Partial Solution Set, Leon §3.6  
Monday 15\textsuperscript{th} October, 2012 at time 12:03

3.6.1 \textbf{(a) Solution:} We want bases for the row space, the column space, and the nullspace of

\[ A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} \]. Elimination transforms \( A \) to \( U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \).

We have one free variable and two nonzero pivots.

A basis for the row space of \( A \) can be either all rows of \( A \) or all rows of \( U \):

\[
\text{basis for row space} = \{(1 3 2), (0 -5 0)\}
\]

(in this case you could also use as a basis for the row space the following: \{(1 3 2), (2 1 4)\} since none of the rows of \( A \) were swapped as row operations were performed).

A basis for the column space of \( A \) consists of the first two columns of \( A \): \( \{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix} \} \)

A basis for the nullspace of \( A \) is \( B = \{ \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \} \). This is not a column of \( A \) nor a column of \( U \), rather the solution of \( A\mathbf{x} = \mathbf{0} \), whose solution is \( x_3 = \alpha, x_2 = 0 \) and \( x_1 = -2\alpha \), which gives the vector \( \mathbf{x} = \alpha \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \). The dimension of this null space is 1 since there is one free variable.

(b) \textbf{Solution:} We want bases for the row space, the column space, and the nullspace of

\[ A = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} \]. Elimination transforms \( A \) to \( U = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \).

We have one free variable and three nonzero pivots.

A basis for the row space of \( A \) can be either all rows of \( A \) or all rows of \( U \):

\[
\text{basis for row space} = \{ [ -3 \ 1 \ 3 \ 4 ], [ 0 \ 7 \ 0 \ -2 ], [ 0 \ 0 \ 1 \ 0 ] \}
\]

(in this case again you could also use as a basis for the row space the following: \{[ -3 \ 1 \ 3 \ 4 ], [ 1 \ 2 \ -1 \ -2 ], [ -3 \ 8 \ 4 \ 2 ]\} since none of the rows of \( A \) were swapped as row operations were performed).
A basis for the column space of $A$ consists of the first three columns of $A$: \[
\begin{bmatrix}
-3 \\
1 \\
-3
\end{bmatrix}, \begin{bmatrix}
1 \\
2 \\
8
\end{bmatrix}, \begin{bmatrix}
3 \\
-1 \\
4
\end{bmatrix}
\]

A basis for the nullspace of $A$ is $B = \{ \begin{bmatrix}
10 \\
2 \\
7
\end{bmatrix} \}$. This is not a column of $A$ nor a column of $U$, rather the solution of $Ax = 0$.

3.6.2 (a) What is the dimension of the subspace of $\mathbb{R}^3$ spanned by \{(1, -2, 2)^T, (2, -2, 4)^T, (-3, 3, 6)^T\}?

\textbf{Solution:} We want dimension of the column space of $A = \begin{bmatrix}
1 & 2 & -3 \\
-2 & -2 & 3 \\
2 & 4 & 6
\end{bmatrix}$. Elimination transforms $A$ to $U = \begin{bmatrix}
1 & 2 & 3 \\
0 & 2 & -3 \\
0 & 0 & 1
\end{bmatrix}$.

We have no free variable and three nonzero pivots. Thus the dimension of the column space (or also the rank of the matrix) is 3, the number of nonzero pivots. Therefore the dimension of the subspace spanned by the three vectors is 3.

(c) What is the dimension of the subspace of $\mathbb{R}^3$ spanned by \{(1, -1, 2)^T, (-2, 2, -4)^T, (3, -2, 5)^T, (2, -1, 3)^T\}?

\textbf{Solution:} The solution is simply the rank of $A = \begin{bmatrix}
1 & -2 & 3 & 2 \\
-1 & 2 & -2 & -1 \\
2 & -4 & 5 & 3
\end{bmatrix}$. We use Gaussian elimination to obtain an equivalent matrix $B = \begin{bmatrix}
1 & -2 & 3 & 2 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}$, from which it is apparent that rank($A$) = 2. Thus the dimension of the subspace in question is 2.

3.6.3 Given $A = \begin{bmatrix}
1 & 2 & 2 & 3 & 1 & 4 \\
2 & 4 & 5 & 5 & 4 & 9 \\
3 & 6 & 7 & 8 & 5 & 9
\end{bmatrix}$,

(a) Compute the reduced row echelon form $U$ of $A$. Which columns of $U$ correspond to the free variables? Write each of these vectors as a linear combination of the column vectors corresponding to the lead variables.
(b) Which columns of $A$ correspond to the lead variables of $U$? These column vectors constitute a basis for $CS(A)$. Write each of the remaining column vectors of $A$ as a linear combination of these basis vectors.

