Ch 6: Eigenvalues

6.7 Positive Definite Matrices

1. Recall that a symmetric matrix $A$ can be

   (a) \textit{positive definite} if $\mathbf{x}^T A \mathbf{x} > 0$, $\forall \mathbf{x} \in \mathbb{R}^n$, i.e. its eigenvalues are all positive
   
   (b) \textit{negative definite} if $\mathbf{x}^T A \mathbf{x} < 0$, $\forall \mathbf{x} \in \mathbb{R}^n$, i.e. its eigenvalues are all negative

2. \textit{the leading principal submatrix} $A_r$ $(1 \leq r \leq n)$ is the matrix obtained from the first $r$ rows and $r$ columns of a symmetric matrix $A$

3. \textit{the leading principal minor} is $\det A_r$ $(1 \leq r \leq n)$

4. if $A$ is a positive definite $\implies$

   (a) $\det A > 0$ (because $\lambda_i > 0$)
   
   (b) $A$ is nonsingular (because $\lambda_i > 0$)

5. $A$ is a symmetric positive definite $\iff$ at least one of the following holds true

   (a) its eigenvalues are all positive
   
   (b) the leading principal submatrices $A_1, A_2, \ldots, A_n$ have positive determinants
   
   (c) $A$ has an L-U factorization: $A$ can be reduced to an upper triangular matrix $U$

      using only the $3^{rd}$ row operation and the pivot elements are all positive
   
   (d) $A$ has a Cholesky Decomposition: if $A$ is symmetric positive definite matrix, then

      $$A = L_1 L_1^T,$$

      where $L_1 = LD^{1/2}$ is a lower diagonal matrix with positive diagonal elements

      (Cholesky Decomposition is used in solving infinite recurrences)

      That is: $U$ can be factored as $U = DU_1$, where

      $D$ is a diagonal matrix whose entries are the diagonal entries of $U$, and

      $U_1$ is an upper triangular one.

      This gives the factorization of $A$ as $A = LDL^T$, where

      $L$ is the lower triangular matrix obtained from the $LU$-factorization, and

      $D$ is a diagonal matrix whose entries are the diagonal entries of $U$.

   (e) $A$ can be factored as $B^T B$ for some nonsingular matrix $B$

6. $A$ is \textit{negative definite} $\iff -A$ is positive definite.

7. Since $\det A = \prod \lambda_i$ and all $\lambda_i < 0$ for a negative definite matrix, it follows that:

   $A$ is negative definite $\iff$ leading principal minors (i.e. the $\det A_r$) alternate in signs.