Diffuse Gaussian Multiple-Shot Patterns

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ABSTRACT

When multiple weapons are fired at a single target, it may not be best to fire all weapons directly at the target on account of errors common to all shots. The probability of hitting the target can sometimes be increased by firing the weapons in a pattern around the target, rather than directly at it. This observation leads naturally to the problem of finding the optimal pattern. The optimization problem is in general difficult because even calculating the hit probability for a given pattern usually requires numerical integrals. One exception is when errors are normally distributed and the damage function takes on a compatible "diffuse Gaussian" form. In that case, the hit probability can be expressed as an analytic function of the pattern's aimpoints, and conventional methods used to optimize it. This paper describes the required mathematics for a general diffuse Gaussian form, thus generalizing previous work.

INTRODUCTION

A marksman who fires several shots at a target, without any feedback between rounds, is sometimes disappointed to find that his shots lie in a tight pattern that is not centered on the target, as in Figure 1.

The tightness of the shot group in Figure 1 indicates a small dispersion error, which is normally desirable, but the group as a whole may nonetheless be ineffective because it is in the wrong place. The group's translation from the vicinity of the target is usually because there is a firing error common to all shots. The error may be due to an alignment problem within the weapon that fires all the shots, to an imperfectly known target location, or to some unanticipated aspect of the environment (wind or current, for example) that changes slowly enough to be in effect constant (albeit randomly so) for all shots in the group. It may even be the sum of several such errors. The effect is the same regardless of the explanation. The problem of firing multiple shots in the presence of a common error occurs in the employment of land and sea-based artillery, in the design and employment of weapons with submunitions, and in bombing.

If the statistical distribution of the common error can be determined beforehand, it may be wise for the marksman to aim his shots in a pattern, rather than all at the target, in the hope that the spread of the pattern might compensate for the common error. The problem of calculating the probability of "killing" the target with such a pattern is the computational problem addressed in this paper. We will maintain the point of view that each shot either kills the target or leaves it undamaged, with the kill probability being some "damage function" of the relative position between target and impact point.

It might seem that the simplest damage function would be one where the target is killed if and only if the distance between target and shot is smaller than some fixed distance, the so-called cookie-cutter damage function. This is not true, especially in evaluating patterns where shots are not aimed directly at the target. If aiming errors are assumed to be normal, as they usually are, the analytically simplest assumption is that the damage function is a "Diffuse Gaussian" (DG) function that resembles the normal distribution in shape.

Depending on the meaning of "kill" and the physical mechanism involved, the DG function can be attacked for being too sloppy or defended for being appropriately so. Regardless of its physical suitability as a model of damage, its analytical merits are undeniable. The mathematics of dealing with the DG damage function is so simple, compared to other alternatives, that a DG analysis is a reasonable first step even when the damage function is not DG. For example, one might find the optimal pattern for DG weapons as a first step in an iterative procedure for weapons of some other kind. The first analysis that took advantage of this simplicity was by John von Neumann in 1941 (Taub (1962)), who employed it to simplify a pattern bombing problem in one dimension. See Eckler and Burr (1972) for other applications.

