3.1.3 We are to show that the set $\mathbb{C}$ of complex numbers, with scalar multiplication defined by $\alpha(a + bi) = \alpha a + \alpha bi$ and addition defined by $(a + bi) + (c + di) = (a + c) + (b + d)i$, satisfies the eight axioms of a vector space. This is only a partial solution.

A1: Let $a + bi, c + di \in \mathbb{C}$. Then

$$(a + bi) + (c + di) = ((a + c) + (b + d)i) \quad \text{(By definition of complex addition)}$$

$$= ((c + a) + (d + b)i) \quad \text{(Real addition is commutative)}$$

$$= (c + di) + (a + bi) \quad \text{(By definition of complex addition)}$$

A2: Similar to A1; pick three complex numbers, use the definition of complex addition as often as necessary, together with the known associativity of real addition, to show that complex addition is associative.

A3: The zero element is $0 = (0 + 0i)$.

A4: To show existence of the additive inverse, choose an arbitrary complex number (say, $x = a + bi$) and construct its additive inverse. This will be made easy by your knowledge of real additive inverses.

A5: We must prove that scalar multiplication distributes over complex addition. Let $a + bi, c + di \in \mathbb{C}$, and let $\alpha \in \mathbb{R}$. Then

$$\alpha((a + bi) + (c + di)) = \alpha((a + c) + (b + d)i) \quad \text{(Def'\n complex addition)}$$

$$= \alpha(a + c) + \alpha(b + d)i \quad \text{(Def'\n of scalar mult. in \mathbb{C})}$$

$$= (\alpha a + \alpha c) + (\alpha b + \alpha d)i \quad \text{(Distributivity in \mathbb{R})}$$

$$= (\alpha a + \alpha bi) + (\alpha c + \alpha di) \quad \text{(Def'\n of complex addition)}$$

$$= \alpha(a + bi) + \alpha(c + di) \quad \text{(Def'\n of scalar mult. in \mathbb{C})}$$

A6: Similar to A5.

A7: Use definition of scalar multiplication in $\mathbb{C}$ and associativity of real multiplication.

A8: Use definition of scalar multiplication in $\mathbb{C}$ and the fact that 1 is the multiplicative identity in $\mathbb{R}$.

3.1.4 Use the solution to 3.1.3 as a template for your solution. The objects are different (matrices rather than complex numbers) and the operations are necessarily defined differently, but these differences have no effect on the structure - $\mathbb{R}^{m \times n}$ is simply another vector space. The challenge is to avoid committing yourself to concrete values of $m$ and/or $n$. 
3.1.7 Show that the element 0 in a vector space is unique.

Note: This is a standard uniqueness argument. We assume that we have two zero elements and then discover that they are identical twins. The proof goes like this:

**Proof**: Let $V$ be a vector space. We know that $V$ contains at least one zero element, since $V$ satisfies the axioms. We must show, then, that $V$ contains at most one zero element. So suppose that $v$ and $w$ are zeros in $V$. Then

$$v = v + w \quad \text{(Since $w$ is a zero)}$$

$$= w + v \quad \text{(Since addition commutes)}$$

$$= w \quad \text{(Since $v$ is a zero)}$$

Thus uniqueness is proven, and it now makes sense to reserve a special symbol ($0$) to denote the zero element.

3.1.8 Let $x$, $y$, and $z$ be vectors in a vector space $V$. Prove the additive cancellation law: if $x + y = x + z$, then $y = z$.

**Proof**: Suppose $x + y = x + z$. Then

$$y = 0 + y \quad \text{(A3, A1)}$$

$$= (-x + x) + y \quad \text{(A4)}$$

$$= -x + (x + y) \quad \text{(A2)}$$

$$= -x + (x + z) \quad \text{(Since $x + y = x + z$)}$$

$$= (-x + x) + z \quad \text{(A2)}$$

$$= 0 + z \quad \text{(A4)}$$

$$= z \quad \text{(A1, A3)}.$$  

3.1.11 Let $V$ be the set of all ordered pairs of real numbers with addition defined in the usual fashion by $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$, but with scalar multiplication defined by $\alpha \circ (x_1, x_2) = (\alpha x_1, x_2)$. Is $V$ a vector space with these operations? Justify your answer.

