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MORE ON EMISSION GAMES

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Abstract

Emission games are two-person zero-sum games where P1 emits a sequence of symbols from some alphabet while P2 watches until he decides to predict what P1 will emit next. P1 does not know either the time or details of the prediction. P2 wins if the next symbols emitted by P1 are in accordance with his prediction. Time is usually considered to be discrete. Two modifications are introduced in this paper. One is to make time continuous in a new game called the communication game. The other is to explore the value of information by making P2 blind, so that P2 does not get to study P1's emissions before making his predictions.

Introduction

I believe that emission games were inspired by an incident that occurred at the beginning of the Battle of Midway. At one point, the Japanese aircraft carrier Hiryu was attacked by a flight of B-17 bombers that attempted to sink her by dropping bombs from 20,000 feet. None of the bombs scored a hit-it takes on the order of a minute for a bomb to fall from that altitude, and an aircraft carrier can do considerable maneuvering in a minute. This incident and other similar incidents inspired considerable game theoretic work after

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World War II, the object being to describe how ships should maneuver, when and how bombers should aim bombs, and what the net result of the competition would be (Isaacs and Karlin [6]), Dubins [2] and Ferguson [3]). The subject is still relevant. A more recent example occurred during the Cold War before the advent of ballistic missile submarines. The United States was concerned about the possibility of a nuclear first strike by the Soviet Union, and considered putting ballistic missiles onto vehicles that could perpetually move around in the southwestern desert. It takes about half an hour for a missile to get from the USSR to the USA, so the hope was that the vehicles could maneuver enough to survive an attack, just like Hiryu. The delay time and the lethal radius are both greatly enlarged, but the same issues arose as in WWII.

Very little is known about such games. The initial workers quickly realized that considerable abstraction would be necessary in order to have any chance of successfully employing game theory. Their bomber-versus-battleship game is played in one dimension with no time limits, and the ship's only options are to either move forwards or backwards at every discrete time instant. The one-step game was easily solved, the two-step game turned out to be much harder (Dubins [2])), and the three-step game is still unsolved, albeit well approximated (Goodson [4] and Lee and Lee [8]).

Part of the difficulty with the bomber-versus-battleship game is that there are multiple emission strings that can result in the same target location. Matula's [9] emission prediction games are simpler because both sides are concerned with the occurrence of only one particular emission string. P1 emits symbols from some alphabet one at a time. P2 can observe as long as he wants, but must eventually predict that the next symbols will *not* be the string for which the game is named. Using Matula's notation, for example, EP1100 has an alphabet with only two characters, and P1 wins unless P2 successfully predicts that the next four characters emitted will not be 1100. EP11 is trivial because P1 can win by emitting nothing but 1's, but EP1100 is not trivial. In addition to solving the game, Matula shows that no finite memory for the number of symbols already emitted will suffice for P1.

The first game we will consider here can be thought of as a modification of an emission prediction game where time is continuous, rather than discrete. Instead of choosing a single time to act, P2 must choose an open interval of "act time".

The Communication Game (CG)

This game is played on the nonnegative real line, hereafter "time". P1 emits "starts" at any sequence of times that he desires, and wins if any start has two properties, those being

1. The start must be "valid" (see below), and

2. There must be no succeeding start within time *L* of it, *L* being one of the two parameters of the game.

P2 can watch P1 make starts as long as he wishes, but must eventually choose some open subset of time within which subsequent starts will be valid. That subset can itself consist of subsets and P2 can rearrange it in the future as he wishes, but the total measure of the set of open times must eventually be at least *T*, the other parameter of the game. P1 cannot observe any of P2's actions. Both players know both parameters.

To be complete, we must describe the payoff in extreme cases. If P1 emits only a finite number of starts, none of which are valid, he loses even if P2 does not allocate all of *T*. Otherwise P1 wins if P2 does not allocate all of *T*.

I call CG the communication game because P1 might be trying to send a message of length *L* over a channel guarded by P2, with P2 being forced for some reason to relax his guard (open the channel) for an amount of time *T*. To succeed, P1 must start his message at one of the "valid" times left open by P2. Only the start of P1's message needs to be in a valid interval, perhaps because it includes a key that is required to decode the rest of the message. If P1 interrupts himself by making another start too soon after a valid start, the message received on the other end of the channel will be incomplete, hence the second rule above. The nomenclature of sending messages will be useful in the sequel. A message will be called "complete" if it is not interrupted, and "successful" if it is both valid and complete. P1 wins if any of his starts are successful.

