Two sample induction problems

1. Find a formula for $1 + 4 + 7 + \ldots + (3n - 2)$ for positive integers $n$, and then verify your formula by mathematical induction.

First we find the formula. Let

$1 + 4 + 7 + \ldots + (3n - 2) + (3n - 5) + (3n - 8) + \ldots + 7 + 4 + 1 = x,$

$\frac{(3n - 1) + (3n - 1) + (3n - 1) + \ldots + (3n - 1) + (3n - 1) + (3n - 1)}{2} = 2x$

The above line being obtained by adding the two equations together. Note that we have that where there are exactly $n$ copies of $(3n - 1)$ in the sum. Thus $n(3n - 1) = 2x$, so $x = \frac{n(3n - 1)}{2}$.

Next we prove by mathematical induction that for all natural numbers $n$,

$1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n - 1)}{2}$.

**Proof:** We prove by induction that $S_n : 1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n - 1)}{2}$ is true for all natural numbers $n$.

The statement $S_1 : 1 = \frac{1(3-1)}{2}$ is true.

Assume that $S_k : 1 + 4 + 7 + \ldots + (3k - 2) = \frac{k(3k-1)}{2}$ is true and prove that $S_{k+1} : 1 + 4 + 7 + \ldots + (3(k+1) - 2) = \frac{(k+1)(3(k+1)-1)}{2}$ is true. Observe that

$1 + 4 + 7 + \ldots + (3(k+1) - 2) = (1 + 4 + 7 + \ldots + (3n - 2)) + (3k + 1)$

$= \frac{k(3k - 1)}{2} + (3k + 1)$

$= \frac{k(3k - 1) + 2(3k + 1)}{2}$

$= \frac{3k^2 - k + 6k + 2}{2}$

$= \frac{3k^2 + 3k + 2k + 2}{2}$

$= \frac{(k + 1)(3k + 2)}{2}$

$= \frac{(k + 1)(3(k + 1) - 1)}{2}$

Thus by the Principle of Math Induction $S_n$ is true for all natural numbers $n$. ■
2. Prove that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$ for every positive integers $n$.

**Proof:** We prove by induction that $S_n : 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$ is true for all natural numbers $n$.

The statement $S_1 : 1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$ is true.
Assume that $S_k : 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k + 1) = \frac{k(k+1)(k+2)}{3}$ is true and prove that $S_{k+1} : 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (k + 1)(k + 2) = \frac{(k+1)(k+2)(k+3)}{3}$ is true. Observe that

$$
1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (k + 1)(k + 2) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k + 1) + (k + 1)(k + 2) \\
= \frac{k(k+1)(k+2)}{3} + (k + 1)(k + 2) \\
= \frac{k(k+1)(k+2) + 3(k + 1)(k + 2)}{3} \\
= \frac{(k + 1)(k+2)(k+3)}{3}
$$

Thus by the Principle of Math Induction $S_n$ is true for all natural numbers $n$. ■