**Solution**: No, this is not a vector space. Axiom 6 fails:

$$(\alpha + \beta)(x_1, x_2) = ((\alpha + \beta)x_1, x_2)$$

$$= (\alpha x_1 + \beta x_1, x_2)$$

$$\neq (\alpha x_1 + \beta x_1 + 2x_2)$$

$$= (\alpha x_1, x_2) + (\beta x_1, x_2)$$

$$= \alpha(x_1, x_2) + \beta(x_1, x_2)$$

unless $x_2 = 0$. 2
Let $\mathbb{R}^+$ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted $\circ$, by $\alpha \circ x = x^\alpha$ for any real $\alpha$ and $x \in \mathbb{R}^+$. Define addition, denoted $\oplus$, by $x \oplus y = x \cdot y$ for all $x, y \in \mathbb{R}^+$. (The dot represents the usual multiplication of reals.) Is $\mathbb{R}^+$ a vector space when equipped with these operations? Prove your answer.

**Solution:** Yes, this is a vector space. To prove this, we must verify that the axioms hold. Here is a partial proof:

A1: Use the definition of $\oplus$, together with the commutativity of ordinary real multiplication.

A2: Use the definition of $\oplus$, together with the associativity of ordinary real multiplication.

A3: The zero element is the number 1, since for any $x \in \mathbb{R}^+$ we have $x \oplus 1 = x \cdot 1 = x$.

A4: The additive inverse in this oddball space is the usual *multiplicative* inverse. That is, for any $x \in \mathbb{R}^+$, $1/x \in \mathbb{R}^+$, and $x \oplus 1/x = x \cdot 1/x = 1$. By the preceding argument, 1 is the zero element.

A5: Let $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}^+$. Then

\[
\alpha \circ (x \oplus y) = \alpha \circ (x \cdot y)
\]

\[
= (x \cdot y)^\alpha
\]

\[
= x^\alpha \cdot y^\alpha
\]

\[
= (\alpha \circ x) \cdot (\alpha \circ y)
\]

\[
= (\alpha \circ x) \oplus (\alpha \circ y)
\]

A6: Let $\alpha, \beta, x \in \mathbb{R}$. Then

\[
(\alpha + \beta)x = x^{\alpha+\beta}
\]

\[
= x^\alpha \cdot x^\beta
\]

\[
= \alpha x \oplus \beta x
\]

A7: Let $\alpha, \beta \in \mathbb{R}$, and $x \in \mathbb{R}^+$. Then

\[
(\alpha \beta) \circ x = x^{\alpha \beta}
\]

\[
= x^{3\alpha}
\]

\[
= (x^\beta)^\alpha
\]

\[
= \alpha \circ (x^\beta)
\]

\[
= \alpha \circ (\beta \circ x)
\]

A8: Let $x \in \mathbb{R}$. Then $1 \cdot x = x^1 = x$, where the first equality is by our local definition of scalar multiplication and the second is by the usual laws of exponents.
3.1.16 We can define a one-to-one correspondence between the elements of $P_n$ and $\mathbb{R}^n$ by

$$p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1} \leftrightarrow (a_1, a_2, \ldots, a_n)^T = \mathbf{a}.$$ 

Show that if $p \leftrightarrow \mathbf{a}$ and $q \leftrightarrow \mathbf{b}$ then

1. $\alpha p \leftrightarrow \alpha \mathbf{a}$ for every scalar $\alpha$.
2. $p + q \leftrightarrow \mathbf{a} + \mathbf{b}$.

**Proof:** For $p(x)$ as given in the introduction to the problem, and for any choice of $\alpha$,

$$\alpha p(x) = \alpha a_1 + \alpha a_2 x + \cdots + \alpha a_n x^{n-1} \leftrightarrow (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)^T = \alpha \mathbf{a}.$$ 

Now given $p(x)$ as above, and given $q(x) = b_1 + b_2 x + \cdots + b_n x^{n-1}$ and its corresponding image $\mathbf{b} = (b_1, b_2, \ldots, b_n)$,

$$(p + q)(x) = p(x) + q(x)
= (a_1 + b_1) + (a_2 + b_2)x + \cdots + (a_n + b_n)x^{n-1}
\leftrightarrow (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)^T
= (a_1, a_2, \ldots, a_n)^T + (b_1, b_2, \ldots, b_n)^T
= \mathbf{a} + \mathbf{b},$$

and we’re done. \hfill \Box