LANCHESTER SYSTEMS

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The utility of these notes will be much enhanced if the reader has available the Excel workbook Lanchester.xls. That workbook has several named sheets in it that will be referred to below at the appropriate place.

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1. Biological Roots

Lanchester sytems are systems of Ordinary Diffential Equations (ODE) where the state variables represent numbers of surviving combat units. They are named for F. W Lanchester (1916), who applied them to aerial combat in WWI. By that time ODE systems had already been used to model various biological phenomena. The simplest of such models is

$$\dot{x} = \alpha x. \tag{1.1}$$

If x is population size at time t, equation 1.1 says that the rate of increase $\dot{x}$ is proportional to current population, with the constant α being a "birth rate." The same equation applies to a bank deposit x continuously compounded at interest rate α. In these notes we will use the convention of indicating time derivatives as $\dot{x}$, rather than $dx(t)/dt$, and of not explicitly showing that state variables depend on time. This economy of notation should cause no confusion because all derivatives will be with respect to time, and because every state variable will have an equation like (1.1) showing its rate of change.

Equation 1.1 is reasonable when population growth is not limited by factors other than fecundity. The solution is

$$x(t) = x_0 \exp(\alpha t), \tag{1.2}$$

where x_0 is the initial value at time 0. This is the law of exponential growth. It can be witnessed for a period of time in populations that are suddenly placed in an enriched environment and freed from predation (man and certain laboratory raised microbe populations being examples).

Equation 1.1 is solvable analytically, but in practice most ODE systems are solved numerically. The simplest numerical method is Euler's method, which replaces equation 1.1 with the iterative equation $x(t+\delta)=x(t)+\delta\ \dot{x}$, in this case $x(t+\delta)=x(t)(1+\alpha\delta)$. To advance time by δ, it suffices to multiply by the factor $(1+\alpha\delta)$. Thus, if α=0.1/year and δ=.5 years and x(0)=100, then x(0.5)=105, x(1)=105(1.05)=1.1025, x(1.5)=1.157625, etc. The procedure is easily automated in a spreadsheet, with the analytic solution being the limit as δ approaches 0.

In a finite environment, any exponentially increasing population must eventually become so dense that its birth rate will decrease. The simplest model of this phenomenon is to make the birth rate $\alpha(1 - x/x_\infty)$, which decreases linearly from α when $x = 0$ to 0 when some limiting population x_∞ is reached. The ODE governing x is then

$$\dot{x} = \alpha\left(1 - x/x_\infty\right)x \,. \tag{1.3}$$

The solution of (1.3) is

$$x(t) = \frac{x_\infty}{1 + \dfrac{x_\infty - x_0}{x_0}\exp(-\alpha t)}, . \tag{1.4}$$

where $x(0) = x_0$. This is the "logistic curve" introduced by P. F. Verhulst in the mid-nineteenth century. Figure 1 shows the actual population of the United States along with a fitted logistic curve. Note that the slope of the curve has stopped increasing, but has not yet started to decrease. If the logistic fit is as good for the future as it is for the past, then the asymptotic population will be x_∞ = 420,000,000.

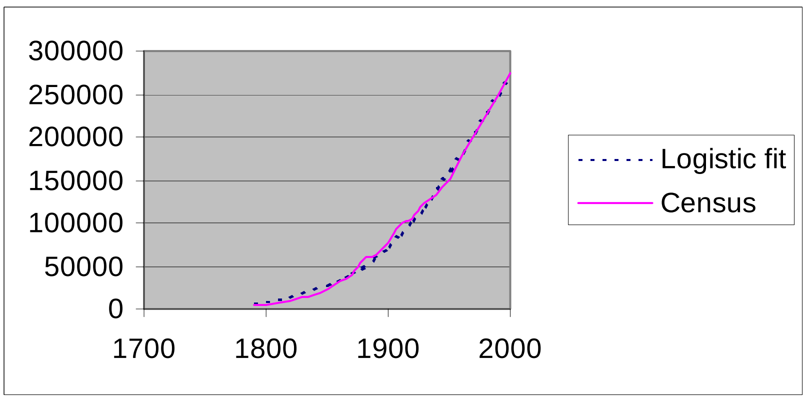


Figure 1. A logistic fit to the population of the United States

One can also model populations that interact with each other using ODE. For example, the Lotka-Volterra ODE for a predator-prey relationship are

$$\begin{aligned} \dot{x} &= \quad rx - axy;\ x > 0 \\ \dot{y} &= -sy + bxy;\ y > 0 \end{aligned} \tag{1.5}$$

In equations 1.5, the rate at which predators catch their prey is proportional both to the prey population (*x*) and the predator population (*y*). The biological assumption is that a predator's main activity is searching, with the success rate being proportional to y. This is why there is a prey decrease term proportional to *xy* and a predator increase term proportional to *xy*. In addition, there is a birth rate *r* for the prey (if predators were wiped out x would grow exponentially with rate *r*), and a "natural hazards plus deaths" rate *s* for the predators (without such a term the equations would show the prey being eventually wiped out and the predators growing to some equilibrium level and staying there). The solution of these ODE is periodic—first the prey population increases, then the predators increase until they start to wipe out the prey population, after which the predator population decreases due to lack of prey, after which the prey increases due to lack of predators, ad infinitum. It is not necessary to postulate variations in the environment to explain these cycles; they are a natural property of the system in the same way that swinging back and forth is natural to a pendulum.

More than two different populations can be discussed simultaneously. In modeling the herring population in the ocean, for example, one might have to simultaneously model the populations of diatoms, copepods and salmon as other important elements of the food chain that herring are involved in. The systems of equations quickly reach the point where analytic solutions are not available, but this fact is of little consequence given the ease of solving ODE numerically. There continues to be a significant amount of effort devoted to the modeling of ecological systems, some including man, using systems of ODE. The DYNAMO computer language is specifically oriented toward constructing such models.