It is not true that P1 should never interrupt himself. If he were to adopt any emission strategy with that property and announce it to P2, as he is free to do if the strategy is optimal, P2 could simply wait for an invalid start and then safely allocate an open interval within the next L units of time, perhaps of length $L/2$. By repeating that action with subsequent invalid starts, P2 could dispose of the entirety of T without ever risking a valid start. Occasionally interrupting himself has got to be part of P1's optimal strategy.

It turns out that P1's optimal strategy is to make starts in a Poisson process. A complete proof of this can be found in Appendix A, but the rest of this paragraph may suffice. If λ is the rate of P1's Poisson process, then the average number of valid starts will be λT regardless of how P2 distributes T (P2's ability to monitor P1's starts is of no use to P2 because of the lack of memory of the exponential distribution that separates starts). Each of P1's starts will be a successful transmission with probability $\exp(-\lambda L)$, so the average number of successful starts is $\lambda T \exp(-\lambda L)$. To maximize this quantity, P1 should choose λ to be $1/L$, and the maximized average number of successful transmissions is then $T/(eL)$, where e is the base of natural logarithms. If the actual number of successful transmissions is a Poisson random variable, the probability of at least one such is $1-\exp(-T/(eL))$, which we will show to be the value of the game. The hypothesis that the number of successful transmissions is a Poisson random variable is actually false, hence the need for Appendix A, but the conclusion is nonetheless correct.

Player 2's optimal strategy is more complex than P1's. To describe it, begin by partitioning time into cycles marked by the end of a completed message, with the first cycle starting at time 0. Whether complete messages are successful is irrelevant as far as cycles are concerned. Any given cycle might include several incomplete messages, but by definition will include only one complete message. The cycling structure is entirely under P1's control.

If a cycle ends at time (say) t, P2 selects a random variable X from a carefully chosen density $f(\,)$ that will be described later, and, if the next cycle does not end before $t + X$, opens the channel for a small valid time interval δ just after $t + X$. Call this event A for "act". Event A happens if and only if the next cycle does not end before $t + X$. If A does not happen, then P2 does nothing in the cycle that begins at t, and will independently draw another X for the following cycle. At some time $t + y$, P1 will start his next complete message. Event A is the event that $y + L > X$. Note that P2 does not need to know y in order to act, since all he needs to observe is that the current cycle has not ended. If and only if $t + y$ lies in the open interval δ just after $t + X$, as it will with probability $\delta f(y)$ if P2 acts, then P1 wins in the current cycle, call that event W for "win". The conditional probability that P1 wins in the current cycle, given that P2 acts, is therefore $P(W \mid A) = \delta f(y)/F(y + L)$, where $F(\,)$ is the cumulative distribution function corresponding to density $f(\,)$. P1 controls y and will maximize the ratio involved in $P(W \mid A)$. For P2, the problem therefore arises of finding the density $f(\,)$ that makes the maximum value of that ratio be as small as possible.

The density that P2 desires is one of a class called "distribution constrained densities" by Matula [9]. It is useful to introduce an intermediate function $G(u)$ in describing them. $G(u)$ is the solution of an ordinary differential equation $G'(u) = -\beta G(u-1)$ for $u > 0$, where β is a parameter and $G(u)$ is defined to be 1 for $u \le 0$. The solution is $G(u) = 1 - \beta u$ for $0 < u \le 1$, then a quadratic function for $1 < u \le 2$, then cubic, etc., in successive unit intervals. The details are in Appendix B, but it turns out that $G(u) > 0$ for all u as long as $\beta \le \beta_0 \equiv 1/e$. If $\beta > \beta_0$, $G(u)$ becomes negative at some point u_0. Figure 1 plots $G(u)$ for two values of β. When $\beta = 0.5$, u_0 is approximately 2.7. When $\beta = \beta_0$, $G(u)$ never quite becomes negative, hence the term "boundary function". It will be shown in Appendix B that the area under the boundary function is $e - 1 \approx 1.7$.

