# Ratio Game on a Network

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ABSTRACT: This paper was originally motivated by a military problem where one side plants mines on a network of roads while the other side clears them in an extended conflict. The two-person zero-sum game considered here is a possible model. The essential features of the game are that the payoff on each segment of the selected path depends only on the ratio of the two sides' allocations of effort, and that the total payoff is a sum over the arcs of that path. We solve the game, presenting mathematical programs for the two sides. One side uses a mixed strategy, while the other does not.

APPLICATION AREAS: Land and expeditionary warfare
OR METHODOLOGIES: Decision theory, Stochastic processes

## INTRODUCTION

IEDs (Improvised Explosive Devices) have been responsible for many of the casualties in recent Middle Eastern wars (USA Today, 2013; Atkinson, 2004). An essential feature of what we call IED warfare is that it is extended in time, rather than the conventional short battle where one side plants mines while the other either quickly suffers from them or quickly clears them. There are three types of decision in IED warfare:

- the miner must decide when, where and with what intensity to plant mines on a network of roads,
- the clearance forces must similarly decide how to allocate clearance effort to the network, and
- logistic traffic must decide what path to take through the network.

DeGregory (1997) and Washburn and Ewing (2014) describe models of IED warfare where the first and third decisions are taken as given when the clearance forces make the second decision. This is a one-sided decision problem for the clearance forces. All three decisions will be modeled in this paper, so we have a two-person zero-sum game. The clearance forces and the logistic traffic are assumed to be on the same side and to have no communication problem, so one of the players will make both of those decisions. As will be seen, much depends on the order in which the three decisions are made.

Bold symbols represent vectors, with the same symbol italicized representing its components. Thus the *n*-vector **x** represents $(x_1,...,x_n)$. The symbol $\equiv$ means "by definition". If limits are not given for a summation index, "for all" should be understood.

## THE MODEL

Our ratio game is a special case of the game described in Theorem 2 below. Since the allocations of the two sides are continuous variables, we must first establish the existence of a solution (Theorem 1) before investigating its characteristics (Theorem 2).

**Theorem 1**: Let $B$ and $C$ be closed, bounded, nonempty convex subsets of finite-dimensional Euclidean space, and let $A_k(\mathbf{x},\mathbf{y})$ be a continuous function on $B \times C$

for $k = 1,...,K$ . The two-person zero-sum game where the maximizer chooses **x** and the minimizer chooses (*k*,**y**) has a value, and both sides have optimal, potentially mixed strategies.

The proof of Theorem 1 is a modification of Burger's proof (Burger, 1963) for the case where $K = 1$. The proof would be conventional if *B* and *C* were finite, so the essential task is to show that various limits exist as continuous space is more and more finely subdivided. We omit the details.

**Theorem 2**: In addition to the assumptions of Theorem 1, assume that $A_k(\mathbf{x}, \mathbf{y})$ is concave-convex for all *k*; that is, assume that $A_k(\mathbf{x}, \mathbf{y})$ is a concave function of **x** for all (*k*,**y**) and a convex function of **y** for all (*k*,**x**). Then player 1 has an optimal pure strategy, and player 2's optimal strategy consists of *K* "atoms" $(p_k, \mathbf{y}_k)$, where $\mathbf{p} \equiv (p_1,..., p_K)$ is a probability distribution and $\mathbf{y}_k \in C$. Player 2 first selects an index *k* according to **p**, and then chooses $\mathbf{y} = \mathbf{y}_k$. If $p_k = 0$, $\mathbf{y}_k$ can be any point in *C*.

Proof: Theorem 2 is known to be true for the case $K = 1$ (Washburn, 2014), and the general proof for Player 1 can follow the same line here because the minimum of a finite number of continuous, concave functions is still a continuous, concave function. The proof for player 2 amounts to showing that his choice of **y**, given that his index choice is *k*, need not be random. Toward that end let random variable $\mathbf{Y}_k$ be his optimal choice, and let $\mathbf{y}_k = E(\mathbf{Y}_k)$ be its mean. The mean $\mathbf{y}_k \in C$ exists because the distribution of $\mathbf{Y}_k$ is supported by *C*. Since $A_k(\mathbf{x}, \mathbf{y})$ is a convex function of **y** for all **x**, by Jensen's inequality,

$$A_k(\mathbf{x}, \mathbf{y}_k) \le E(A_k(\mathbf{x}, \mathbf{Y}_k)).$$

If **p** is optimal, we therefore have

$$\sum_k p_k A_k(\mathbf{x}, \mathbf{y}_k) \le \sum_k p_k E(A_k(\mathbf{x}, \mathbf{Y}_k)) \le v \quad ,$$

where *v* is the game value. The right-hand inequality is true because **p** and **Y** have been assumed optimal and player 2's optimal strategy must guarantee at most *v* regardless of **x**. The left-hand inequality then shows that the optimal $\mathbf{Y}_k$ can always be replaced by its mean without harm to player 2, as was to be shown. QED

