(a) Straight forward computations

(b) We have \( u_1 = \left( \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{1}{3} \right)^T \), \( u_2 = \left( \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right)^T \), and \( u_3 = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)^T \). Let \( x = (1, 1, 1)^T \). Write \( x \) as a linear combination of \( u_1, u_2, \) and \( u_3 \), and use Parseval’s formula to compute \( ||x|| \).

Solution: We know from part (a) that \([u_1, u_2, u_3]\) is an orthonormal basis for \( \mathbb{R}^3 \). By Theorem 5.5.2, we know that

\[
x = (x^T u_1) u_1 + (x^T u_2) u_2 + (x^T u_3) u_3
\]

\[
= -\frac{2}{3\sqrt{2}} u_1 + \frac{5}{3} u_2 + 0 u_3
\]

\[
= -\frac{2}{3\sqrt{2}} u_1 + \frac{5}{3} u_2
\]

By Parseval’s formula, \( ||x|| = \left( \frac{4}{18} + \frac{25}{9} \right)^{1/2} = \sqrt{3} \).

5.5.3 We are given \( S \), the subspace spanned by \( u_2 \) and \( u_3 \) of the preceding exercise, and \( x = (1, 2, 2)^T \). We are to find the projection \( p \) of \( x \) onto \( S \), and to verify that \( p - x \in S^\perp \).

Solution: The projection is

\[
p = (x^T u_2) u_2 + (x^T u_3) u_3
\]

\[
= \frac{8}{3} u_2 - \frac{1}{\sqrt{2}} u_3
\]

\[
= \left( \frac{23}{18}, \frac{41}{18}, \frac{8}{9} \right)^T
\]

So \( p - x = \left( \frac{5}{18}, \frac{5}{18}, -\frac{10}{9} \right)^T \). It is easy to show that \( p - x \in S^\perp \), by showing that it is orthogonal to each of \( u_2, u_3 \).

Note: A close look at the computation by which the projection was obtained is consistent with the observation (Corollary 5.5.9) that the projection operator is \( U U^T \), where \( U \) in this case is the matrix whose columns are \( u_1 \) and \( u_2 \).

5.5.5 Let \( u_1 \) and \( u_2 \) form an orthonormal basis for \( \mathbb{R}^2 \), and let \( u \) be a unit vector in \( \mathbb{R}^2 \). If \( u^T u_1 = \frac{1}{2} \), determine the value of \( |u^T u_2| \).

Solution: Since \( u \) is a unit vector, and since \( u_1 \) and \( u_2 \) form an orthonormal basis for \( \mathbb{R}^2 \), then by Parseval’s formula we know that \( (u^T u_1)^2 + (u^T u_2)^2 = 1 \). Given \( u^T u_1 = \frac{1}{2} \), it follows that \( (u^T u_2)^2 = \frac{3}{4} \), so \( |u^T u_2| = \frac{\sqrt{3}}{2} \).
5.5.6 Let \( \{u_1, u_2, u_3\} \) be an orthonormal basis for an inner product space \( V \), and let
\[
u = u_1 + 2u_2 + 2u_3 \quad \text{and} \quad v = u_1 + 7u_3.
\]

Determine the value of each of the following:

(a) \( \langle u, v \rangle \)
(b) \( ||u|| \) and \( ||v|| \)
(c) The angle \( \theta \) between \( u \) and \( v \).

Solution:

(a) By Corollary 5.5.3, \( \langle u, v \rangle = 1 + 0 + 14 = 15 \).
(b) By Parseval’s formula, \( ||u|| = (1 + 4 + 4)^{1/2} = 3 \), and \( ||v|| = (1 + 0 + 49)^{1/2} = 5\sqrt{2} \).
(c) Using our results from (a) and (b), we have
\[
\theta = \arccos \frac{15}{15\sqrt{2}} = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}.
\]

5.5.14 Let \( u \) be a unit vector in \( \mathbb{R}^n \), and let \( H = I - 2uu^T \). Show that \( H \) is both orthogonal and symmetric and hence is its own inverse.

Proof: The symmetry of \( H \) follows from the symmetry of \( I \) and the symmetry of \( uu^T \), i.e., \( (uu^T)^T = u^T T u^T = uu^T \), along with the fact that the sum of symmetric matrices is symmetric:
\[
H^T = (I - 2uu^T)^T = I^T - 2u^T T u^T = I - 2uu^T + H.
\]

To show that \( H \) is orthogonal, we show that \( H^TH = I \):
\[
H^TH = ((I - 2uu^T)^T) (I - 2uu^T)
= I^T I - 4uu^T + 4uu^T uu^T
= I^2 - 4uu^T + 4u (u^T u) u^T
= I - 4uu^T + 4uu^T
= I.
\]

But if \( H \) is both orthogonal and symmetric, then \( H^{-1} = H^T = H \). \( \square \)
5.5.21. (b.ii) Let \( A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \).

Solve the least squares problem \( Ax = b \) for \( b = (1, 2, 3, 4)^T \).

**Solution:** Since the columns of \( A \) constitute an orthonormal set, it follows that \( A^T A = I \), and the normal equations reduce to

\[
\hat{x} = A^T b = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.
\]