2. Deterministic Lanchester Systems

2.1 Historic Battles

In Lanchester systems, the positive terms are generally due to resupply efforts, since most military systems have rates of increase unrelated to populations currently in the field. For example, Engel (1954) applied the ODE

$$\begin{aligned} \dot{x} &= \quad r(t) - \beta y;\ x > 0 \\ \dot{y} &= -\alpha x;\ y > 0 \end{aligned} \tag{2.1}$$

to the battle of Iwo Jima, where x and y are U.S. and Japanese troops on the island and $r(t)$ is the rate at which U.S. troops were landed. The parameters α and β have units of "opposing casualties per man day of combat," and were chosen to fit the record of what actually happened. If x is defined to be "U.S. troops not killed, wounded, or missing," the best fit is with β= .0544 U.S. casualties per Japanese man day and α = .0106 Japanese casualties per U.S. man day, for which choices the theoretical and actual graphs of x vs time are shown in Figure 2. The upper curve in Figure 2 also shows a different comparison where x is defined to be "U.S. troops alive." The fit in both cases is quite good, although different parameters α and β were needed to fit the upper curve.

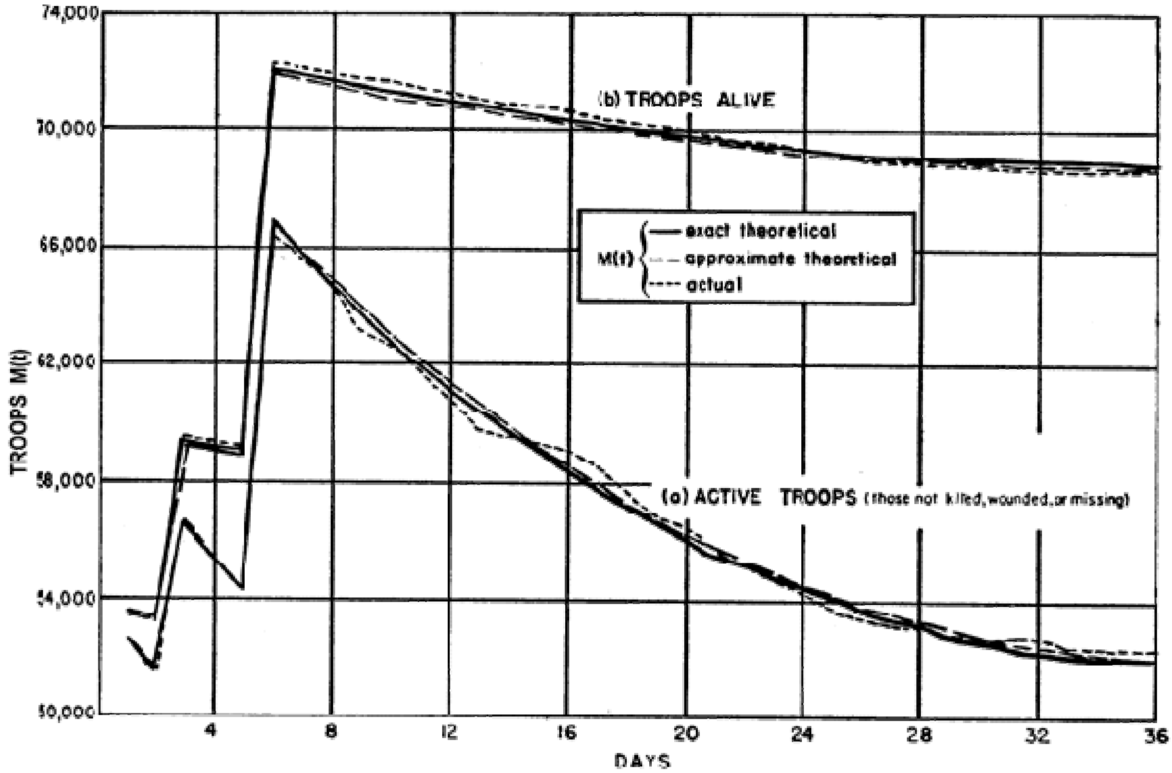


Figure 2. Lanchester model of the Iwo Jima battle

There were initially 20,000 Japanese troops on Iwo Jima, all of whom were killed. The number of U.S. casualties was also about 20,000, of which only about 25% were killed. It is somewhat surprising that total casualties on the two sides were equal when a Japanese soldier behind prepared defenses had about five time the lethality rate ($\beta/\alpha = 5.1$) as an attacking U.S. soldier. The explanation is that U.S. troops substantially outnumbered Japanese troops during most of the battle, and there is a big advantage to numerical superiority when equations 2.1 govern the dynamics. We will return to this topic when discussing Lanchester's Square Law. Sheet *IwoJima* of *Lanchester.xls* contains a numerical solution of Engel's Iwo Jima equations. Implemenation in a spreadsheet makes it easy to experiment with different parameters α and β, or different reinforcement schedules. Try increasing the initial number of Japanese troops to 30,000, for example.

Engel also did some similar computations for the battle of Crete, concluding that there were something like .104 Allied casualties per German man day and .0162 German casualties per Allied man day, although he had considerable trouble deciding what an Allied troop was (some were unarmed, some were wounded before the start of the battle, etc.). The lethality coefficient for the German troops is particularly high, probably because the German troops were crack paratroops attacking a poorly organized Allied force. Even so, the Crete parameters are not grossly different from the Iwo Jima parameters that held at a different place at a different time in a different part of the world. There does seem to be a certain degree of predictability about what will happen in land combat. Further historical material can be found in Willard (1962).