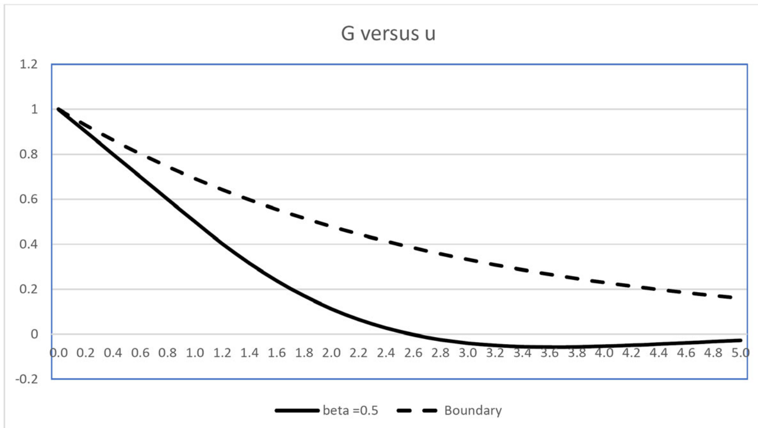

Figure 1. The function $G(u)$ for two values of β. The boundary curve is for $\beta = \beta_0$.

As long as $\beta > \beta_0$, P2 can use $G(u)$ to make sure that the critical ratio is β/L. P2 does this by setting $F(x) = G(u_0 - x/L)$ for $x \geq 0$. That function rises monotonically from 0 to 1 as x increases from 0 to Lu_0 and is 1 for larger values of x, so it is a legitimate differentiable cumulative distribution function. The corresponding density is $f(x) = -G'(u_0 - x/L)/L = \beta F(x + L)/L$, hence the critical ratio is β/L.

In order to win, P2 must act T/δ times without any of those independent acts leading to a successful transmission. The probability of this is $(1 - \delta\beta/L)^{T/\delta}$, and the limit of this expression as δ approaches 0 is $\exp(-\beta T/L)$. Thus this strategy for P2 guarantees, in the limit where P2 makes β nearly β_0 and δ nearly 0, that the probability that P1 wins cannot exceed $1 - \exp(-(T/eL))$. This is the same value that P1 can guarantee by using a Poisson process of starts, so CG is solved.

The communication game would be boring to watch, especially when P2 is conservative. P2's samples of X in the successive cycles tend to be very large when β is not much larger than β_0; the parameter u_0 becomes very large, and the amount subtracted from u_0 in the process of obtaining X cannot on the average exceed 1.7, the area under the boundary curve. In most cycles, a conservative P2 will not act. Even when P2 acts, in most cycles nothing will happen because δ is so small that P1 is unlikely to make a start in the valid interval. None of this dalliance should be surprising, since information is free to P2 in CG and, as every statistician knows, there is always something to be said for gathering more data.

Assessing the Value of P2's Information Advantage

CG without information

The communication game is still well defined even if P2 does not have the privilege of monitoring P1's emissions before deciding what set of times to leave open, and solution of the revised game (call it IG for "Intersection Game") would be illuminating as to the value of information. In IG, the argument that P1 should never interrupt himself is correct, since any start that is subsequently interrupted should simply be deleted because it cannot succeed and might itself serve as an interruption. In fact, it is conceptually simpler to omit the idea of interruption and simply require that P1 chooses a set of starts that are all separated by at least L. P2 chooses a set of open times with measure at least T, and P1 wins if and only if the sets intersect. By comparing the values of CG and IG, the value of P2's information in CG can be determined.

P2 cannot use his CG strategy in IG because he cannot observe P1's starts. One reasonable strategy for P2 would be a "confetti" strategy where the open times come in the form of widely distributed, randomly placed small bits. P2 picks any long interval of time of length S, independently locates n random points within S and surrounds each point with an open interval of length $\delta \equiv T/n$. The probability that there is no overlap is

$\prod_{i=0}^{n-1}(1 - i\delta/S)$, which P2 can make approach 1 by making *S* large, as he will want to do anyway. P1 can make at most $m \equiv 1 + [S/L]$ starts within *S*, each of which will win for P1 if it starts within one of the open intervals. Each invalid start establishes an additional interval of length δ that is not open, so the probability that all attempts are invalid is $P \equiv \prod_{j=0}^{m-1}(1 - T/(S - j\delta))$. Since δ can be made small by increasing *n*, we have $\lim_{S\to\infty} \lim_{n\to\infty} P$ $= \lim_{S\to\infty}(1 - T/S)^{S/L} = \exp(-T/L)$. This strategy for P2 with large values for *n* and *S* thus establishes an upper bound of $1 - \exp(-T/L)$ on the value of IG.

