11.8 Power Series

1. a standard power series: \[ \sum_{n=1}^{\infty} c_n x^{n-1} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \ldots \]
   or
   \[ \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \ldots \]

   where \( x \) is the variable and \( c_n \)'s are constants, like the geometric series \( \sum_{n=0}^{\infty} r^n = \sum_{n=0}^{\infty} x^n \)

2. the power series can be about \( a \) (or also called a power series in \( (x - a) \)):
   \[ \sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + c_3 (x - a)^3 \ldots \]

3. a power series can
   (a) diverge
   (b) converge just at \( x = a \)
   (c) converge if \( |x - a| < R \), for some \( R \)

4. the above \( R \) is called the radius of convergence

5. use ratio or root test to find the interval of convergence (or absolute convergence since you use ration/root tests) or to find the radius of convergence.

6. use a different test to find if the series converges at the endpoints of the interval of convergence, since the ration/root tests are inconclusive at the endpoints.
11.9 Representation of Functions as Power Series

1. functions can be represented as power series, for example \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \), for \( |x| < 1 \). This particular one worked just because the function happened to be the limit of the geometric series.

2. other functions can be represented as power series with their interval of convergence (see examples 1 – 3 page 729).

3. otherwise, term-by-term differentiation and integration can be used: Given \( \sum_{n=0}^{\infty} c_n (x - a)^n \) has radius of convergence \( R > 0 \), then the function

\[
  f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 \ldots = \sum_{n=0}^{\infty} c_n (x - a)^n
\]

is continuous and differentiable, and so

\[
  f'(x) = c_1 + c_2(x - a) + c_3(x - a)^2 + c_4(x - a)^3 \ldots = \sum_{n=1}^{\infty} c_n (x - a)^{n-1} = \sum_{n=0}^{\infty} c_{n+1}(x - a)^n
\]

and

\[
  \int f(x) \, dx = C + c_0(x - a) + c_1 \frac{(x - a)^2}{2} + c_2 \frac{(x - a)^3}{3} \ldots = C + \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}
\]

with the same radius of convergence \( R \).