Engel was not the first to estimate Lanchester parameters from combat results. The Russian M. Osipov had already done considerable work along these lines for land battles at about the time when Lanchester was applying the idea to air warfare (see the translation in Helmbold and Rehm (1995)).

2.2 Lanchester's Square Law (Aimed Fire)

The ODE system

$$\begin{aligned} \dot{x} &= -\beta y;\ x > 0 \\ \dot{y} &= -\alpha x;\ y > 0 \end{aligned} \tag{2.2}$$

represents a situation where attrition to each side is proportional to the number of units remaining on the other, and there are no reinforcements. It should be understood that no state variable can become negative in (2.1) or in any other Lanchester system; that is, that the rate of decrease is zero rather than negative when the subject state variable is 0. Equations 2.1 are consistent with the idea that each unit has a fixed rate of fire, with each shot having a certain probability of eliminating the opposing unit at which it is aimed (but no other unit), hence the "aimed fire" description. Except for the U.S. reinforcements, it is the model Engel applied to Iwo Jima. A typical time history for $x_0 = 1000$, $y_0 = 800$, $\alpha = .8$, and $\beta = .9$ is shown in Figure

3, where it can be seen that y is eliminated entirely at time 1.5 in spite of the fact that $\beta > \alpha$. Figure 3 is taken from the *SquareLaw* sheet, which permits easy variation of parameters to explore sensitivity to initial conditions and firing rates.

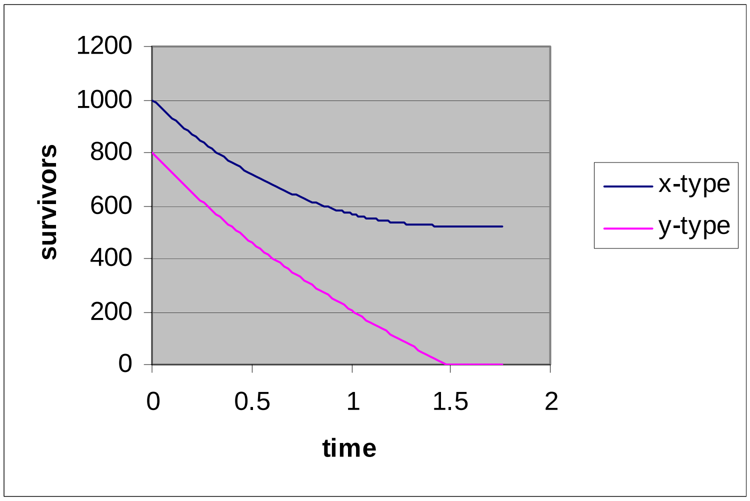


Figure 3. A square law battle

Equations 2.2 can actually be solved explicitly for x and y as functions of time, but it is simpler to eliminate time by dividing the second equation by the first to obtain y as a function of x. The result is $\frac{dy}{dx} = \frac{\alpha}{\beta}\frac{x}{y}$ which has the solution

$$\beta\left(y^2 - y_0^2\right) = \alpha\left(x^2 - x_0^2\right) \tag{2.3}$$

as long as x and y are both non-negative. If it is presumed that the battle will proceed until one side or the other is reduced to 0, and if x_f or y_f is the final number of survivors on the other side, then, according to (2.3),

$$\begin{aligned} y_f &= 0 \quad \text{and} \quad x_f = \sqrt{x_0^2 - \frac{\beta}{\alpha} y_0^2} \quad \text{if} \quad \beta y_0^2 \le \alpha x_0^2 \\ x_f &= 0 \quad \text{and} \quad y_f = \sqrt{y_0^2 - \frac{\alpha}{\beta} x_0^2} \quad \text{if} \quad \beta y_0^2 \ge \alpha x_0^2. \end{aligned} \tag{2.4}$$

In the example in Figure 3, βy_0^2 = 576,000 and αx_0^2 = 800,000, so $y_f = 0$ and $x_f = 529$.

Since the outcome in a square law battle is determined by a comparison of βy_0^2 with αx_0^2, these quantities are sometimes referred to as "fighting strengths." It is important to note that fighting strength is proportional to the lethality coefficient and to the *square* of the number of units. This heavy dependence on numbers is reasonable if one considers that there are *two* distinct reasons for introducing a new unit into a square law battle:

1) the new unit fires at the enemy
2) the new unit dilutes the enemy's fire against the units already in battle.

The introduction of better weapons does not have a dilution effect, which is why the lethality coefficient enters fighting strength only as the first power.

One consequence of the heavy emphasis on numbers is that an attacking force can easily make up for an inferior lethality coefficient by having superiority in numbers, Iwo Jima and guerilla warfare being examples.

2.3 Lanchester's Linear Law (Unaimed Fire)

If x's fire is merely directed into y's operating area, rather than being aimed at a specific y unit, then the attrition rate for y will be proportional to y, as well as to x. If the number of targets is doubled, then the number hit will also be doubled. In the simplest situation, each x unit fires at rate r, each shot eliminates all y units within some lethal area a, and y units are uniformly distributed over an area A, in which case $\dot{y} = -(ra/A)xy$. Identifying ra/A as α and the corresponding parameter for the other side as β, we are led to the equations for Lanchester's linear law:

$$\begin{aligned} \dot{x} &= -\beta xy;\ x > 0 \\ \dot{y} &= -\alpha xy;\ y > 0. \end{aligned} \tag{2.5}$$

By eliminating time and proceeding as before, we obtain

$$\alpha(x - x_0) = \beta(y - y_0) \quad \text{for} \quad x, y > 0 \tag{2.6}$$

and also

$$y_f = 0 \quad \text{and} \quad x_f = x_0 - \frac{\beta}{\alpha} y_0 \quad \text{if} \quad \alpha x_0 \geq \beta y_0$$
$$x_f = 0 \quad \text{and} \quad y_f = y_0 - \frac{\alpha}{\beta} x_0 \quad \text{if} \quad \alpha x_0 \leq \beta y_0. \tag{2.7}$$

The definition of fighting strength is now αx_0 or βy_0; that is, the product of lethality coefficient and initial force level. There is no "dilution effect" for the linear law, since fire is unaimed, and consequently a lesser influence of numbers on fighting strength than in the square law.