P1 might be attracted by the "comb" strategy of emitting a start every *L* units of time, since doing so maximizes the density of starts. To incorporate unpredictability, the first start could be made uniformly at random in the interval $[0, L]$. However, the win probability guaranteed by that strategy is 0. P2's counter would be to use a comb strategy himself, with T/δ small intervals of length δ separated by *L*. The two combs will intersect with probability δ/L, which P2 can make as small as he likes, so P1 cannot guarantee a win probability of anything greater than 0 using the comb strategy. P1 has a conflict between start density and unpredictability, and the comb strategy is not sufficiently unpredictable. P1 could still use his CG Poisson process, improving it after generation by removing interruptions as described above. Doing so would guarantee at least the CG value as a lower bound, but this still leaves a significant gap between the upper and lower bounds. I do not know the solution of IG. I hypothesize that the best strategy for P1 is actually a renewal process where all of the independent interstart times are *L* plus a uniform random variable. I also hypothesize that the confetti strategy is not optimal for P2. Both of these hypotheses are prompted by the solution of the discrete equivalent introduced next.

A Discrete Version of IG

The discrete version of IG is played on the set of positive integers, rather than the nonnegative real line, and P2 is required to select a certain number of integers, none of which he hopes will be included in the set selected by P1. Call the game G_{kj} if P1 is required to have the absolute difference between any two of his selected integers be at least *j* and if P2 is required to select *k* integers.

The value of G_{0j} is zero because P2 can choose the empty set, and the value of G_{k1} is 1 for $k > 0$ because P1 can include every integer. G_{12} is almost as simple: P1 selects all the odd integers or all the even integers on a coin flip, P2 chooses either 1 or 2 on a coin flip (any other pair of consecutive integers would do as well), and the value is $1/2$. In fact, the value of G_{1j} is $1/j$, with P2's strategy being to randomize over any *j* consecutive integers. Since P1 cannot include more than one of those integers in his chosen set, he can win at most one time in *j*. P1's optimal strategy is also simple: P1's lowest integer *i* is equally likely to be any of the first *j* integers, and the rest of his set consists of $i + j, i + 2j, i + 3j, \ldots$. Every positive integer is included in one of those *j* possible sets, so P1 can guarantee the same value as P2. P1's strategy is the discrete version of the comb strategy that is not optimal for P1 in IG, but it is optimal in G_{1j}. P1 does not have a conflict between density and randomness in G_{1j}; his chosen strategy is always maximally dense, always including the fraction $1/j$ of the positive integers, and the randomness of his initial integer is sufficient. We will show below that this is not the case for $k = 2$.

It is of interest to compare G_{1j} with an Emission Prediction game EP_j where the crucial character string is a 1 followed by $j - 1$ zeros. The only significant difference between the two games is the information that P2 gets as he watches P1's emissions in EP_j. If the rules of EP_j are modified so that P2 cannot observe P1's sequence before making his prediction, we have a game that is essentially G_{1j} because P1 would lose by interrupting the string once it is begun; that is, every one of P1's 1's should be followed by at least $j-1$ 0's. The comparison thus shows the value of P2's information in EP_j. Matula [9] shows that the value of EP_j is $(j-1)^{j-1}/j^j$, considerably smaller than the value of G_{1j}. Since $\lim_{j\to\infty} j(j-1)^{j-1}/j^j = 1/e$, it might be said that the value of information in this circumstance is a factor of about 2.7 in the value of the game.

The game G_{kj} gets more complicated if $k > 1$. In G_{1j}, P1's selected integers always differ by exactly *j*, thus making the selected set be as dense as the rules permit. If P1 were to do that in G_{2j}, then P2 could exploit the predictability of that tactic by selecting two integers that also differ by exactly *j*, thus holding P1 to winning only one time in *j*, just as in G_{1j}. As in IG, P2's comb would defeat P1's comb. When P2 is required to select more than one integer, P1 must introduce some additional variety into his selection, even at the cost of reducing its density. The following theorem gives the solution of G_{2j}, with an optimizing strategy for P1 included in the proof.

Theorem. *The value of* G_{2j} *is* $V \equiv (2K+1)/((j+K/2)(K+1))$, *where* K *is the largest integer in* $\sqrt{j}$.

Proof. We will show that both P1 and P2 can guarantee *V*, considering first P1 and then P2.