ANALYSIS

We consider the problem of analytically evaluating the probability that at least one of several DG shots kills a single, point target located in Euclidean m-space. Except for a common error, all shots will be assumed to be independent, with each shot's impact point differing from its aim point by a normal (Gaussian) dispersion error. All integrals will be taken over the entirety of Euclidean m-space in this paper, so limits will not be shown explicitly. We begin by considering a single shot, and then generalize to patterns of n shots.
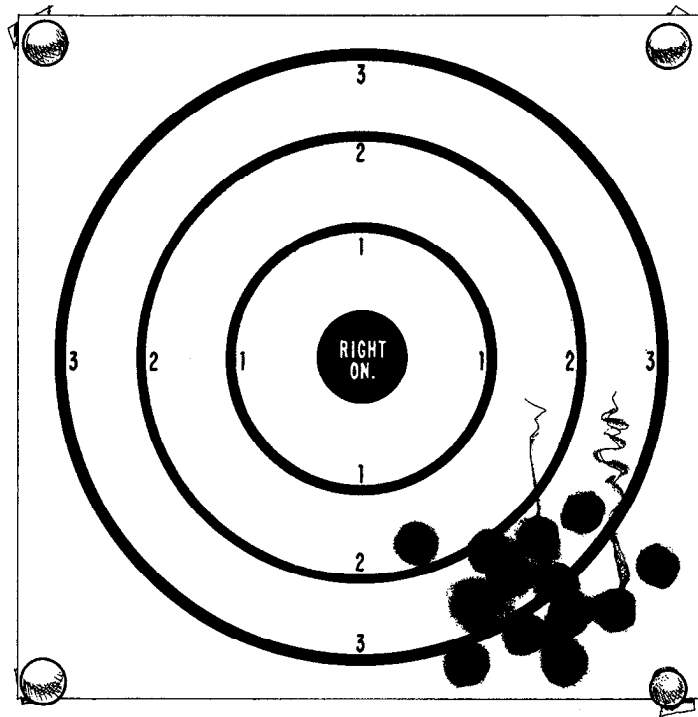
Figure 1. Illustrating a tight grouping of shots that might have been more effective if a pattern had been used.

We first observe that, if Σ is a positive definite, symmetric m-by-m matrix, and if x and μ are appropriately dimensioned column-vectors, then

$$\int \exp\left(-\frac{1}{2}(x - \mu)'\Sigma^{-1}(x - \mu)\right)dx$$

$$= (2\pi)^{m/2}\sqrt{|\Sigma|}, \quad (1)$$

where $|\Sigma|$ denotes the determinant of the matrix and x' is the transpose of x. Equation (1) is true because the multivariate normal distribution integrates to unity (DeGroot (1970)).

We will take the DG damage function to be $D(x) \equiv \rho\exp(-0.5x'Sx)$, where ρ is a reliability and S is any symmetric, positive definite matrix. The matrix S determines the lethality of a reliable weapon. The contours of constant lethality are all ellipsoids: for example, the ellipsoid $x'Sx = 1$ (the "one-sigma ellipsoid") determines those points x that will be killed with probability $\exp(-0.5) = .607$ by a reliable weapon. Various special cases of this lethality function have been considered by other authors. von Neumann (Taub (1962)) employs the

function $\exp(-x^2/w^2)$ in one dimension, a special case where $S = 1/w^2$. Grubbs (1968), Helgert (1971) and Bressel (1971) use the function $\exp(-0.5(x^2/\sigma_x^2 + y^2/\sigma_y^2))$ in two dimensions. Fraser (1951) employed essentially the same function in his study of the effects of aim point wander. The generalized form used here allows for a multiplicative factor ρ and permits the lethality ellipse to be oriented in an arbitrary direction in an arbitrary number of dimensions. Although we will consistently refer to ρ as a reliability, it might also incorporate other factors useful in fitting the form to the damage function of a reliable weapon. The mathematics for handling this generalized form is the primary contribution of this paper.

Let L be the inverse of S, and call it the "lethality matrix". Since L has the properties of a covariance matrix, Equation (1) applies and the lethal volume of the weapon is

$$\int D(x)dx = \rho(2\pi)^{m/2}\sqrt{|L|}. \quad (2)$$

Thus, powerful weapons have large lethality matrices, in the sense that the size of a matrix is measured by its determinant.