**Solution:**

(a) The reduced row echelon form of $A$ is

$$U = \begin{bmatrix} 1 & 2 & 0 & 5 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. $$

The columns corresponding to free variables are $u_2$, $u_4$, and $u_5$. Because of the simplicity of the column vectors corresponding to the lead variables, it is easy to see that $u_2 = 2u_1$, $u_4 = 5u_1 - u_3$, and $u_5 = -3u_1 + 2u_3$.

(b) The column vectors of $A$ corresponding to the lead variables of $U$ are $a_1$, $a_3$, and $a_6$. What is perhaps not immediately obvious is that the dependencies among the columns of $A$ are precisely the same as those among the columns of $U$ (this is why Gaussian elimination works!). To see that this is so, let’s suppose that $A$ is an $m \times n$ matrix and that some column of $A$, say $a_n$, is a linear combination of the remaining columns. We can then regard $A$ as an augmented matrix representing the $m \times (n-1)$ system of equations with right-hand side $a_n$. As the elimination proceeds, the dependency of the right-hand side upon the columns of $A$ is revealed, but the coefficients that describe the dependency cannot change. This enables us to use back-substitution on the reduced system to discover the dependencies in the original system. So the bottom line for problem 3.6.3 (b) is that $a_2 = 2a_1$, $a_4 = 5a_1 - a_3$, and $a_5 = -3a_1 + 2a_3$, same relation as the one in part (a).

3.6.6: How many solutions will the linear system $Ax = b$ have if $b$ is in the column space of $A$ and the column vectors of $A$ are linearly dependent?

**Solution:** Infinitely many. Why? Since $b$ is in the column space of $A$, it follows that $Ax = b$ is consistent. Since the columns of $A$ are linearly dependent, it follows that $A$ has a nontrivial nullspace, i.e., the homogeneous equation $Ax = 0$ has infinitely many solutions. So let $b$ be an element of the column space of $A$, and suppose $Ax = b$. For each $z \in N(A) - \{0\}$, we have $x + z \neq x$, but

$$A(x + z) = Ax + Az = b + 0 = b,$$

where $Az = 0$ above since $z \in N(A)$. So there are infinitely many solutions to $Ax = b$. □

3.6.8: Let $A$ be an $m \times n$ matrix, with $m > n$. Let $b \in \mathbb{R}^m$, and suppose that $N(A) = \{0\}$. 
(a) What can you conclude about the column vectors of $A$? Are they linearly independent? Do they span $\mathbb{R}^m$? Explain.

**Solution:** Since $N(A) = \{\mathbf{0}\}$, we know that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, which tells us that the columns of $A$ are linearly independent. But they cannot span $\mathbb{R}^m$, since the column space of $A$ has dimension $n < m$.

(b) How many solutions will the system $A\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is not in the column space of $A$? How many solutions will there be if $\mathbf{b}$ is in the column space of $A$? Explain.

**Solution:** If $\mathbf{b} \notin CS(A)$, then obviously $A\mathbf{x} = \mathbf{b}$ has no solutions. Suppose, then, that $\mathbf{b} \in CS(A)$. Clearly $A\mathbf{x} = \mathbf{b}$ has at least one solution, by definition of $CS(A)$. So assume that

$$A\mathbf{x} = Ay = \mathbf{b}.$$ 

Then

$$A(\mathbf{x} - y) = Ax - Ay = b - b = 0 - 0 = 0,$$

so $\mathbf{x} - y \in N(A)$. Since $N(A) = \{\mathbf{0}\}$, it follows that $\mathbf{x} = y$ and the solution is unique.

How does the answer to the second question change if we assume that $A$ has a nontrivial nullspace? See exercise 3.6.6, above.

**3.6.9** Let $A$ and $B$ be $6 \times 5$ matrices. If $\dim N(A) = 2$, what is the rank of $A$? If the rank of $B$ is 4, what is $\dim N(B)$?

**Solution:** $\text{rank}(A) = 3$, and $\dim N(B) = 1$.

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Note: Problems 12 and 22 are very closely related. Before reading the solution to 12, you might take a look at the statement to be proven in 22(a). If you took that statement (namely 22(a)) as a lemma (to be proven later), you would have the key to 12(a) in hand, or you can prove the statement (again 22(a)) now.

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**3.6.12** Let $A$ and $B$ be row-equivalent matrices.
(a) Show that the dimension of the column space of $A$ is equal to the dimension of the column space of $B$.