It is convenient to imagine P1 selecting positive integers in an increasing sequence because that sequence can be described as a Markov chain, even though it is only P1's chosen set (not the order of choice) that matters in the payoff. We will use "state" to mean the largest integer generated so far in that sequence. From state *i*, P1 advances to each of the states $i+j, i+j+1, \ldots, i+j+K$ with equal probability, that probability necessarily being $Q \equiv 1/(K+1)$. The average advance is $j + K/2$, so only a fraction $f = 1/(j + K/2)$ of the integers will appear in this sequence. P1's object is to make every positive integer appear in this sequence with probability *f* by carefully choosing the initial state. Specifically, P1 initially chooses states $1, \ldots, j$ with probability *f* each. The total initial probability for those states is only *jf*, which is less than 1, but P1 also initially chooses state $j + i$ with probability $f(K - i + 1)/(K + 1)$ for $1 \le i \le K$. This makes the total initial probability be 1, and also makes the total visitation probability be f for every positive integer. State $j + 1$, for example, is initially chosen with probability $fK/(K + 1)$, but is also visited from state 1 on the first advance with probability *fQ*, making the total visitation probability *f*. It is only the total visitation probability that matters - the advance on which the visit occurs is irrelevant as far as the payoff is concerned.

Against this strategy for P1, suppose P2 selects the two distinct integers *m* and *n*, with $m > n$, let $k \equiv n - m$, and let E_m and E_n be the events that those integers occur in P1's sequence. The event that P1 wins is $E_m \cup E_n$, the union of the two events. P1's strategy makes the probability of both events be *f* each, but the probability of winning is less than $2f$ because E_m and E_n are not mutually exclusive. We must subtract the probability of the intersection to find P1's win probability. Let $p_i \equiv P(E_{m+i} \mid E_m)$ for $i > 0$, which is independent of *m*. Then $p_i = 0$ for $0 < i < j$, since the state must always advance by at least *j*. Also, $p_i = Q$ for $j \le i \le j + K$ because the only way to advance by i in that range is by a single transition. The sequence subsequently satisfies the recursion $p_i = (p_{i-j} + p_{i-j-1} + \cdots + p_{i-j-K})/(K + 1)$ for $i > j + K$. The largest number in this sequence is Q by induction, since every calculation is an average of numbers that do not exceed *Q*. Since $P(E_m) = f$, $P(E_m \cap E_n) = fp_k$, which cannot exceed *fQ*. Therefore,

$$P(E_m \cup E_n) \geq f + f - fQ = f(2 - 1/(K+1))$$
$$= (2K+1)/((j + K/2)(K+1)) = V.$$

This expression is maximized by making K the greatest integer in $\sqrt{j}$, as stated in the theorem, but it is not necessary to establish that here because it follows from the fact that P2 can guarantee the same value, as will be shown next.

P2's strategy is to select one integer m uniformly at random from the set $S = \{1, \ldots, M\}$, where M is a large number that can be revealed to P1, and then to randomly choose an advance i in the interval $[j, j + K]$ to obtain his second number $n = m + i$, except that P2's second number is $n = m + i - M$ should $m + i$ exceed M. The probability distribution that P2 uses to advance is not uniform like P1's. Instead P2 uses a distribution **q** that makes the differences $2 - q_i$ be proportional to i. If C is the proportionality constant, then necessarily $\sum_{i=j}^{j+K} Ci = C(K+1)(j + K/2)$. But the same sum must also be $\sum_{i=j}^{j+K}(2 - q_i), = 2(K+1) - 1 = 2K+1$, so $C = (2K+1)/((j + K/2)(K+1))$. Since C equals V, we will refer only to V from here on.

To develop P1's optimal response to P2's strategy, we first observe that P1 must choose at least one number within S, since otherwise he will lose. Let his sequence of choices in S be $s_1, \ldots, s_N$ in ascending order, and define the advance from s_k to be $s_{k+1} - s_k$ except that the advance from s_N is defined to be $M + s_1 - s_N$. The sum of all N advances is M.