Now suppose that the weapon impacts at a random point X with respect to the target, where X is multivariate normal with mean μ and covariance matrix Σ. We interpret μ as the aim point and Σ as the covariance matrix of the dispersion error. Let the probability of killing the target be $P(\rho, \mu, \Sigma, S)$. Then $P(\rho, \mu, \Sigma, S) = E(D(X))$, the expected value of the damage function. Letting the right-hand-side of (1) be C, this is

$$P(\rho, \mu, \Sigma, S) = \rho C^{-1} \int \exp(-Q(x)/2) dx, \quad (3)$$

where $Q(x) = (x - \mu)'\Sigma^{-1}(x - \mu) + x'Sx$. After completing the square, it can be shown that, $Q(x) = (x - m)'(\Sigma^{-1} + S)(x - m) + K$, where $m'(\Sigma^{-1} + S) = \mu'\Sigma^{-1}$. It follows from the latter equation that $m = (I + \Sigma S)^{-1}\mu$, where I is the identity matrix. The constant K is

$$
\begin{aligned}
K &= \mu'\Sigma^{-1}\mu - m'(\Sigma^{-1} + S)m \\
&= \mu'\Sigma^{-1}(\mu - m) \\
&= \mu'(\Sigma^{-1} - \Sigma^{-1}(I + \Sigma S)^{-1})\mu \\
&= \mu'S(I + \Sigma S)^{-1}\mu.
\end{aligned}
$$

According to (1) and the definition of C,

$$
C^{-1} \int \exp\left(-\frac{1}{2}(Q(x) - K)\right) dx
$$
$$
= 1/\sqrt{|\Sigma||\Sigma^{-1} + S|}. \quad (4)
$$

Letting $R \equiv S(I + \Sigma S)^{-1}$, we therefore have

$$P(\rho, \mu, \Sigma, S) = \frac{\rho}{\sqrt{|I + \Sigma S|}} \exp\left(-\frac{1}{2}\mu'R\mu\right), \quad (5)$$

a simple expression for the probability of killing the target with one shot. When specialized to one dimension, (5) reduces to the formula that von Neumann first derived and employed in 1941. The important thing about (5) is that it

has the same form as $D(x)$, except that x has been replaced by μ.

Anticipating a salvo of several shots, we now introduce an aiming error Y that is common to all of them. The kill probability of the typical shot is then $P(\rho, Y + \mu, \Sigma, S)$, since the common error has the effect of shifting the aim point. If there are actually n shots, then attach subscripts to ρ, μ, Σ, S, and R, retaining the unsubscripted symbols for n-vectors. All n shots are assumed to be independent, conditional on Y being given, so the probability that all n shots miss is

$$Q(\mu) = E\left(\prod_{i=1}^{n}(1 - P(\rho_i, \mu_i + Y, \Sigma_i, R_i))\right),$$

$$(6)$$

where the expectation is with respect to the distribution of Y. The notation stresses dependence on μ because μ is controllable by the marksman; indeed, the purpose of the whole development is generally to choose the aim points μ in order to minimize $Q(\mu)$.

Except for multiplicative constants, (5) has the same Gaussian form as the damage function. If Y is itself Gaussian, the expectation in (6) can be therefore be accomplished analytically. First, it is necessary to expand the product in (6). Let N be the set of all subsets of $\{1, \ldots, n\}$, a set of 2^n subsets, and let U be an element of N. Then (6) can be written

$$Q(\mu) = \sum_{U \in N} E\left(\prod_{i \in U}(-P(\rho_i, \mu_i + Y, \Sigma_i, R_i))\right)$$

$$= \sum_{U \in N}\left\{\prod_{i \in U}\left(-\frac{\rho_i}{\sqrt{|I + \Sigma_i S_i|}}\right)\right\}$$

$$E\left(\exp\left(-\frac{1}{2}\sum_{i \in U}(\mu_i + Y)'R_i(\mu_i + Y)\right)\right)$$

$$\equiv \sum_{U \in N} C_U E_U, \quad (7)$$

where C_U is the factor in braces in (7) and E_U is the expectation. To evaluate E_U, first define the moments

$$M_U^2 \equiv \sum_{i \in U} \mu_i' R_i \mu_i;$$

$$M_U^1 \equiv \sum_{i \in U} \mu_i' R_i; \quad M_U^0 \equiv \sum_{i \in U} R_i. \quad (8)$$

