6.4 Hermitian Matrices

We consider matrices with complex entries \( (a_{i,j} \in \mathbb{C}) \) versus real entries \( (a_{i,j} \in \mathbb{R}) \).

1. in \( \mathbb{R} \) the length of a real number \( x \) is \( |x| = \text{the length from the origin to the number} \) (either in the positive or the negative direction).

2. in \( \mathbb{C} \) the length of a complex number \( z = a + bi \) is \( |z| = \sqrt{a^2 + b^2} = \text{the length of the vector } [a, b] \) (from the origin to the point \( (a, b) \)). Also \( z^2 = a^2 + b^2 \).

3. in \( \mathbb{C} \) the scalars are complex numbers \( z \), addition of complex numbers: \( (a + bi) + (c + di) = (a + c) + (b + d)i \) and product of complex numbers: \( (a + bi)(c + di) = (ac - bd) + (ad + bc)i \), and \( \mathbb{C} \) with + and \( \cdot \) is a vector space.

4. Particularly \( \mathbb{C} \) is a normed vector space with the vectors \( z = (z_1, z_2, \ldots, z_n)^T \), and the norm \( |z| = \sqrt{(\bar{z}^T z)} = \sqrt{z_1 \bar{z}_1 + z_2 \bar{z}_2 + \ldots + z_n \bar{z}_n} \), which is the square root of the inner product, thus a real number. Here \( \bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)^T \) is the conjugate of \( z \). We denote by \( z^H = \bar{z}^T \), and so \( \|z\| = \sqrt{z^H z} \).

5. the inner product of \( z \) and \( w \) is the complex number \( \langle z, w \rangle = w^H z \).

6. if \( z \) is a vector in the complex vector space with the orthonormal basis \( \{w_1, w_2, \ldots, w_n\} \), then we can write \( z \) as a linear combination of the vector basis as \( z = \sum_{i=1}^n \langle z, w_i \rangle w_i \) (see Exercise 2 of Homework)

7. if \( A \) is a matrix in \( \mathbb{C}^{n \times n} \), then \( A^H \) is the matrix whose every entry is the conjugate of the corresponding entry of \( A \).

For example, if \( A = \begin{bmatrix} 2 - i & 1 & -2i \\ 5i & 0 & 5 - i \\ 0 & 5 & -5i \end{bmatrix} \) then \( A^H = \begin{bmatrix} 2 + i & 1 & +2i \\ -5i & 0 & 5 + i \\ 0 & 5 & 5i \end{bmatrix} \).

8. vectors in \( \mathbb{R}^n \) versus in \( \mathbb{C}^n \) and matrices in \( \mathbb{R}^{n \times n} \) versus in \( \mathbb{C}^{n \times n} \):

\[
\begin{array}{|c|c|c|}
\hline
\mathbb{R}^n & \mathbb{C}^n & \\
\hline
\langle x, y \rangle = y^T x & \langle z, w \rangle = w^H z & \\
\langle x, x \rangle \geq 0 \text{ with equality iff } x = 0 & \langle z, z \rangle \geq 0 \text{ with equality iff } z = 0 & \\
\langle x, y \rangle = \langle y, x \rangle & \langle z, w \rangle = \langle w, z \rangle & \\
\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle & \langle \alpha z + \beta u, w \rangle = \alpha \langle z, w \rangle + \beta \langle u, w \rangle & \\
\|x\|^2 = \langle x, x \rangle = x^T x & \|z\|^2 = \langle z, z \rangle = z^H z & \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\mathbb{R}^{n \times n} & \mathbb{C}^{n \times n} & \\
\hline
(A^T)^T = A & (A^H)^H = A & \\
(\alpha A + \beta B)^T = \alpha A^T + \beta B^T (\alpha, \beta \in \mathbb{R}) & (\alpha A + \beta B)^H = \bar{\alpha} A^H + \bar{\beta} B^H (\alpha, \beta \in \mathbb{C}) & \\
(AB)^T = B^T A^T & (AB)^H = B^H A^H & \\
\text{if } A^T = A \iff A \text{ is symmetric} & \text{if } A^H = A \iff A \text{ is Hermitian} & \\
\hline
\end{array}
\]
9. A matrix $A$ is a **Hermitian matrix** if $A^H = A$ (they are ideal matrices in $\mathbb{C}$ since properties that one would expect for matrices will probably hold).