2.4 Extensions

There are circumstances where one side aims and the other does not. Equations corresponding to equations 2.6 and 2.7 are left as exercise 5. One can also investigate power laws other than the first and second, make the lethality coefficients functions of time, etc. Two other extensions are the multivariable case (section 2.4.1 below) and the introduction of dynamic control (2.4.2).

2.4.1 More than two state variables

Military forces are generally made up of a variety of participants, all with different parameters. This is no barrier to the construction of Lanchester models as long as an ODE can be written for each state variable. Numerical solution of the resulting system will generally be required, but such solutions are not difficult. Three examples follow.

Mine Warfare: The following system of equations represents a hypothetical situation where mines (*M*) are laid in a shipping channel by minelayers (*ML*) to cause shipping losses (*L*). The minelayers are opposed by a counterforce (*CL*) that is itself subject to attrition from the mines. The mines are gradually swept by sweepers (*S*), as well as by other ships that strike mines accidentally. The mines also have a tendency to fail all by themselves (.02 of them every day). The minelayers are reinforced at a rate of one per day, and each lays 10 mines per day as long as it survives; otherwise, there is no reinforcement.

$$
\begin{aligned}
\dot{M} &= 10\,ML - .1(S)(M) - .04\,M - .0001(CL)(M) \\
\dot{CL} &= -.0001(CL)(M) \\
\dot{S} &= -.001(S)(M) \\
\dot{L} &= .02\,M \\
\dot{ML} &= -.02(CL)(ML) + 1.
\end{aligned}
\tag{2.8}
$$

The equations provide a concise summary of the description, as well as making it more precise. Given initial values, they can be solved numerically to provide records of the five state variables as functions of time. It is inevitable that *CL* and *S* will eventually approach 0, since they are not reinforced, after which *ML*, *M*, and *L* will become very large. The equations could be used to explore the impact of parameter changes, as well as sensitivity to initial numbers (see Problem 6, or the *Minelayer* sheet).

Flak Suppression: Here is another example. This time, the four state variables are attack aircraft (*A*), suppression aircraft (*S*), cumulative bombs dropped on target (*C*), and flak sites (*F*). The flak sites shoot at both types of aircraft, and also have the effect of decreasing the accuracy of the attack aircraft in attempting to drop bombs on target. The flak sites are themselves killed by suppression aircraft. We make the following assumptions:

1) one mission per day
2) the number of flak sites *N* shooting at attack aircraft depends on the number of suppression aircraft per flak site; specifically, $N \equiv F \exp(-S/F)$ flak sites fire at attack aircraft.
3) One flak site firing at an attack aircraft shoots it down with probability .05. Attack aircraft are resupplied at .1 per day.
4) Each suppression aircraft knocks out a flak site with probability .1 per mission.
5) Each flak site shoots down a suppression aircraft with probability .02.

6) The probability that an attacker's single bomb will be accurately delivered is $1 - \exp(-.5(r_0/\sigma)^2)$, where r_0 = 100m. and $\sigma = 100(1 + .5N/A)$. Note that aircraft bombing accuracy suffers as the number of flak sites firing per aircraft increases.

The resulting differential equations are:

$$\begin{aligned} \dot{A} &= -.05N + .1;\ A > 0 \\ \dot{S} &= -.02F;\ S > 0 \\ \dot{C} &= A\left(1 - \exp\left\{-.5\big/(1 + .5N/A)^2\right\}\right);\ C > 0 \\ \dot{F} &= -.1S;\ F > 0 \end{aligned} \tag{2.9}$$

Note that N is not a state variable, but rather a quantity determined by F and S that could be eliminated by substituting into (2.9). Either the flak sites or the suppression aircraft will eventually be wiped out, since they are fighting their own little Square law battle. See the *FlakBomb* sheet.

It should be evident that most of the work involved in using ODE to model warfare is not in solving them, but in writing them down in the first place. Assumptions 1-6 in this problem would have to be based on operational experience or other forms of testing. Note the "square law" type of assumptions in this model as opposed to the "linear law" assumptions in the mine example.

Battle of the Atlantic: A roughly realistic model of the WWII U-boat war in the Atlantic is outlined below. It assumes that submarines operate only in "wolf packs" and that all ships sail in escorted convoys. Morse and Kimball (1950) make the approximation that $5n/c$ and $nc/100$ merchant ships and U-boats, respectively, will be lost when a convoy defended by c escorts is attacked by n submarines. Since the exchange ratio of U-boats to escorts was about 5/1, we will also assume that the average number of escorts lost is $nc/500$. Other numbers assumed below were typical of the time. Let

M ≡ cumulative merchant ships sunk

S ≡ remaining submarines

E = remaining escorts

p_c = probability that a given convoy is attacked by a given patrolling wolf pack on the way over = .01

t_c = time a convoy must be escorted counting trips in both directions = 30 days

T_c = cycle time for an escort = 50 days

f_c = fraction of escorts escorting convoys = t_c/T_c = .6

t_s = patrol time for a submarine = 20 days

T_s = cycle time for a submarine = 50 days

f_s = fraction of submarines on patrol = t_s/T_s = .4

r = rate at which convoys leave = 1/day

m = convoy size = 40

n = wolf pack size = 4

p_s = probability of loss per cycle for a submarine due to causes other than escorts = .04