Then

$$E_U = E\left(\exp\left(-\frac{1}{2}\{M_U^2 + M_U^1 Y + Y' M_U^1{}' \right.\right.$$

$$\left.\left. + Y' M_U^0 Y\}\right)\right). \quad (9)$$

Equation (9) is the expectation of another exponentiated quadratic expression. Therefore, if we now assume that Y is multivariate normal with mean 0 and covariance matrix T, (9) can be evaluated by a process paralleling the derivation of (5). The result is

$$E_U = \exp\left(-\frac{1}{2}\{M_U^2 - M_U^1{}' T\right.$$

$$\left. \cdot (I + M_U^0 T)^{-1} M_U^1\}\right) / \sqrt{|I + M_U^0 T|}. \quad (10)$$

With this expression inserted into (7), we finally have an analytic expression for $Q(\mu)$. The empty subset $U = \phi$ need not be included in the sum if the actual object is to compute the kill probability $1 - Q(\mu)$; since $C_\phi E_\phi = 1$, the two unity terms cancel.

Equation (10) is valid even if Σ or T is a singular covariance matrix, or if S is a singular precision matrix. Note that a weapon with infinite lethal volume according to (2) need not have a unity kill probability, even if its reliability is 1. The same thing is true of cookie-cutter damage functions, when generalized to include ellipses.

The probability that the number of "killing" shots is zero is given by (7). A similar method can be used to find the probability mass function of the number of killing shots (Fraser (1951)).

Except for generalizing to the case $m > 1$, the above derivation is essentially identical to von Neumann's (Taub (1962)).

COMPUTATION

The moments defined in (8) are all sums, efficiently calculated by considering the shots one-by-one. A list of subsets is initialized to include only the empty set, and then repeatedly doubled in length by both including and not including the next shot in each of the current subsets. When the next shot is included in a subset, the three moments of the child subsets are augmented by adding the appropriate term to the moments of the parent. At the end, the list will include all 2^n subsets.

Formula (7) is an alternating series in the sense that a given subset has a different sign than all of its children. Alternating series are notorious for poor convergence properties, so inaccuracy should be expected when n is "large." Assuming that double precision arithmetic is used, all numbers being summed should be accurate to about 10^{-13}. If all 2^n roundoff errors are that large and have the same sign, the resulting total error should be about $2^n 10^{-13}$, which is 10^{-4} for $n = 30$. One might therefore expect computations based on (7) to be suspect for $n > 30$, approximately. Breaux and Moler (1968) expand the function $(1 - z)^n$ in Jacobi polynomials to avoid summing an alternating series, successfully testing their procedure on problems where $50 \leq n \leq 1000$. Pattern problems where n is large are also subject to accurate confetti approximations (Washburn (1974)).

Our own experience has been that the time required to compute $Q(\mu)$ from (7) becomes excessive (several seconds for $n = 20$ on a 1.6 GHZ Wintel machine) long before any problem with accuracy emerges. For smaller patterns, a fraction of a second suffices. For patterns of 10 or fewer shots, (7) is fast enough to enable local optimization of the aimpoints. An Excel workbook DG.xls that optimizes the aimpoints using the embedded Solver routine can be downloaded from http://diana.gl.nps.navy.mil/~washburn/.

EXAMPLES

Example 1: Suppose that there are three weapons all aimed at the origin, each with a reliability of $\rho = \frac{1}{2}$. Suppose further that S, T, and Σ are all $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the identity matrix I. Then $\sqrt{|I + \Sigma_i S_i|} = 2$ for all i, and $\sqrt{|I + M_U^0 T|} = 1 + \#(U)/2$, where $\#(U)$ is the number of elements in the set U. Since all weapons are aimed at the origin, M_U^1 and M_U^2 are both 0 for

Table 1. Showing the C and E factors of equation (7) for each of the subsets in a firing problem with three weapons

Subset U	C_U	E_U
{1}	−1/4	2/3
{2}	−1/4	2/3
{3}	−1/4	2/3
{1,2}	1/16	1/2
{1,3}	1/16	1/2
{2,3}	1/16	1/2
{1,2,3}	−1/64	1/5

all U. There are seven nonempty subsets of the three weapons, with the factors C_U and E_U as shown in Table 1. The sum of the $C_U E_U$ products is −33/80, so the kill probability is $1 - Q(0) = 33/80 = 0.4125$. This figure cannot be improved; the optimal pattern is to fire all three shots at the origin. This is not unusual in problems with unreliable weapons.

