3.6.1b We want bases for the row space, the column space, and the nullspace of \( A = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} \). Elimination transforms \( A \) to \( U = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \). We have one free variable and three nonzero pivots. A basis for the row space of \( A \) can be either all rows of \( A \) or all rows of \( U \). A basis for the column space of \( A \) consists of the first three columns of \( A \). A basis for the nullspace of \( A \) is \( B = \{ (10, 2, 0, 7)^T \} \).

3.6.2c What is the dimension of the subspace of \( \mathbb{R}^3 \) spanned by \( \{(1, -1, 2)^T, (-2, 2, -4)^T, (3, -2, 5)^T, (2, -1, 3)^T \} \)?

**Solution:** The solution is simply the rank of \( A = \begin{bmatrix} 1 & -2 & 3 & 2 \\ -1 & 2 & -2 & -1 \\ 2 & -4 & 5 & 3 \end{bmatrix} \). We use Gaussian elimination to obtain an equivalent matrix \( B = \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \), from which it is apparent that \( \text{rank}(A) = 2 \). Thus the dimension of the subspace in question is 2.

3.6.3 Given \( A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{bmatrix} \),

(a) Compute the reduced row echelon form \( U \) of \( A \). Which columns of \( U \) correspond to the free variables? Write each of these vectors as a linear combination of the column vectors corresponding to the lead variables.

(b) Which columns of \( A \) correspond to the lead variables of \( U \)? These column vectors constitute a basis for \( \text{CS}(A) \). Write each of the remaining column vectors of \( A \) as a linear combination of these basis vectors.

**Solution:**

(a) The reduced row echelon form of \( A \) is

\[
U = \begin{bmatrix} 1 & 2 & 0 & 5 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\]

The columns corresponding to free variables are \( u_2, u_4, \) and \( u_5 \). Because of the simplicity of the column vectors corresponding to the lead variables, it is easy to see that \( u_2 = 2u_1, u_4 = 5u_1 - u_3, \) and \( u_5 = -3u_1 + 2u_3. \)
(b) The column vectors of $A$ corresponding to the lead variables of $U$ are $a_1$, $a_3$, and $a_6$. What is perhaps not immediately obvious is that the dependencies among the columns of $A$ are precisely the same as those among the columns of $U$ (this is why Gaussian elimination works!). To see that this is so, let’s suppose that $A$ is an $m \times n$ matrix and that some column of $A$, say $a_n$, is a linear combination of the remaining columns. We can then regard $A$ as an augmented matrix representing the $m \times (n-1)$ system of equations with right-hand side $a_n$. As the elimination proceeds, the the dependency of the right-hand side upon the columns of $A$ is revealed, but the coefficients that describe the dependency cannot change. This enables us to use back-substitution on the reduced system to discover the dependencies in the original system. So the bottom line for problem 3.6.3 is that $a_2 = 2a_1$, $a_4 = 5a_1 - a_3$, and $a_5 = -3a_1 + 2a_3$.

3.6.6: How many solutions will the linear system $Ax = b$ have if $b$ is in the column space of $A$ and the column vectors of $A$ are linearly dependent?

Solution: Infinitely many. Why? Since the columns of $A$ are linearly dependent, it follows that $A$ has a nontrivial nullspace, i.e., the homogeneous equation $Ax = 0$ has infinitely many solutions. So let $b$ be an element of the column space of $A$, and suppose $Ax = b$. For each $z \in N(A) - \{0\}$, we have $x + z \neq x$, but

\[
A(x + z) = Ax + Az \\
= b + 0 \\
= b,
\]

so there are infinitely many solutions to $Ax = b$. \qed

3.6.7: Let $A$ be an $m \times n$ matrix, with $m > n$. Let $b \in \mathbb{R}^m$, and suppose that $N(A) = \{0\}$.

(a) What can you conclude about the column vectors of $A$? Are they linearly independent? Do they span $\mathbb{R}^m$? Explain.

Solution: Since $N(A) = \{0\}$, we know that $Ax = 0$ has only the trivial solution $x = 0$, which tells us that the columns of $A$ are linearly independent. But they cannot span $\mathbb{R}^m$, since the column space of $A$ has dimension $n < m$.

(b) How many solutions will the system $Ax = b$ have if $b$ is not in the column space of $A$? How many solutions will there be if $b$ is in the column space of $A$? Explain.

Solution: If $b \notin CS(A)$, then obviously $Ax = b$ has no solutions. Suppose, then, that $b \in CS(A)$. Clearly $Ax = b$ has at least one solution, by definition of $CS(A)$. So assume that

\[
Ax = Ay = b.
\]
Then
\[ A(x - y) = Ax - Ay = b - b = 0 - 0 = 0, \]

so \( x - y \in N(A) \). Since \( N(A) = \{0\} \), it follows that \( x = y \) and the solution is unique.

How does the answer to the second question change if we assume that \( A \) has a nontrivial nullspace? See exercise 3.6.6, above.

**3.6.8** Let \( A \) and \( B \) be \( 6 \times 5 \) matrices. If \( \dim N(A) = 2 \), what is the rank of \( A \)? If the rank of \( B \) is 4, what is \( \dim N(B) \)?

**Solution:** \( \text{rank}(A) = 3 \), and \( \dim N(B) = 1 \).

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Note: Problems 9 and 16 are very closely related. Before reading the solution to 9, you might take a look at the statement to be proven in 16(a). If you took that statement as a lemma (to be proven later), you would have the key to 9(a) in hand.

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**3.6.9** Let \( A \) and \( B \) be row-equivalent matrices.

(a) Show that the dimension of the column space of \( A \) is equal to the dimension of the column space of \( B \).