Let x_i be the number of times that P1's advance is i. Then $\sum_i ix_i = M$ and the total number of integers in P1's sequence is $X \equiv \sum_i x_i$. Since P2's first choice m is selected uniformly at random within S, $P(E_m) = X/M$, and for the same reason, it is also true that $P(E_n) = X/M$. The probability that both events occur is x_i/M, since it is merely a question of whether n is within the set of P1's integers wherein the advance is i. The probability distribution required for removing the condition on P2's advance is known, so we can write

$$v \equiv P(E_m \cup E_n) = P(E_m) + P(E_n) - P(E_m \cap E_n) = \left(2X - \sum_{i=j}^{j+K} q_i x_i\right)/M.$$

The vector **x** must consist of nonnegative integers, at least one of which is positive. With one exception, x_i must be 0 for $i < j$. The exception is because the advance from s_N can be as small as 1 (s_1 could be 1 and s_N could be M). P1 can therefore set x_1 to 1, rather than 0, which he should do if v is to be maximized. Taking account of this and recalling that $2 - q_i = Vi$, we can rewrite the equation for v as

$$Mv = 2 + \sum_{i=j}^{j+K} x_i(2 - q_i) + 2\sum\nolimits_{i > j+K} x_i = 2 + V\sum_{i=j}^{j+K} ix_i + 2\sum\nolimits_{i > j+K} x_i.$$

But $\sum_{i=j}^{j+K} ix_i = M - 1 - \sum_{i > j+K} ix_i$. Substituting, we have $Mv = 2 + V(M-1)$ $+ \sum_{i > j+K} x_i(2 - Vi)$. The factor $(2 - Vi)$ is never positive within the sum, even when the index i is at its lowest value $j + K + 1$. Some elementary algebra reveals that the factor is positive if and only if $j > (K+1)^2$, which is false because K is the greatest integer in $\sqrt{j}$. Therefore, P1 can do no better than to make $x_i = 0$ for $i > j + K$, and the maximized v is $2/M + V(M-1)/M$. Since M is P2's choice, he can make v be as close to V as he wishes by making M large, and therefore V is the value of the game. □

The theorem gives an asymptotically optimal strategy for P2, but there are cases where P2 can guarantee V without approximation. Consider G_{22}.

In that game P2 can randomly choose a pair of integers from the set {13, 24, 35, 46, 25}, each pair having probability $1/5$. After listing all of P1's possible sequences up to integer 6, one can observe that P1 has no sequence that will intersect more than three of those pairs, so he cannot win with a probability exceeding $3/5$, which is *V*.

It may even be true that P2 *always* has an optimal strategy, but, if he does, it is not easy to discover. Consider making *M* be a parameter of the game, rather than a choice for P2, so that both players are required to choose integers not exceeding *M*. If *M* is required to be anything larger than 5 in G_{22}, the game value is still $3/5$ and P2's optimal strategy does not change. Similarly there is a mixed strategy for P2 in G_{23} that will assure *V* (which is $3/7$) for any $M \geq 10$. However, I have been unable to discover a value for *M* that is large enough that the value of the modified G_{24} is *V* as stated in the theorem. The largest value considered so far is $M = 17$, where one can enumerate all of the strategies for both sides and solve the game by linear programming. The value of that game is $121/360$, slightly larger than $V = 1/3$, and the optimal strategies are as complicated as the denominator 360 would suggest. This situation is reminiscent of Johnson's [7] comments about a different game played on the first *M* integers where P1 selects an integer and P2 makes a sequence of guesses that are reported to be either high or low until he finally finds P1's integer. Johnson solves this game for $M \leq 11$, commenting that the case $M = 11$ is more complicated than the others.

There is a hint here that the confetti strategy evaluated earlier for P2 is probably not optimal. The strategy of selecting two integers independently at random from some large set is certainly not optimal for P2 in G_{2j}. P1 could counter with a comb and win with probability $1-(1-1/j)^2$, which is larger than the game value.

Summary

The communications game CG is a kind of emission game in continuous time. CG has been solved for all values of its two parameters. Seemingly, it would be simplifying to remove P2's information advantage in CG, thus enabling an assessment of the importance of information, but the resulting game IG has not been solved. The solution of some discrete versions of IG may nonetheless be useful to others who are interested in the subject.

Appendix A. Player 1's winning strategy in the communications game

Here we will show that P1 will win with a probability that is at least the value of the game by emitting starts in a Poisson process with rate $\lambda = 1/L$. The only fallacy in the earlier argument in the text is the assumption that the number of successful starts is a Poisson random variable. If $T < L$ and if P2 utilizes a single open interval of length T, for example, then the number of successful starts is 0 if the number of starts in that interval is 0, or otherwise 1.