Example 2: A damage function may be elliptical because the weapon actually distributes a pattern of submunitions, with the pattern being stretched out in the direction from which the weapon arrives. It is sometimes asserted that the marksman would be better off if weapons arrive from multiple directions, instead of all from the same direction. To test this, consider a case where there are no dispersion errors ($\Sigma = 0$), the target location covariance matrix is $T = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$, and where the weapon lethality matrix $L \equiv (L_{ij})$ for each of four weapons is as shown in Table 2. The lethality matrices are also pictured in the form of one-sigma ellipses in the left side of Figure 2. Two of the weapons arrive from a Southwest direction (positive L_{12}), while the other two arrive from a Southeast direction (negative L_{12}). All four weapons are assumed to be perfectly reliable.

The optimal pattern is unknown, but the best locally-optimal pattern known to the au-

Table 2. Columns x and y contain locally optimal aim points for four weapons. The lethality matrix L is shown in the next three columns. The last column is the reliability.

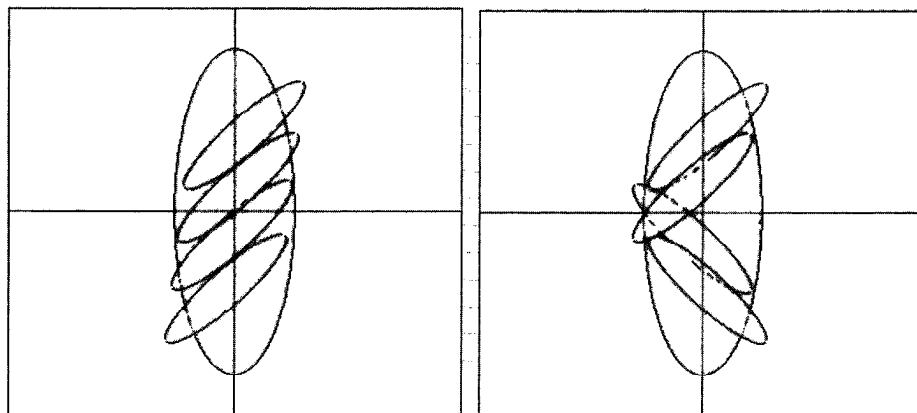| Index	Aim x	Aim y	L_{11}	L_{22}	L_{12} or L_{21}	ρ
1	−0.15	−2.82	4	4	−3.6	1
2	0.37	−0.95	4	4	−3.6	1
3	−0.15	2.82	4	4	3.6	1
4	0.37	0.95	4	4	3.6	1

Figure 2. Both sides show the target as a two-sigma ellipse, and all four weapons as one-sigma ellipses centered on the aim points shown in Table 2. The side of the square in the background has dimension 15. The pattern on the left, where all weapons come from the same direction, has a slightly higher kill probability.

thor is as given in the x and y columns of Table 2 and illustrated in the right side of Figure 2. That pattern achieves a kill probability of .640. If L_{12} is set to 3.6 for all four weapons, the best known pattern is as shown in the left side of Figure 2, with a slightly larger kill probability of .664. In this case, at least, it appears that the marksman is better off if the weapons all come from the same direction.

The reader can confirm the above calculations by employing the spreadsheet *DG.xls* mentioned above.

SUMMARY

An analytic method for computing the probability that at least one of several shots kills a point target is derived. The method relies on the assumption that all shots have a DG damage function, as well as Gaussian errors for both dispersion and the common error. It is a generalization of previous work in that lethality and error ellipses are permitted to have arbitrary orientations; that is, the Euclidean components are not assumed to be independent. The method is most suitable when the number of shots n is small, since it employs a sum that has 2^n terms.

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