(b) Are the column spaces of \( A \) and \( B \) necessarily the same? Justify your answer.

**Solution:** See problem 11(a) for a clue, and maybe try this again before reading the rest of the solution. The simplest way to prove part (a) is to take a slightly indirect route. Since \( A \) and \( B \) are row-equivalent, they are certainly the same size, say \( m \times n \). If we could show that \( N(A) = N(B) \), the result would follow from the fact that
\[ \dim(CS(A)) = m - \dim(N(A)) = m - \dim(N(B)) = \dim(CS(B)). \]

Now since \( A \) and \( B \) are row-equivalent, then there exists some \( m \times m \) matrix \( E \), a product of elementary matrices, with the property that \( B = EA \). Since \( E \) is a product of elementary matrices, then \( E \) is nonsingular.
We now show that $N(A) = N(B)$, by showing first that $N(A) \subseteq N(B)$ and then $N(B) \subseteq N(A)$. Suppose first, that $x \in N(A)$, and we show that $x \in N(B)$. Since $x \in N(A)$, it follows that $Ax = 0$. Then $Bx = EAx = E0 = 0$, so $N(A) \subseteq N(B)$. Conversely, to show that $N(B) \subseteq N(A)$, suppose that $x \notin N(A)$ (contrapositive). Then $Bx = EAx = E(Ax) \neq 0$, since $Ey = 0$ (where $y = Ax$) has only the trivial solution as the $E$ is invertible as a product of elementary matrices. This means that $x \notin N(A)$ and so $N(B) \subseteq N(A)$. Thus $N(A) = N(B)$, and the result follows.

The answer in part (b) is no. Row-equivalent matrices may or may not have identical column spaces. For example, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A$ and $B$ are row-equivalent, but clearly have different column spaces: the column space of $A$ is the line $y = x$ in the plane, while the column space of $B$ is the $x$-axis.

3.6.16 Let $A$ be an $m \times n$ matrix.

(a) If $B$ is a nonsingular $m \times m$ matrix, show that $BA$ and $A$ have the same nullspace and therefore the same rank.

(b) If $C$ is a nonsingular $n \times n$ matrix, show that $AC$ and $A$ have the same rank.

Solution:

(a) See the proof of 9(a).

(b) (Hint: how is the rank of a matrix related to that of its transpose?) Here is the solution: Since the rank of an arbitrary matrix $X$ is the $\dim \text{row} S(X) = \dim \text{col} S(X)$, it follows that $\text{rank}(X) = \text{rank}(X^T)$. And so

$$\text{rank}(AC) = \text{rank}((AC)^T) = \text{rank}((C^T A^T)) \overset{\text{by(a)}}{=} \text{rank}(A^T) = \text{rank}(A).$$

3.6.20 Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and let $x_0$ be a particular solution to the system $Ax = b$. Prove:

(a) A vector $y \in \mathbb{R}^n$ will be a solution to $Ax = b$ if and only if $y = x_0 + z$, where $z \in N(A)$.

Proof: First suppose that $y = x_0 + z$, where $z \in N(A)$. Then

$$Ay = A(x_0 + z) = Ax_0 + Az = b + 0 = b.$$
Conversely, assume, to the contrary, that \( y \) is not of the form \( y = x_0 + z \), where \( z \in N(A) \). Let \( w = y - x_0 \), where \( w \notin N(A) \). Then \( Aw = c \) for some \( c \neq 0 \). But then
\[
Ay = A(x_0 + w) \\
= Ax_0 + Aw \\
= b + c \\
\neq b.
\]

(b) If \( N(A) = \{0\} \), then the solution \( x_0 \) is unique.

**Proof:** Suppose \( N(A) = \{0\} \). If \( z \in N(A) \), then \( Az = 0 \). Since \( y \) is a solution to \( Ax = b \) we have \( Ay = b \). Similarly, since \( x_0 \) is a particular solution to \( Ax = b \) we have \( Ax_0 = b \). Then
\[
A(x_0 - y) = Ax_0 - Ay = b - b = 0,
\]
so \( x_0 - y \in N(A) \). Since \( N(A) = \{0\} \), then \( x_0 - y = 0 \), but then \( x_0 = y \) and the solution is unique. \( \square \)

3.6.22 Let \( A \in \mathbb{R}^{m \times n} \), \( B \in \mathbb{R}^{n \times r} \), and \( C = AB \). Show that

1. The column space of \( C \) is a subspace of the column space of \( A \).
2. The row space of \( C \) is a subspace of the row space of \( B \).
3. \( \text{rank}(C) \leq \min\{\text{rank}(A), \text{rank}(B)\} \).

**Solution:**

1. Letting \( C = \left[ \begin{array}{ccc} c_1 & c_2 & \cdots & c_r \end{array} \right] \) and \( B = \left[ \begin{array}{ccc} b_1 & b_2 & \cdots & b_r \end{array} \right] \), we see that
\[
C = AB = \left[ \begin{array}{ccc} Ab_1 & Ab_2 & \cdots & Ab_r \end{array} \right].
\]
It follows that each column of \( C \) is a linear combination of the columns of \( A \), so the column space of \( C \) must be a linear combination of the column space of \( A \).

2. By applying the logic from (a) to the column space of \( C^T = B^T A^T \), we see that the column space of \( C^T \) is a subspace of the column space of \( B^T \), but then the row space of \( C \) is a subspace of the row space of \( B \). This can also be done directly, by observing that each row of \( C \) is a linear combination of the rows of \( B \).

3. This follows from (a) and (b), and from the fact that \( \text{rank}(C) = \dim R(C) = \dim R(C^T) \).