It is convenient to generalize so that the amount of time remaining for P2 to allocate is t, with $0 \leq t \leq T$. P2 will then face a Markov decision process with a state that includes t and whether the most recent start is valid or invalid. The lack of memory of the exponential distribution that separates the starts in P1's Poisson process justifies that statement, and also means that we do not need to incorporate the amount of time expired since the last start as part of the process state. Let $P(t)$ and $Q(t)$ be P1's win probabilities when the most recent start is valid or invalid, respectively. In either case, P2 will open the channel for an interval of length δ. Again because of the memoryless property of the interstart times, the open interval can be assumed to start immediately after the most recent start. Let X be the time from P1's most recent start to the next start, an exponential random variable with density $f(x) \equiv \lambda \exp(-\lambda x)$. If the most recent start is invalid, then the succeeding start will be valid only if X is in the interval $(0, \delta]$, and the amount of time remaining for P2 to allocate will in that case be decreased by

X. Otherwise, P2 will rejoice because the next start will also be invalid and he will have δ less time to leave open. In other words, P1's conditional win probability is $Q(t-X)$ if $X \le \delta$ or $P(t-\delta)$ if $X > \delta$. Removing the condition, we have a functional equation for $P(t)$:

$$P(t) = \inf_{0<\delta\le t}\left\{\int_0^{\delta} f(x)Q(t-x)\,dx + \exp(-\lambda\delta)P(t-\delta)\right\}.$$

We use "inf" rather than "min" because P2 is not free to make the open interval null. If he did, the situation would repeat indefinitely and P2 would never open the channel, thus losing to P1.

The situation is similar if the most recent start is valid, except that P2 is free to use an empty interval and P1 wins if X exceeds L, as will happen with probability $q \equiv \exp(-\lambda L)$. The corresponding functional equation for $Q(t)$, after noting that $\int_{\delta}^{L} f(x)\,dx = \exp(-\lambda\delta) - q$, is

$$Q(t) = q + \min_{0\le\delta\le\min(t,L)}\left\{\int_0^{\delta} f(x)Q(t-x)\,dx + (\exp(-\lambda\delta) - q)P(t-\delta)\right\}.$$

Making δ larger than L would be pointless after a valid start because P1 will win if the next start advances by more than L, hence the restriction on δ in the equation. P2 is free to make $\delta = 0$ after a valid start, since there is no danger of the situation repeating. If he does, we have $Q(t) = q + (1-q)P(t)$. We will show, in fact, that the only differentiable solution of the two functional equations has P2 doing exactly that, with δ being any positive number after an invalid start.

If δ is infinitesimal after an invalid start, the equation for $P(t)$ becomes, to terms of first order with $P'(t)$ being the derivative of $P(t)$,

$$P(t) = \delta f(0)(q + (1-q)P(t)) + (1-\lambda\delta)(P(t) + \delta P'(t)).$$

After cancelling second order terms and then cancelling δ, this reduces to $P'(t) = \lambda q P(t)$. Since $P(0) = 0$, the only solution is $P(t) = 1 - \exp(-\lambda q t)$.

It remains to be shown that this solution is in fact minimizing. When the analytic expressions for $P(t)$ and $Q(t)$ are substituted into the right-hand-side of the functional equation for $P(t)$, it evaluates to $1 - \exp(-\lambda qt)$; that is, P2's choice of δ is immaterial after an invalid start, as long as δ is positive. The RHS of the functional equation for $Q(t)$ evaluates to $1 - \exp(-\lambda qt) + q\exp(-\lambda q(t - \delta))$. This is an increasing function of δ, confirming the assumption that P2 should make $\delta = 0$ after a valid start.

Appendix B. Distribution constrained densities and the function $G(u)$

In this appendix, we describe the function $G(u)$ defined in the text as the solution of the ordinary differential equation $G'(u) = -\beta G(u - 1)$ for $u \geq 0$, with $G(u) \equiv 1$ for $u \leq 0$. The parameter β can be either positive or negative and it is slightly easier to deal with nonnegative quantities, so for immediate purposes, define $\alpha \equiv -\beta$. For an explicit solution of the differential equation, first define $G_{-1} \equiv 1$ and then the sequence of functions $g_n(x) = \sum_{j=0}^{n} G_{n-j-1}(\alpha x)^j/j!$. If we take $G_n = g_n(1)$, then the sequence $G_0, G_1, \ldots$ is defined iteratively by evaluating $g_0(1), g_1(1), \ldots$ in that order. It is easily verified that $g'_n(x) = \alpha g_{n-1}(x)$ for $n > 0$ and that $g_n(0) = G_{n-1}$ for $n \geq 0$. Thus $g_n(x)$ advances from G_{n-1} to G_n as x advances from 0 to 1. The function $G(u)$ can now be defined in intervals as $G(u) = g_{n+1}(u - n)$; $u \geq 0$, where n is the greatest integer in u. Note that the intermediate functions $g_n(x)$ can be dispensed with if it suffices to know $G(u)$ at only nonnegative integer values n, since $G(n) = G_n$ and $G_n = \sum_{j=0}^{n} G_{n-j-1}(\alpha)^j/j!$.