R_s = rate of replacement of submarines = .7/day

R_E = rate of replacement of escorts = .5/day

Sf_s/n = packs on patrol

e = engagement rate counting engagements both ways = $2r\, p_c S\, f_s/n$

c = escorts/convoy = $(Ef_c)/(rt_c) = E/rT_c$

Given all the above assumptions,

$$\dot{M} = e(5n/c) = 10\text{r}^2\, T_c p_c f_s\, (S/E)\ ;\ \text{M}>0$$

$$\dot{S} = -e(nc/100) - p_s\, S/T_s + R_s$$
$$= -\left(\left[2P_c\, f_s/100T_c\right]E + p_s/T_s\right)S + R_s\ ;\text{S}>0$$

$$\dot{E} = -e(nc/500) + R_E$$
$$= -[2P_c f_s/500T_c]ES + R_E\ ;\text{E}>0$$

Note that the rate of loss of merchant ships is proportional to the square of the sailing rate, and has nothing to do with the size of the convoy. If one were to double the convoy size and halve the sailing rate, merchant losses would go down sharply because there would be fewer engagements and also fewer losses per engagement because of increased escort protection. This was basically the observation that led to increased convoy sizes in WWII. The wolf pack size also does not appear in any equation, which is superficially at odds with histories of the battle that report devastating effects of wolf pack introduction. However, while the effect of encountering a wolfpack was indeed devastating for the unlucky convoy, the effect is exactly balanced in these equations by the encounter rate's inverse proportionality to wolf pack size. McCue (1990) inlcudes a more in-depth analysis of this issue.

Substituting numbers, with time in days,

$$\dot{M} = 2(S/E);\ M > 0$$

$$\dot{S} = -(1.6 \times 10^{-6}E + 8 \times 10^{-4})S + .7;\ S > 0$$

$$\dot{E} = 3.2 \times 10^{-7}\, SE + .5;\ E > 0$$

Figure 4 shows the solution of these equations for the initial conditions $M(0) = 0$, $S(0) = 50$, $E(0) = 100$, except that merchant losses are divided by 2 to keep all 3 curves on the same scale. During the 2000 day period, there are 2 × 1250 = 2500 merchant ships sunk. The number of escorts grows at approximately .5/day throughout the period, since escort attrition is negligible. The number of submarines eventually reaches a peak and declines, but not as catastrophically or as fast as actually happened. A major reason for this is that p_s actually increased strongly with time in WWII, whereas it has been held constant here.

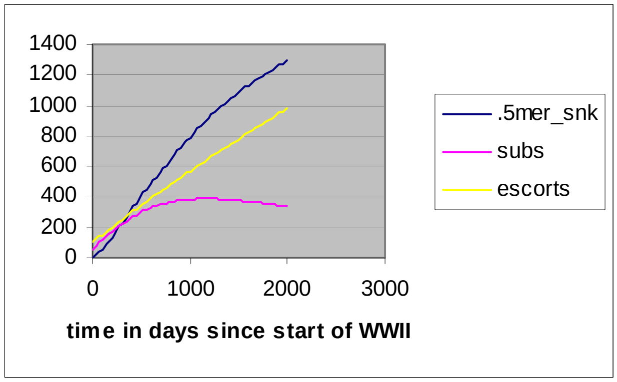


Figure 4. The Battle of the Atlantic

The *Atlantic* sheet (from which Figure 4 is taken) permits the user to experiment with different initial conditions or parameters. A great many excursions are possible; e.g., change p_c to indicate better submarine sensors, change R_E to illustrate the benefit of increased construction capacity, etc. As long as convoy engagements are well approximated by the Morse and Kimball equations, however, the submarines are doomed in the long run because they can't sink escorts fast enough.

2.4.2 The possibility of control

A Lanchester model, like any other model, should be constructed with some sort of decision in mind. The decision might typically involve armament or numbers, with the model being used to show sensitivity of some MOE to a coefficient or initial quantity. However, tactical questions can also be explored within the Lanchester framework. For example, consider a hypothetical pack of S submarines attacking a convoy of M merchant ships defended by E escorts. The submarines have to decide whether to attack the destroyers first and then the merchant ships, or to attack the merchant ships directly. Allocation of fire questions such as this can be formulated as Lanchester systems with a control variable; we might also have based an example on whether artillery should fire at an advancing enemy or at the artillery supporting him. In the present example we might have

$$\begin{aligned} \dot{M} &= -2S\,u(t);\ M > 0 \\ \dot{S} &= -E;\ S > 0 \\ \dot{E} &= -.5S(1-u(t));\ E > 0 \end{aligned} \tag{2.10}$$

where $u(t)$ is the fraction of the submarine forces' effort devoted to sinking merchant ships at time t, with time measured in hours. Note that submarines are 3 times as effective against merchant ships as they are against escorts. Suppose that long-range ASW aircraft will arrive in 2 hours, after which the submarines will have to break off, and that the submarine MOE is based entirely on survivors at that point in time. Specifically, suppose the MOE is $3\,S(2) - M(2) - E(2)$; that is, the submarine commander essentially gets 1 point for each enemy ship sunk, and loses 3 points for each sub sunk. Suppose further that the initial values are 50, 15 and 8 for M, S, and E. Each control function $u(t)$, $0 \le t \le 2$, can now be associated with a definite value of the MOE, and we can therefore consider the question "Which is the best possible control function?" One might suppose that the submarines should spend all their time shooting at merchant ships, since they are easier to sink than destroyers and count just as much. This would not have the planned effect, however, since the submarines would take rather heavy losses from the escorts. The solution turns out to be that the submarines should initially spend all of their effort shooting at escorts, and then suddenly turn their attention to the merchant ships. See the *Control* sheet to experiment with different control programs or parameters. A control variable might also be introduced into the flak suppression example by assuming that any aircraft may be designated "attack" or "suppression" at any time, and letting $u(t)$ be the proportion devoted to attack (see Exercise 9).