(b) Are the column spaces of $A$ and $B$ necessarily the same? Justify your answer.

**Solution:** See problem 22 for a clue, and maybe try this again before reading the rest of the solution. The simplest way to prove part (a) is to take a slightly indirect route. Since $A$ and $B$ are row-equivalent, they are certainly the same size, say $m \times n$. If we could show that $N(A) = N(B)$, the result would follow from the fact that

$$\dim(CS(A)) = n - \dim(N(A)) = n - \dim(N(B)) = \dim(CS(B)).$$

Now since $A$ and $B$ are row-equivalent, then there exists some $m \times m$ matrix $E$, a product of elementary matrices, with the property that $B = EA$. Since $E$ is a product of elementary matrices, then $E$ is nonsingular (so this $E$ is the nonsingular matrix of problem 22 if you want to use that problem as a shortcut here).

We now show that $N(A) = N(B)$, by showing first that $N(A) \subseteq N(B)$ and then $N(B) \subseteq N(A)$.

Suppose first, that $x \in N(A)$, and we show that $x \in N(B)$. Since $x \in N(A)$, it follows that $Ax = 0$. Then $Bx = EAx = E0 = 0$, so $N(A) \subseteq N(B)$.

Conversely, to show that $N(B) \subseteq N(A)$, suppose that $x \notin N(A)$ (contrapositive proof). Then $Bx = EAx = E(Ax) \neq 0$, since $Ey = 0$ (where $y = Ax$) has only the trivial solution as the $E$ is invertible as a product of elementary matrices. This means that $x \notin N(A)$ and so $N(B) \subseteq N(B)$. Thus $N(A) = N(B)$, and the result follows. \(\square\)

(Note: The conversely part above could be done by a direct proof as well: Let $x \in N(B) \Rightarrow Bx = 0$. Premultiplying by $E^{-1}$, which exists as $E$ is an elementary matrix (or product of them) gives $E^{-1}Bx = E^{-1}0$, which is equivalent to $Ax = 0$ since $B = EA$ and $A = E^{-1}B$.)

The answer in part (b) is no. Row-equivalent matrices may or may not have identical column spaces. For example, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A$ and $B$ are row-equivalent, but clearly have different column spaces: the column space of $A$ is the identity line in the plane, while the column space of $B$ is the $x_1$-axis.

3.6.22 Let $A$ be an $m \times n$ matrix.

(a) If $B$ is a nonsingular $m \times m$ matrix, show that $BA$ and $A$ have the same nullspace and therefore the same rank.
(b) If $C$ is a nonsingular $n \times n$ matrix, show that $AC$ and $A$ have the same rank.

**Solution:**

(a) We show that $N(A) = N(BA)$, by showing first that $N(A) \subseteq N(BA)$ and then $N(BA) \subseteq N(A)$.

Suppose first, that $a \in N(A)$, and we show that $a \in N(BA)$. Since $a \in N(A)$, it follows that $Aa = 0$. Then premultiplying by $B$ we have $BAa = B0$ or $BAa = 0$ so $a \in N(BA)$. Thus $N(A) \subseteq N(BA)$.

Conversely, to show that $N(BA) \subseteq N(A)$, suppose that $b \not\in N(AB)$. Then $BAb = 0$, since $B$ is nonsingular/invertible it follows that $B^{-1}$ exists. This means that premultiplying by $B^{-1}$ we obtain $B^{-1}BAb = B^{-1}0$ or $Ab = 0$. Thus $b \in N(A)$ and so $N(BA) \subseteq N(A)$. Thus $N(A) = N(BA)$, and the result follows. □

(b) Hint: how is the rank of a matrix related to that of its transpose?

**Solution:** Since the rank of an arbitrary matrix $X$ is the dim row$S(X) = \dim$ col$S(X)$, it follows that rank($X$) = rank($X^T$). And so

$$\text{rank}(AC) = \text{rank}((AC)^T) = \text{rank}((CTA)^T) \subseteq \text{rank}(AT) = \text{rank}(A).$$

### 3.6.26

Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, and let $x_0$ be a particular solution to the system $Ax = b$.

**Prove:**

(a) A vector $y \in \mathbb{R}^n$ will be a solution to $Ax = b$ if and only if $y = x_0 + z$, where $z \in N(A)$.

