b*) is an impossible combination, this function will return a very large positive number.

For *Y* given *U* and *B*, use Ysample(*u*,*v*,*b*). If (*u*,*v*,*b*) is impossible, this function will return a very large positive number.

For both *B* and *C* given *U*, use subroutine BCSample(*u*,*v*,*b*,*c*).

These functions are not exhaustive, but others can be based on densities derived above. For example, (1.7) could be used to give a joint sample of (*R*,*B*,*U*,*C*).

7. THE WORKBOOK MEETINGS.XLS

This workbook includes several sheets named after density functions described in Section 6. Each of these sheets is a graphical or computational exploration of the function with its name. Sheet “leading” permits the approximate computation of $P(E)$ as described in Section 4.6. These are all “Section 4” sheets; that is, no reference is made to a prior distribution for (R,U).

The five “range” sheets whose names start with R are all controlled by the sheet named “RUBZ”, which includes a command button that updates all five. The range sheets empower the exploration of the extent to which inferences about R can be made, based on observations of B and possibly X or Z. Specifically,

sheet RUprior shows the prior density of (R,U) before meeting;
sheet RU shows the density for met targets;
sheet RUB shows the density conditioned on knowing B;
sheet RUBX shows the density conditioned on knowing B and X; and
sheet RUBZ shows the density conditioned on knowing B and Z.

The joint distribution of (R,U) is, in all cases, displayed digitally after multiplying by a large constant (100) to improve readability. Numbers larger than 0.1 are colored to get a graphical idea of the region in which density is “significant”. In all cases, the marginal distribution of R is computed by summing, then used to approximate E(R) under the specified condition, and then copied to the controlling sheet RUBZ to permit easy comparisons. In some cases, the exact value of $E(R)$ is also known and used for comparison.

The range pages have many inputs, so many experiments are possible. One experiment begins with the “Standard” scenario (Tools menu), exploring the consequences of modifying conditions on B and Z. The *a priori* mean detection range upon meeting in this scenario is 1.52 nm. Table 1 shows the mean detection range with various observations.

		Bearing (B) in radians			
Bearing rate (Z) in radians per hour		0	0.5	1	1.5
	Null	1.40	1.41	1.48	1.67
	0	2.03	1.72	2.75	3.35
	1	2.09	1.88	2.36	2.80
	2	1.50	1.74	2.13	2.34
	3	4.27	1.40	1.76	1.84
	4	0.77	1.12	1.45	1.49

Table 1: The mean range in nm for several assumptions about bearing and bearing rate, “Null” meaning that no observation of Z is made.

Table 1 displays an anomaly when $Z = 3$ and $B = 0$, where the expected range is clearly not related to the values in neighboring cells. The anomaly is easily explained, but nonetheless is something to beware of in computations such as these involving unbounded densities on a grid, which in this case includes values of R and U that are multiples of 0.1. When $Z = 3$ and $B = 0$, it happens that the point $(R,U) = (4.3, 12.9)$ is nearly on a singularity of the joint density, to the extent that it ends up being the only possibility with any significant probability. This one point takes over the numerical computation of the mean value of R, forcing it to be about 4.3. An operational version of these computations would recognize and protect against such anomalies, but our simple grid-based computation does not.

Except for the anomaly, Table 1 shows a decrease in expected range as the bearing rate increases. This should be intuitive because Z is inversely proportional to R. However, the decrease is not as fast as one might expect, and is certainly not inversely proportional to Z. The reason for this mushiness is that there are other explanations for a high bearing rate: the target might be going fast, or the target's relative course ψ might be nearly $\pi/2$. Since neither U nor ψ has been observed, it makes intuitive sense to lend some credence to both of those possibilities, in addition to the possibility that R might be small.

Table 1 (the Null row) also shows that the expected range increases with the observed bearing. Intuitive or not, that is the truth of the matter. Although not shown in Table 1, each of the quantities displayed there is surrounded by a widespread distribution. Figure 1 displays three distributions associated with the Standard scenario, all of which exhibit significant variance.

The results in this section are typical, and demonstrate the difficulty that ownship must inevitably have when meeting a new target, prior to making any maneuver. Instant TMA will not normally be definitive, so considerable risk will remain if decisions must be made quickly. To accurately predict the target's future movement, it will be necessary to maneuver and accept the implied time delay.

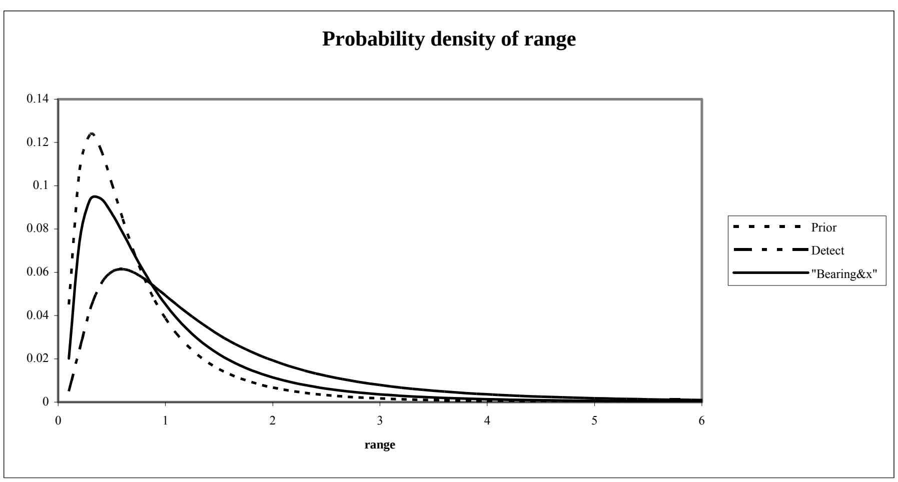

Figure 8: Marginal densities of range under three variations of the Standard scenario. The "Detect" curve assumes only detection, and the "Bearing&x" curve is conditioned on knowing the bearing and range rate. All three exhibit significant variance.

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