3. Probabilistic Lanchester Systems

In cases where the number of units on each side is small, it is difficult to interpret the continuous quantities involved in Lanchester's equations because the number of surviving units must actually be an integer. Furthermore, one expects intuitively that "luck" should play a significant role, since the presence or absence of a single unit may make a substantial difference in the outcome. Both of these problems with the ordinary interpretation disappear if Lanchester's equations are interpreted probabilistically.

3.1 Monte Carlo Simulation

Specifically, suppose the state equations are $\dot{x} = -f(x,y)$ and $\dot{y} = -g(x,y)$. In terms of a small quantity of time Δ, the equations are $x(t+\Delta) = x(t) - f(x(t),y(t))\Delta$ and $y(t+\Delta) = y(t) - g(x(t),y(t))\Delta$. Now, actually, x and y will probably not change *at all* in a small time Δ, and will decrease by exactly one unit if there is a change. To better indicate the integer nature of the forces involved, replace x and y by m and n, and think of (m,n) as the "state" of the battle. Then there are only three possible changes in the state over a small time Δ: either $(m,n) \xrightarrow{\Delta} (m-1,n)$ or $(m,n) \xrightarrow{\Delta} (m,n-1)$ or $(m,n) \xrightarrow{\Delta} (m,n)$, where $s_1 \xrightarrow{\Delta} s_2$ means s_2 follows s_1 after time Δ. The probabilities involved should be

$$
\begin{aligned}
&P\left\{(m,n) \xrightarrow{\Delta} (m-1,n)\right\} = f(m,n)\Delta \\
&P\left\{(m,n) \xrightarrow{\Delta} (m,n-1)\right\} = g(m,n)\Delta \qquad (3.1) \\
&P\left\{(m,n) \xrightarrow{\Delta} (m,n)\right\} = 1 - f(m,n)\Delta - g(m,n)\Delta
\end{aligned}
$$

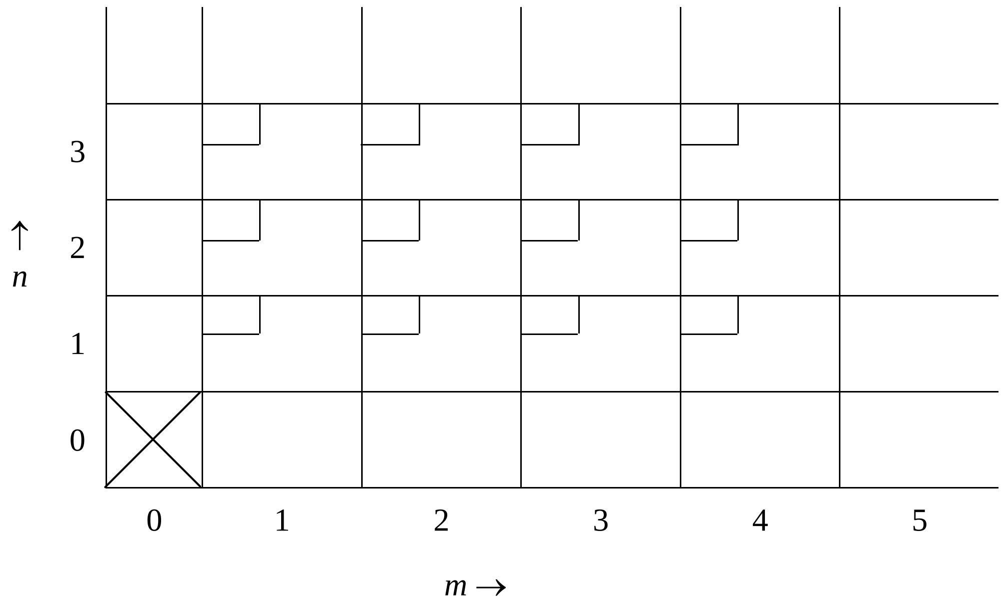
Equations 3.1 are an interpretation of the Lanchester system as a continuous-time Markov chain (Ross, 1997). The most direct implementation of this idea is a Monte Carlo simulation where equation 3.1 govern transitions made in a long succession of small time intervals. The *SquareLawRnd* sheet does this for Lanchester's Square Law. Results can be surprisingly variable, with sometimes one side winning and sometimes the other. The winner of a Square law battle generally does so resoundingly, and as a result a histogram of the net difference in survivors is actually bimodal (the *SimSheet* sheet will draw the histogram if its A1 cell references the appropriate net difference cell of SquareLawRnd).