One could also imagine a different game where the maximizer chooses **x**, the minimizer chooses **y**, and the payoff is $A(\mathbf{x}, \mathbf{y}) \equiv \min_k A_k(\mathbf{x}, \mathbf{y})$. This game would in general be more favorable to player 2 because the choice of index can depend on **x**, as well as **y**. However, in the concave-convex case the two players can still use the same strategies to enforce the same value as in the game we have in mind. The two games are therefore tactically identical in the concave-convex case, and we will make no further reference to this possible modification.

The only application of theorems 1 and 2 in this paper will be to a game played on a network of *J* arcs where player 2's index choice corresponds to a feasible path connecting source $s$ to destination $t$. Every arc is assumed to be part of some feasible path — other arcs can be eliminated because neither player will allocate anything to them. The set of feasible paths is known to both sides. A nonnegative vector $\mathbf{c}$ is given, and, if $r_k$ is the set of arcs in the $k^{\text{th}}$ path, the payoff when player 2 chooses that path is

$$A_k(\mathbf{x},\mathbf{y}) \equiv \sum_{j\in r_k} x_j c_j^2 / y_j \qquad (1)$$

Here $\mathbf{x}$ and $\mathbf{y}$ are both nonnegative vectors with $\sum_j x_j = \sum_j y_j = 1$. Our reason for referring to the square of $c_j$ at this point will emerge later. The payoff depends only on the ratio of allocations to arcs, increasing with $x_j$ and decreasing with $y_j$. The game could also be described as one where player 1 chooses a single arc, with $\mathbf{x}$ being a mixed strategy, but we will persist in identifying $\mathbf{x}$ as the strategy because of the motivating problem.

Equation (1) does not meet the requirements of Theorem 1 because the payoff is discontinuous (potentially infinite) when $y_j = 0$. We could restore continuity by requiring all of player 2's allocations to exceed some small positive quantity, but will forego doing so because player 2 is not motivated to make payoffs large. The ratio 0/0 will be interpreted as 0 in evaluating (1).

In the motivating problem, the components of $\mathbf{x}$ represent the rates at which IEDs are planted on the arcs of the network, with the emplantment rates being constrained by $\sum_j x_j = X$, the given total rate of emplantment. The components of $\mathbf{y}$ are assumed to be the rates at which clearance teams visit the arcs, similarly constrained by $\sum_j y_j = Y$. In both cases "rate" means the rate of a time-homogeneous Poisson process; that is, there is no predictable schedule of either mine plants or clearance visits that either side can exploit. It will be shown that player 1's choice of $\mathbf{x}$ does not depend on $Y$ and player 2's choice of $\mathbf{y}$ does not depend on $X$, so it is not restrictive to assume (as we do) that both players know both $X$ and $Y$.

One way of obtaining equation (1) is to assume that a visit by a clearance team to arc $j$ will clear each mine on arc $j$ with probability $q_j$, independently of all other mines, where $q_j$ depends on jointly known parameters such as the physical location of the arc. Let $N_j$ be the number of IEDs on arc $j$, a stochastic process that goes up by 1 with every mine planted and down by 1 with every mine cleared. This is an M/M/∞ queue where the IEDs have arrival rate $x_j$ and are "serviced" by the clearance teams with service rate $q_j y_j$ per mine, so the mean number of IEDs on the arc is $E(N_j) = x_j / (q_j y_j)$ (Ross, 2000). If we define $c_j^2 \equiv 1/q_j$ and sum on the arcs in path $k$, the average number of IEDs on path $k$ is $A_k(\mathbf{x},\mathbf{y})$. It is equivalent to solve a standard game with $X = Y = 1$ and then simply multiply the standard value by $X/Y$ to get the average number of IEDs actually encountered. When $X = Y = 1$, the components of $\mathbf{x}$ and $\mathbf{y}$ in (1) can be interpreted as fractions of the total rate committed to the various arcs.