**Proof:** ($\Leftarrow$) First suppose that $y = x_0 + z$, where $z \in N(A)$. Then

$$Ay = A(x_0 + z) = Ax_0 + Az = b + 0 = b.$$

($\Rightarrow$) Conversely, we prove that if $y \in \mathbb{R}^n$ is a solution to $Ax = b$, then $y$ is of the form $x_0$ plus some vector in the null space of $A$. Here are two methods:

**Method 1: (Direct proof)** Suppose that $y$ is a solution of $Ax = b$, so in particular $Ay = b$. Let $t$ be the vector that is the difference between $y$ and the given solution $x_0$, so $t = y - x_0$. Then

$$At = A(y - x_0) = Ay - Ax_0 = b - b = 0.$$
This means that \( t \in N(A) \) and so indeed \( y \) is obtained from \( x_0 \) by adding vectors from the null space of \( A \), and you can call them \( w \) or \( t \). □

**Method 2:** *(Contradiction proof)* Suppose that \( y \) is a solution of \( Ax = b \) that is not of the form \( y = x_0 + z \), where \( z \in N(A) \), (i.e. assume that \( y = x_0 + w \), where \( w \not\in N(A) \)). Since \( w \not\in N(A) \), it follows that \( A_w = c \) for some \( c \neq 0 \). But then

\[
Ay = A(x_0 + w) = Ax_0 + Aw = b + c \neq b,
\]

which is a contradiction to the fact that \( y \) is a solution of \( Ax = b \). □

(b) If \( N(A) = \{0\} \), then the solution \( x_0 \) is unique.

**Proof:** Suppose \( N(A) = \{0\} \). If \( z \in N(A) \), then \( Az = 0 \). Since \( y \) is a solution to \( Ax = b \) we have \( Ay = b \). Similarly, since \( x_0 \) is a particular solution to \( Ax = b \) we have \( Ax_0 = b \). Then \( A(x_0 - y) = Ax_0 - Ay = b - b = 0 \), so \( x_0 - y \in N(A) \). Since \( N(A) = \{0\} \), then \( x_0 - y = 0 \), but then \( x_0 = y \) and the solution is unique. □

### 3.6.28

Let \( A \in \mathbb{R}^{m \times n} \), \( B \in \mathbb{R}^{n \times r} \), and \( C = AB \). Show that

1. The column space of \( C \) is a subspace of the column space of \( A \).
2. The row space of \( C \) is a subspace of the row space of \( B \).
3. \( \text{rank}(C) \leq \min\{\text{rank}(A), \text{rank}(B)\} \).

**Solution:**

1. Here are two methods:

   **Method 1:** We show \( \text{CS}(C) \subseteq \text{CS}(A) \) by taking an arbitrary element \( t \in \text{CS}(C) \) and showing it is in \( \text{CS}(A) \):

   \[
   \begin{align*}
   t \in \text{CS}(C) \quad \Rightarrow \quad Cx &= t, \exists x \\
   \Rightarrow \quad Ay &= t, \exists y \quad \Rightarrow \quad Ax &= t \in \text{CS}(A).
   \end{align*}
   \]

   **Method 2:** Letting \( C = [ \begin{bmatrix} c_1 & c_2 & \cdots & c_r \end{bmatrix} ] \) and \( B = [ \begin{bmatrix} b_1 & b_2 & \cdots & b_r \end{bmatrix} ] \), we see that

   \[
   C = AB = [ \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_r \end{bmatrix} ].
   \]
It follows that $c_i = Ab_i (i \in \{1, 2, \ldots, r\})$. This means that each column of $C$ is a linear combination of the columns of $A$, the linear combination being given by $b$. So each vector in the column space of $C$ is in the column space of $A$ as it is a linear combination of the columns of $A$.

2. Here are two methods:

Method 1: We show $RS(C) \subseteq RS(B)$ by taking an arbitrary element $r \in RS(C)$ and showing it is in $RS(B)$:

$$r \in RS(C) \Rightarrow r^T \in CS(C^T) \Rightarrow r^T \in CS((AB)^T) \Rightarrow r^T \in CS(B^TA^T) \Rightarrow$$

and it follows by part (1) above that

$$\Rightarrow r^T \in CS(B^T) \Rightarrow r \in RS(B).$$

Method 2: By applying the logic from (a) to the column space of $C^T = B^TA^T$, we see that the column space of $C^T$ is a subspace of the column space of $B^T$. But since the column space of $C^T$ is the row space of $C$, and similarly the column space of $B^T$ is the row space of $B$, it then follows that the row space of $C$ is a subspace of the row space of $B$. (This can also be done directly, by observing that each row of $C$ is a linear combination of the rows of $B$.)

3. This follows from (a) and (b), and from the fact that $\text{rank}(C) = \dim RS(C) = \dim RS(C^T)$. 