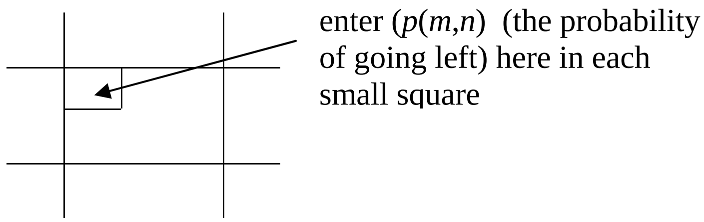
3.2 Payoffs Based on the Terminal State

A time-step Monte Carlo simulation such as the one in SquareLawRnd is computationally wasteful because in most time intervals a great deal of trouble is taken to conclude that nothing happens, particularly when Δ is made small to ensure accuracy. If the state transitions are of interest, but not their times, this problem can be avoided by requiring that the transition should not be null. If we introduce $s_1 \to s_2$ as meaning that the next actual state change is from s_1 to s_2, without regard to when the change occurs, then it follows from (3.1) that

$$
\begin{aligned}
P\{(m,n) \longrightarrow (m-1,n)\} &= \frac{f(m,n)}{f(m,n)+g(m,n)} \equiv p(m,n) \\
P\{(m,n) \longrightarrow (m,n-1)\} &= \frac{g(m,n)}{f(m,n)+g(m,n)} \equiv 1-p(m,n)
\end{aligned}
\tag{3.2}
$$

The likelihoods of the transitions from one state to another can be shown on a transition diagram like the one below, where all transitions are either “down” or “left,” and $p(m,n)$ is the probability of going “left”:





The transition diagram might be the basis of a different kind of Monte Carlo simulation where random numbers are compared to $p(m,n)$ to determine whether the next transition is "down" or "left." In some cases, however, it may be possible to compute the quantity of interest without doing any simulation at all. If the battle is presumed to continue until either $m = 0$ or $n = 0$, and if some MOE is given that depends only on the terminal state, then the terminal values of the MOE can be written in the $m = 0$ column and the $n = 0$ row of the transition diagram. The expected value of the MOE can then be computed for any large square in the diagram, using the principle that the MOE in square (m,n) can be found if it is already known in $(m,n-1)$ and $(m-1,n)$. The required formula is the formula for conditional expectations:

$$\text{MOE}(m,n) = p(m,n)\,\text{MOE}(m-1,n) + (1 - p(m,n))\,\text{MOE}(m,n-1). \qquad (3.3)$$

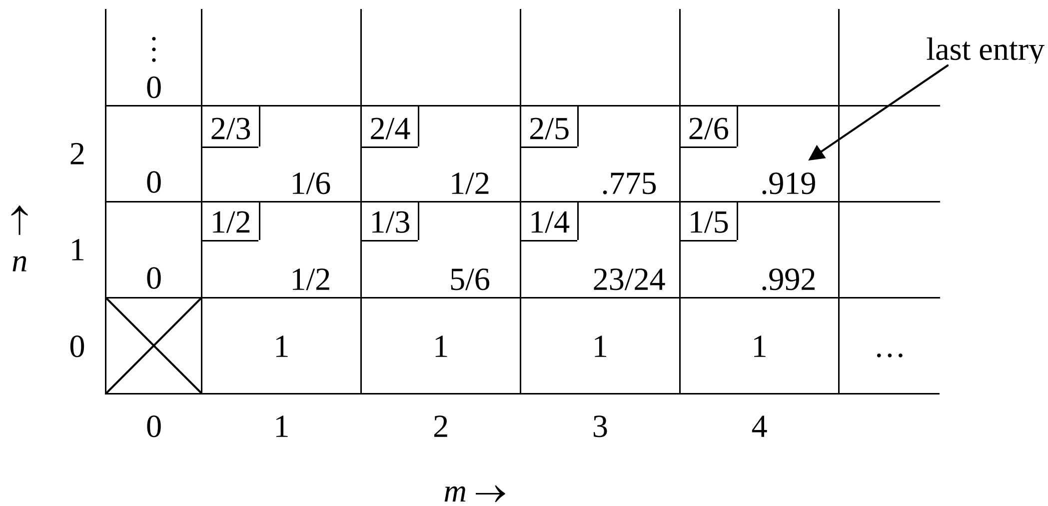
Example: $f(m,n) = \beta n$, $g(m,n) = \alpha m$, $\alpha = \beta$.

This is the "square law", and $p(m,n) = n/(n+m)$.

Let

$$\text{MOE} = \begin{cases} 1 & \text{if } n = 0 \text{ before } m \\ 0 & \text{if } m = 0 \text{ before } n. \end{cases}$$

The calculations are shown below



The last entry is (2/6)(.775) + (4/6)(.992) = .922, which is the probability that "m" will win, given $(m,n) = (4,2)$ initially. The meaning of the entry in each square is the probability that m will win if the state of the battle should ever correspond to the square. Note that the deterministic law would predict a win probability of 1.0 for m from square (4,2), rather than .922. It would also predict $\sqrt{4^2 - 2^2} =$ 3.46 survivors. The number of survivors is actually a random variable. The probability that there are 0 m-survivors is $1 - .922 =$ .078. The probability that there are exactly 2 survivors could be computed by replacing the four l's in row 0 with (0,1,0,0) and repeating the calculations. Or the average number of m-survivors could be computed by replacing the four l's with (1,2,3,4). Or the average difference (m-type – n-type) survivors could be computed by replacing the four l's by (1,2,3,4) and the two 0's in the first column by (-l,-2); the only limitation to the technique is that the MOE must be defined on the state of the battle *when it ends*, regardless of how it gets there. The computational method is always repeated application of (3.3), a formula that is well suited to computer implementation. The method is well suited to spreadsheet implementation-see the *Endgame* sheet for a Square law example.

The same type of approach would also work for more than two types of unit, but the amount of computation involved will grow quickly with the number of unit types. One has to do a calculation in each of $n_1, n_2, \ldots, n_k$ cells if there initially are n_1 units of the i^{th} type. The difficulty is that the MOE for a large battle can only be computed by first computing the MOE in all conceivable smaller battles, however unlikely. Even for only two unit types, the amount of computation can be very large. To construct even an oversimplified model of Iwo Jima would require $20{,}000 \times 50{,}000$ = one billion cells. Fortunately, the deterministic interpretation of the equations is usually adequate when the forces involved are large; it is problems with a small number of units that are most in need of a probabilistic treatment.