There may be other assumptions that would lead to a game based on (1), but let the derivation above suffice. For ease of interpretation we will continue to describe **x** and **y** as above, and to the components of **c** as "vulnerability parameters".

We now turn to the analysis of the standard game. Hereafter player 1 is referred to as P1 (the game's maximizer) and player 2 as P2 (the minimizer).

## PURE OPTIMAL STRATEGY FOR PLAYER 1

Since P1's strategy is pure according to theorem 2 and can safely be announced (like all optimal strategies) to P2, we can formulate a mathematical program to determine P1's optimal strategy and the game value *v*.

Whatever path *k* P2 chooses, he will want to minimize (1) using **y**. When **x** is known, this is an elementary minimization problem that can be solved using Lagrange multipliers. Doing this we find that the (negative) of the derivative of each term of (1) with respect to $y_j$, which is $x_j c_j^2 / y_j^2$, should be a constant when **y** is chosen optimally, the constant being chosen to make $\sum_{j \in r_k} y_j = 1$ (note that the sum is over $r_k$, with no clearance effort devoted to arcs not in the chosen path). Thus P2's optimal choice for **y** is to select *k* and then

$$y_j(\mathbf{x}) \equiv \frac{c_j \sqrt{x_j}}{\sum_{i \in r_k} c_i \sqrt{x_i}}; \; j \in r_k, \tag{2}$$

with other components of **y** being 0. Inserting this into (1) and summing, we find

$$A_k(\mathbf{x}, \mathbf{y}(\mathbf{x})) = (\sum_{j \in r_k} c_j \sqrt{x_j})^2 , \tag{3}$$

which P1 must maximize, or equivalently P1 must maximize the positive square root $\sum_{j \in r_k} c_j \sqrt{x_j}$ . Since P1 does not know *k*, P1 must consider every possible path. This consideration leads to a nonlinear optimization problem that we will call NLP1. Letting $u_j^2 \equiv x_j$, NLP1 is

$$\begin{aligned} &\max Z \\ &\text{subject to } \sum_j u_j^2 = 1, \text{ and} \\ &\sum_{j \in r_k} c_j u_j \ge Z \text{ for all } k \end{aligned} \tag{4}$$

NLP1 has one nonlinear (quadratic) constraint and one linear constraint for each possible path. After solving it, P1 can guarantee that $A_k(\mathbf{x}, \mathbf{y}) \ge Z^2$, regardless of what P2 does with **y** and *k*, by setting $x_j = u_j^2$ for all *j*. The value of the game is therefore $v = Z^2$. The form of NLP1 may explain our initial reference to $c_j^2$ in (1), rather than $c_j$.

As long as every path includes at least one arc with a positive vulnerability parameter (the alternative is that the value of the game is 0), a slightly simpler version of NLP1 is NLP2 with variables $z$ and $\mathbf{h} = (h_1, ..., h_J)$ :

$$\min z \equiv \sum_j h_j^2$$
$$\text{subject to } \sum_{j \in r_k} c_j h_j \geq 1 \text{ for all } k \tag{5}$$

NLP2 is quadratic program with one variable for every arc (**h**) and one linear constraint for every path. If the solution of NLP2 is ($z$,**h**), then one can set $Z = 1/\sqrt{z}$ and $\mathbf{u} = \mathbf{h}Z$ to obtain a solution of NLP1.

NLP2 is our simplest mathematical program for determination of the game value and P1's optimal strategy. P2's optimal strategy will be considered in a later section.

## DEFINITIONS, SPECIAL CASES AND GAME VARIATIONS

If a path $k$ exists for which $c_j = 0$ for all $j \in r_k$, then P2 will always use it and the value of the game is 0. This is the "null" case. In all other cases the value is positive, since P1 can guarantee that by making a positive allocation to every arc.

It is convenient to define

$$A(k) \equiv \sum_{j \in r_k} c_j^2 = A_k(\mathbf{1},\mathbf{1}), \tag{6}$$

and also to define $k$* as the path index that minimizes $A(k)$.

