1 Chapter 1: Matrices and Systems of Equations

1.4 Elementary Matrices

1. Elementary matrices, denoted by $E$, are matrices obtained by performing one elementary row operation on $I$. There are three types:

- **[Type I]** interchanging two rows:
  $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- **[Type II]** constant multiple of a row:
  $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- **[Type III]** adding a multiple of a row to another row:
  $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

2. **post multiplying** a matrix $B$ by an elementary matrix $E$ is equivalent to performing that particular column operation on $B$

3. **pre multiplying** a matrix $B$ by an elementary matrix $E$ is equivalent to performing that particular row operation on $B$

4. that is, every elementary matrix has an inverse which is also an elementary matrix:

   - (a) the inverse of a matrix of type $I$ is also of type $I$, actually it is its own inverse

   - (b) the inverse of a matrix of type $II$ is also of type $II$:
     $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   - (c) the inverse of a matrix of type $III$ is also of type $III$:
     $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$
5. $B$ is row equivalent to $A$ if multiplying $A$ by a series of elementary matrices we get $B$ (note that every multiplication by an elementary matrix can be viewed as a step in Gaussian Elimination method).

6. The following are equivalent:
   
   (a) $A$ is nonsingular
   
   (b) $Ax = 0$ has only one solution: $x = 0$
   
   (c) $A$ is row equivalent to $I$.

7. $Ax = 0$ has a unique solution (namely 0) $\iff A$ is nonsingular

8. finding $A^{-1}$ using the elementary row operations (page 66).

9. diagonal matrix: if the entries of the diagonal are the only possible nonzero entries of the matrix

10. upper triangular matrix: if the entries below the diagonal are zero (some of the other entries could be zero as well)

11. lower triangular matrix: if the entries above the diagonal are zero (some of the other entries could be zero as well)

12. triangular matrix: if it is either upper or lower triangular matrix (recall strict triangular matrix).

13. skip Triangular Factorization.