EXERCISES

1) Construct a "crossplot" of y as a function of x from Figure 3; that is, eliminate time and plot y as a function of x. Show that the crossplot and Formula 2.3 give the same value for y when $x = 800$.

2) The square law holds, $x_0 = 200$, $y_0 = 300$, $\alpha = .04$, and $\beta = .01$. Assuming the battle continues until one side is wiped out, who wins, and how many survivors are there? You may wish to use the *SquareLaw* sheet. Ans.: x wins with 132 survivors.

3) Same as problem 2 except that the question is "What is x when $x = y$?" Ans.: $x = 153$.

4) The y side has 1000 men in each of several forts, with $\beta = .02$. The x side has 4100 men in a single group, with $\alpha = .01$. The x side attacks the forts in succession, with the survivors of one battle entering the next. How many forts are taken before x is wiped

out, assuming each battle proceeds until one side is exhausted? Assume first that the square law holds and next that the linear law holds. Ans.: 8,2.

5) Suppose $\dot{x} = -\beta y$ and $\dot{y} = -\alpha xy$, with $\alpha = .01$, $\beta = .1$, $x_0 = 100$, and $y_0 = 100$. Solve for y as a function of x and graph the function.

6) Use the *Minelayer* sheet to solve the minelayer problem of section 2.4.1 when the initial numbers are (M,CL,S,L,ML) = (0,30,6,30,20). If you were on the side of the ships and could change any parameter by a factor of 2, which one would your choose?

7) The *FlakBomb* sheet solves eqs. 2.9 numerically for 30 days, assuming initial values for A, S, C, and F are 20, 20, 0, and 30. If you had 40 aircraft in total and the goal of dropping as many bombs as possible on target by time 30, how would you like them to be divided at time 0 between attack and supression aircraft?

8) Use the *Control* sheet to optimize the submarines' division of effort in the convoy engagement model. The Solver feature will find the optimal solution. See if you can find an easy-to-remember control that does almost as well.

9) Modify eqs. 2.9 to reflect the idea that aircraft may easily be converted from "attack" to "suppression" or vice versa, and then construct a spreadsheet to investigate different rules for deciding what fraction u of all aircraft should be devoted to attack at time t, where $0 \le t \le 30$. Examples of rules are

   a) $u = t/30$

   b) $u = 0$ for $t \le 15$, $u = 1$ for $t > 15$

   c) $u = 0$ as long as $F > 0$.

   Which of the three is better? See if you can find a rule that makes C at time 30 larger than any of them.

10) Suppose the m side gets 10 points for winning the battle, one point for each n-unit eliminated, and -l point for each m-unit eliminated. The *Endgame* sheet calculates the expected MOE from various initial states when $\beta = 2$ and $\alpha = 1$, the Square law

interpreted probabilistically. Change β to 20 and α to 10, observe what happens, and explain why.

11) What is the probability that *m* wins when the initial state is $(m,n) = (9,10)$. Assume square law with $\alpha = \beta$, interpreted probabilistically. Use the *Endgame* sheet.

12) There are two battles to be fought, each square law with $\alpha = \beta$. There will be 2 Blue *n*-units in each battle. There are a total of 5 Red *m*-units available. How should the Red units be split up between the two battles, assuming that the goal is to maximize the average number of battles won? Interpret the square law probabilistically.

13) The equations $\dot{x} = -k_1$ and $\dot{y} = -k_2$ for $x, y > 0$ could be said to hold in a "gladiator" situation where there are successive one-on-one combats. Compare the deterministic and probabilistic solutions when $k_1 = k_2 = 1$ and $(x_0,y_0) = (3,2)$.

14) Consider a Square law battle where $(x_0, y_0, \alpha, \beta) = (7, 5, .8, .9)$, interpreted probabilistically. Compute the average net difference of survivors $x - y$ at the end of the battle. Use the *Endgame* sheet to calculate it analytically, and also use the *SquareLawRnd* and *SimSheet* sheets to compute it by Monte Carlo simulation. The results will not be exactly the same. Explain why. Caution: You can do up to 65,535 iterations with *SimSheet* if you have several minutes available to wait for the output, but it is good practice to do only a small number before saving the worksheet. Otherwise *SimSheet* (or any spreadsheet with lots of rows or columns) will needlessly consume megabytes of storage space.

REFERENCES

Engel, J., 1954, "A Verification of Lanchester's Law," *Operations Research*, **2**, pp. 163-171.

Helmbold, R. and Rehm, A., 1995, translation of "The Influence of the Numerical Strength of Engaged Forces in Their Casualties," by M. Osipov, *Naval Research Logistics*, **42**, pp. 435-490.

Morse, P. and Kimball, G., 1950, *Methods of Operations Research,* Wiley.

Operations Research Department, 1999, "Aggregated Combat Models," Naval Postgraduate School.

Lanchester, F., 1916, *Aircraft in Warfare: the Dawn of the Fourth Arm*, Constable and Co. Ltd, London.

McCue, B., 1990, *U-Boats in the Bay of Biscay*, National Defense University Press.

Ross, s., 1997, *Introduction to Probability Models* (6^{th} ed), chapter 6, Academic Press.

Taylor, J., 1983, *Lanchester Models of Warfare*, Operations Research Society of America.

Willard, D., 1962, "Lanchester as Force in History: An Analysis of Land Battles of the Years 1618-l905," Research Analysis Corporation, RAC-TP-74.