In the special case where there is a unique path $k$* from $s$ to $t$, $A(k^*)$ is the value of the game. The solution of NLP1 is $x_j = u_j^2 = c_j^2 / A(k^*)$ and $Z^2 = A(k^*)$. P2 can guarantee the same value by setting $y_j = c_j^2 / A(k^*)$, thus presenting P1 with the problem of maximizing $\sum_{j \in r_{k^*}} x_j A(k^*)$. The value of this sum is $A(k^*)$ when all of P1's effort is devoted to $k$*, and in no case can exceed that. Thus both sides employ the same pure strategy, and the common strategy emphasizes those arcs in $k$* with high vulnerability.

We can also consider some variations on the rules of the game, all of which benefit P1. If P1 knows the path selection $k$ before choosing **x**, then it is easily established that the best path selection for P2 is $k$*. If on the other hand P1 knows **y**, but not $k$, then P1 can simply imitate P2 by setting $\mathbf{x} = \mathbf{y}$, which will make the payoff $A(k)$. The best choice of $k$ for P2 will again be $k$*. Since P2 can guarantee that the payoff will not exceed $A(k^*)$ in either case, or even in the case where P1 knows both $k$ and **y** before choosing **x**, this is the value of the game in all three variations.

This leaves the main, non-null case where P1 knows neither $k$ nor **y** when he chooses **x**. P1's optimal strategy in that case is pure, as established earlier. We next consider the optimal strategy for P2, which in general is mixed.

## THE GENERAL CASE FOR PLAYER 2

According to Theorem 2, P2's optimal strategy consists of $K$ atoms $(p_k, \mathbf{y}_k)$, with **p** a probability distribution over the possible paths from $s$ to $t$. If P2 uses such a mixed strategy, then the expected payoff when P1 employs **x** is

$$\sum_k p_k \sum_{j \in r_k} c_j^2 x_j / y_{jk} \quad .$$

This is a linear function of **x**, so P2 must choose his strategy so that the coefficient of $x_j$ in that expression does not exceed the game value *v* on any arc. This observation leads to nonlinear program NLP3, where all variables are nonnegative and $s_j$ is by definition the set of path indexes *k* for which $r_k$ includes arc *j*:

$$\begin{aligned}
&\text{minimize } v \\
&\text{subject to } c_j^2 \sum_{k \in s_j} p_k / y_{jk} \le v \text{ for all } j \\
&\sum_k p_k = 1, \\
&\sum_{j \in r_k} y_{jk} = 1 \text{ for all } k.
\end{aligned} \tag{7}$$

It should be understood that $y_{jk} = 0$ unless $k \in s_j$ (or equivalently $j \in r_k$ ). Variable *v* can be 0 in the null case because P2 can make $p_k = 1$ on the null path, which makes either the sum or $c_j^2$ be 0 in the inequality constraints. Otherwise *v* must be positive.

If the game is not null, variable *v* and one constraint can be eliminated by defining $t_k \equiv p_k / v$ . Since $\sum_k t_k = 1/v$ , NLP 3 is equivalent to maximizing that sum. We thus have the equivalent NLP4 with one fewer variable and one fewer constraint than NLP3:

$$\begin{aligned}
&\text{maximize } \sum_k t_k \\
&\text{subject to } c_j^2 \sum_{k \in s_j} t_k / y_{jk} \le 1 \text{ for all } j \\
&\sum_{j \in \mathbf{r}_k} y_{jk} = 1 \text{ for all } k.
\end{aligned} \tag{8}$$

The objective function of NLP4 is unbounded in the null case, but not otherwise. Let $\mu_j^2$ and $\lambda_k^2$ be nonnegative Lagrange multipliers for the first and second sets of constraints. Both $\mu_j$ and $\lambda_k$ are assumed to be positive if $j \in r_k$ . The Lagrangian function to be maximized is

$$L(\mathbf{t}, \mathbf{y}) \equiv \sum_k (t_k - \lambda_k^2 \sum_{j \in r_k} y_{jk}) + \sum_j \mu_j^2 c_j^2 \sum_{k \in s_j} t_k / y_{jk}$$