For example $A = \begin{bmatrix} 1 & 2 - i \\ 2 + i & 0 \end{bmatrix}$ is Hermitian since $\bar{A} = \begin{bmatrix} 1 & 2 + i \\ 2 - i & 0 \end{bmatrix}$ and so $A^H = A^T = \begin{bmatrix} 1 & 2 - i \\ 2 + i & 0 \end{bmatrix} = A$.

10. if $A$ is Hermitian, then $A$ is symmetric. However the converse fails, and here is a counterexample: $A = \begin{bmatrix} 1 & 2 - i \\ 2 - i & 0 \end{bmatrix}$. However if $A \in \mathbb{R}^{n \times n}$ is symmetric, then it is Hermitian.

<table>
<thead>
<tr>
<th>Symmetric and orthogonal matrices in $\mathbb{R}^{n \times n}$</th>
<th>Hermitian and unitary matrices in $\mathbb{C}^{n \times n}$</th>
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<td>Defn: if $A^T = A \iff A$ is symmetric</td>
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<tr>
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<td>$A$ symmetric $\implies \lambda_i \in \mathbb{R}, \forall i$</td>
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<td>$Q$ is <strong>orthogonal</strong> if its column vectors form an orthonormal set (i.e. $Q^TQ = I = QQ^T$)</td>
<td>$U$ is <strong>unitary</strong> if its column vectors form an orthonormal set (i.e. $U^HU = I = UU^H$)</td>
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<td>$Q$ is orthogonal $\implies Q^{-1} = Q^T$</td>
<td>$U$ is unitary $\implies U^{-1} = U^H$</td>
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11. These three give the last row in the table above:

   (a) if the eigenvalues of an Hermitian matrix $A$ are all distinct, then $\exists U$ that is unitary and it diagonalizes $A$. In this case $U$ has as columns the normalized eigenvectors of $A$

   (b) Schur’s Theorem: If $A$ is $n \times n$, then $\exists U$ a unitary matrix such that $T = U^H A U$ is upper triangular matrix.

   (c) Spectral Theorem: If $A$ is Hermitian, then $\exists U$ a unitary matrix such that $U^H A U$ is a diagonal matrix.

   Note that if some eigenvalue $\lambda_j$ has algebraic multiplicity $\geq 2$, then the eigenvectors corresponding to $\lambda_j$ are not orthonormal, and so we use Gram-Schmidt to normalize them (we use Gram Schmidt for each set of eigenvectors that correspond to each repeated eigenvalue)
\[ \langle x, y \rangle = y^T x \]
\[ \langle x, x \rangle \geq 0 \text{ with equality iff } x = 0 \]
\[ \langle x, y \rangle = \langle y, x \rangle \]
\[ \langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \]
\[ ||x||^2 = \langle x, x \rangle = x^T x \]
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### Symmetric and orthogonal matrices in \( \mathbb{R}^{n \times n} \)

- Defn: if \( A^T = A \iff A = \text{symmetric} \)
- \( A = \text{symmetric} \implies A \) is a square matrix
- \( A = \text{symmetric} \implies \lambda_i \in \mathbb{R}, \forall i \)
- \( A = \text{symmetric} \implies \) eigenvectors belonging to distinct eigenvalues are orthogonal
- \( Q \) is orthogonal if its column vectors form an orthonormal set
  (i.e. \( Q^T Q = I = Q Q^T \))
- \( Q \) is orthogonal \( \iff Q^{-1} = Q^T \)

### Hermitian and unitary matrices in \( \mathbb{C}^{n \times n} \)

- Defn: if \( A^H = A \iff A = \text{Hermitian} \)
- \( A = \text{Hermitian} \implies A \) is a square matrix
- \( A = \text{Hermitian} \implies \lambda_i \in \mathbb{R}, \forall i \)
- \( A = \text{Hermitian} \implies \) eigenvectors belonging to distinct eigenvalues are orthogonal
  (see #5 page 353)
- \( U \) is unitary if its column vectors form an orthonormal set
  (i.e. \( U^H U = I = U U^H \))
- \( U \) is unitary \( \iff U^{-1} = U^H \)
- if \( A \) is an Hermitian matrix \( A \), \( \iff \exists U \) = unitary and it diagonalizes \( A \)
  (i.e. the diagonal matrix \( T \) is \( T = U^H A U \) or \( A = U T U^H \))
  \( T \) is first shown to be upper triangular in Thm 6.4.3
  and then that it is shown to be diagonal in Thm 6.4.4

(i.e. the diagonal matrix \( D \) is \( D = X^{-1} A X \) or \( A = X D X^{-1} \))
(see Remark 3 page 308)