Although there is no simple closed-form expression for $G(u)$, simple closed-form transforms can be determined. Consider the power series

$F(z) \equiv \sum_{n=0}^{\infty} G_n z^n$. To determine $F(z)$, first note that $G_{n+1} = \alpha^{n+1}/(n+1)!$ $+ \sum_{j=0}^{n} G_{n-j}\alpha^j/j!$. The summation is in the form of a convolution, and the power series for $\alpha^j/j!$ is $\exp(\alpha z)$, so upon multiplying both sides by z^n and summing, we get $(F(z)-1)/z = (\exp(\alpha z)-1)/z + \exp(\alpha z)F(z)$. The solution is $F(z) = 1/(\exp(-\alpha z) - z)$. The Laplace transform of $G(u)$ can also be similarly obtained. It is

$$\Gamma(s) \equiv \int_0^{\infty} G(u)\exp(-su)\,du = (1 + \alpha(1 - \exp(-s))/s)/(s - \alpha\exp(-s)).$$

For the area under the boundary function, take $s \to 0$ and $\alpha = -1/e$ to find that the area is $e - 1 \approx 1.7$.

It is crucial for P2 whether $G(u)$ ever becomes negative for some value of *u*, since in that case (and only in that case), the function can be converted into the distribution function that P2 needs. The claim in the text is that this will happen if and only if $\beta > 1/e$. Matula [9] essentially proves this when he shows that, as long as $\beta > \beta_M \equiv (M-1)^{M-1}/M^M$, where M is a large integer, there exists a probability distribution $x_1, x_2, \ldots$ such that $x_i \le \beta \sum_{j=1}^{i+M-1} x_j$, for all $i > 0$. To prove the corresponding continuous result, divide time into small increments of length $\delta \equiv 1/M$ and define the density function $f(x) = x_i/\delta$ for x in the small interval $\delta(i-1) \le x \le \delta i$. For x in that interval, the corresponding cumulative density function is $F(x) = \sum_{j=1}^{i-1} x_j + yx_i$, where *y* is the nonnegative amount that *x* exceeds the lower bound of the interval. Therefore, since $1 = M\delta$,

$$F(x+1) \geq \sum_{j=1}^{i+M-1} x_j \geq x_i/\beta = \delta f(x)/\beta,$$

as long as $\beta > \beta_M$, where the middle inequality is Matula's. But $\delta = 1/M$ and $\lim_{M\to\infty} M\beta_M = 1/e$, so we have established the existence of a density function satisfying $f(x) \leq \beta F(x+1)$ for all positive x, as long as $\beta > 1/e$. That is possible only if the function $G(u)$ becomes negative, as described in the text.

References

[1] D. Blackwell, The prediction of sequences, Internat. J. Game Theory 31(2) (2003), 245-251.

[2] L. E. Dubins, A discrete evasion game, Contributions to the Theory of Games 3 (1957), 231-255.

[3] T. S. Ferguson, On discrete evasion games with a two-move information lag, Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Vol. 1, 1967, pp. 453-462.

[4] S. W. Goodson, Near-optimal finite solutions to the three and four step discrete evasion games, Naval Postgraduate School, Monterey, CA, 1983.

[5] R. Isaacs, The problem of aiming and evasion (No. RAND-P-642), RAND Corporation, Santa Monica, CA, 1955.

[6] R. Isaacs and S. Karlin, A Game of Aiming and Evasion, The RAND Corporation, Research Memorandum 1316, 1954.

[7] S. M. Johnson, A search game, Advances in Game Theory 52 (1964), 39.

[8] L. Y. Lee and K. T. Lee, The discrete evasion game with three-move lag, Naval Res. Logist. 37(6) (1990), 867-874.

[9] D. W. Matula, Games of sequence prediction (No. ORC-66-3), Operations Research Center, University of California, Berkeley, 1966.