According to Everett's Theorem (Everett, 1963), if we can solve the unconstrained problem of maximizing $L(\mathbf{t}, \mathbf{y})$ , and if the results of that maximization satisfy the constraints of (8), then those results are also optimal for (8). Equating the derivative with respect to $y_{jk}$ to 0, we must have $\mu_j^2 c_j^2 t_k / y_{jk}^2 = \lambda_k^2$ as long as $j \in r_k$ . Since $\lambda_k > 0$ , we can solve this to obtain $y_{jk} = \sqrt{y_k}\, \mu_j c_j / \lambda_k$ . Substituting this expression for $y_{jk}$ into (8), P2's problem is to

$$\begin{aligned} &\text{maximize } \sum_k t_k \\ &\text{subject to } c_j \sum_{k \in s_j} \sqrt{t_k}\,\lambda_k \le \mu_j \text{ for all } j \\ &\sqrt{t_k} \sum_{j \in \mathbf{r}_k} \mu_j c_j = \lambda_k \text{ for all } k. \end{aligned} \tag{9}$$

The variables of (9) are all nonnegative and include $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$. The solution of (9) will not be unique because $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ can both be multiplied by a common positive factor without changing the validity of any of the constraints. Furthermore, except in the null case, the unbounded solution where $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are both $\mathbf{0}$ will not meet the requirements of Everett's theorem. If the case is not null, both difficulties can be avoided by adding an additional constraint (e.g. $\mu_1 = 1$).

The equalities in (9) determine $\lambda_k$. Substituting this expression into the inequalities of (9) and letting $w_j \equiv \mu_j c_j$, we finally obtain NLP5, our simplest mathematical program for P2:

$$\begin{aligned} &\text{maximize } \sum_k t_k \\ &\text{subject to } c_j^2 \sum_{k \in s_j} t_k \sum_{i \in r_k} w_i \le w_j \text{ for all } j \end{aligned} \tag{10}$$

NLP5 is a bilinear program with one variable for every arc $(\mathbf{w})$, one variable for every path ($\mathbf{t}$), and one constraint for every arc. As in (9), an additional constraint, perhaps $w_1 = 1$, can be inserted to eliminate the possibility that the optimal solution includes $\mathbf{w} = 0$. Given the solution of NLP5, the game value is $v = 1/\sum_k t_k$ and the NLP3 variables as originally expressed in (7) are $p_k = vt_k$ and $y_{jk} = w_j / \sum_{i \in r_k} w_i$ for $k \in s_j$. Note that, given a specific arc *j* and the set of paths $s_j$ that include that arc, P2's optimal allocation does not depend on *k* as long as $k \in s_j$. That property is exemplified in the columns of Table 1 below.

**TWO EXAMPLES**

Example 1: The case where there is a unique path has been considered earlier, and might be called the "series" case. Here we consider the "parallel" case where every arc is by itself a path between *s* and *t*. In the parallel case the solution of NLP1 is $Z^2 = (\sum_j c_j^{-2})^{-1}$ and $u_j = Z / c_j$ on each arc. Thus P1 can guarantee $Z^2$ by making $x_j$ inversely proportional to $c_j^2$ on each arc. Note that P1 puts most of his effort on the least vulnerable arcs, exactly contrary to his behavior in the series case. The solution of NLP5 (all of the *w*-variables cancel) has $t_j = c_j^{-2}$ and of course the same game value. P2 selects an arc with a probability inversely proportional to $c_j^2$ and then puts all of his clearance capacity on that arc. P2's preference for arcs with low vulnerability makes intuitive sense, and partially explains P1's similar preference—it makes sense that P1 should emphasize the arcs that P2 is most likely to use.

Example 2: Figure 1 shows a bridge network with five numbered arcs where there are four paths from *s* to *t*, those being in order $(12, 135, 45, 432)$. If the data for the arcs is $\mathbf{c} = (2,5,1,3,4)$, then $k^* = 2$ and $A(k^*) = 4+1+16 = 21$. P1 can solve NLP2 to discover that the best **x** is $(0.143, 0.312, 0.006, 0.077, 0.462)$, which guarantees $v = 12.608$. If P1 uses **x**, P2 will find that all paths are equally attractive except for the last, which should be avoided. P2 can guarantee that the payoff will not exceed $A(k^*)$ by choosing $\mathbf{y} = (4,0,1,0,16)/21$, but can do better with the mixed strategy found by NLP5. This strategy has $\mathbf{p} = (0.47, 0.20, 0.33, 0)$ and conditional allocations **y** as shown in Table 1. The last row of that table is labelled "N/A" because P2 never uses that row. The reader desirous of seeing further detail or exploring the impact of changes in **c** can experiment with the Excel ™ (Microsoft, 2008) workbook *NetRatioGame.xlsx*, which uses Solver to solve NLP2 and NLP5. That workbook can be downloaded from http://faculty.nps.edu/awashburn/ .

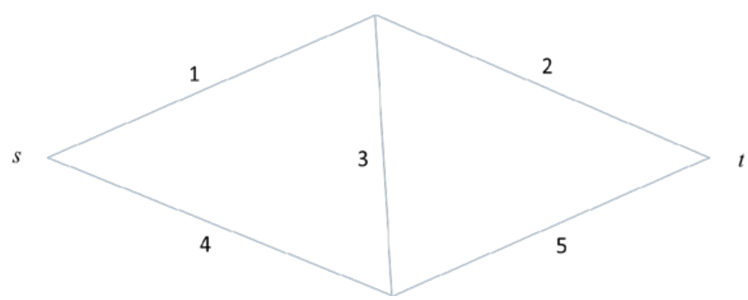


*Figure 1: Five numbered arcs form a bridge network*

*Table 1: P2's optimal allocations in example 2 as a function of path (row) and arc (column)*

| | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| **12** | 0.2129 | 0.7871 | | | |
| **135** | 0.2129 | | 0.0218 | | 0.7653 |
| **45** | | | | 0.2347 | 0.7653 |
| **432** | **N/A** | **N/A** | **N/A** | **N/A** | **N/A** |

## APPLICABILITY TO IED WARFARE

The essential features of (1) are that it is a sum over the arcs of a network path, and that each term of the sum is proportional to the ratio of the two sides' allocations. While there may be applications that do not involve IEDs, the motivating application is IED warfare. In this section we discuss various objections that might be made to the idea of applying a game based on (1) to that application.

In the queuing derivation made earlier, the meaning of (1) is "average number of mines encountered on the selected path". It might be objected that the fate of logistic traffic will more realistically depend on whether the number of mines encountered is 0 (survival) or not (casualty). One response to this objection is that the model is best applied in situations where each mine encountered is only effective with some unknown but small probability $Q$. In that case the number of *effective* mines encountered is approximately a Poisson random variable and the probability of encountering at least one of them is therefore approximately $QA_k(\mathbf{x},\mathbf{y})$, the same small number as the mean number encountered. Since $Q$ is tactically irrelevant, this leads to the game as studied above. The same conclusion can be reached if the ratio $X/Y$ is small, or if the vulnerability vector **c** is small.

Another objection is that the logistic traffic itself will remove some of the mines. Here the reply is that the clearance effect of logistic traffic must be small compared to that of deliberate mine clearance.

A third objection is that the source *s* and destination *t* may not be known to P1, and possibly not even to P2 — logistic traffic must sometimes travel between an unpredictable variety of sources and destinations. If this is the case then a generalized model is needed, perhaps one where a matrix of probabilities that determines *s* and *t* is given. The current model is a special case.

The most serious objection has to do with the information systems employed by the two sides. IED warfare is extended in time, and both sides can usually observe the actions of the other throughout that time, more or less. It does P2 little good to observe P1's choice of **x**, since P1 has no motivation to keep it secret. However, P2 will have difficulty keeping his own choice of **y** secret over any significant period, and P1 can capitalize on knowing it. If P2 must inevitably reveal **y** through his observable clearance actions, then (as pointed out earlier) P1 can guarantee $A(k^*)$ by simply copying P2's allocation. If $A(k^*)$ is significantly greater than $v$, as it is in the above examples and typically will be when there are many paths from *s* to *t*, P2 will be tempted to clear mines "at the last minute" in order to accomplish his clearance before P1 is able to react. This type of clearance is not well modelled as a time-homogeneous Poisson process. In such circumstances the game value derived here should be thought of as a lower bound on what P2 might accomplish through secrecy, with $A(k^*)$ being what P2 should expect if secrecy is not achievable.

## SUMMARY

We have defined a ratio game played on a network where both sides make continuous allocations to the arcs. The game has a solution, and either side can find its value and his own optimal strategy by solving a nonlinear program. The maximizer P1 finds his pure strategy by solving NLP2, and the minimizer P2 finds his mixed strategy by solving NLP5. Depending on circumstances, the game may or may not be applicable to IED warfare.

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