Two Conjectures Involving Diameter and Total Domination in Graphs

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We present two intriguing conjectures involving diameter and total domination in graphs.

The first conjecture, posed independently by Murty and Simon, has origins that can be traced to the work of Ore in the early 1960’s. While the first conjecture is over 50 years old, the second one, posed by Desormeaux, H., and Henning, is brand new.

We will discuss the progress made on each conjecture.
A total dominating set of a graph $G$ with no isolated vertex is a set $S$ of vertices of $G$ such that every vertex of $G$ is adjacent to a vertex in $S$.

The total domination number $\gamma_t(G)$ is the minimum cardinality of any total dominating set of $G$.

Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi and is now well studied in graph theory.
Diameter 2-critical Graphs

A graph $G$ is called *diameter 2-critical* if its diameter is 2, and the deletion of any edge increases the diameter.

Example: Complete Bipartite Graphs
(Murty-Simon Conjecture) If $G$ is a diameter 2-critical graph with order $n$ and size $m$, then $m \leq n^2/4$, with equality if and only if $n$ is even and $G$ is the complete bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$. 
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- 1984: Xu published an incorrect proof.
- 1987: Fan proved the first part of the conjecture for $n \leq 24$ and for $n = 26$. For $n \geq 25$, he obtained
  
  $m < n^2/4 + (n^2 - 16.2n + 56)/320 < .2532n^2$. 
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- 1992: Füredi gave an asymptotic result proving the conjecture is true for large \( n \), that is, for \( n > n_0 \) where \( n_0 \) is a tower of 2’s of height about \( 10^{14} \).
A Different Perspective

“A little perspective, like a little humor, goes a long way.”
Allen Klein

Attacking the conjecture head-on has not solved it. Perhaps it is time for a new approach.
Edge Critical Graphs

Lucas van der Merwe’s Ph.D. Dissertation

A graph $G$ is total domination edge critical if the addition of any edge changes the total domination number.

$k_t$-critical
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The Two Conjectures

The Murty/Simon Conjecture
Total Domination Edge Critical Graphs
The Relationship
Recent Progress

The Connection

“The reverse side also has a reverse side.”
Proverb, Japanese

Seemingly unrelated parameters...

KEY RELATIONSHIP:
Theorem (Hanson and Wang, 2003) A graph is diameter 2-critical if and only if its complement is $3_t$-critical or $4_t$-supercritical.
Examples

EXAMPLE: The self-complementary 5-cycle is both diameter 2-critical and $3_t$-critical.
Examples

Figure: A diameter 2-critical graph $G$ and its $3_t$-critical complement $\overline{G}$. 
Examples

\[ G : \]
\[ \overline{G} : \]

Figure: Edge Removed From \( G \).
Examples

Figure: A $\gamma_t$-set of $\overline{G}$. 
Examples

Figure: Edge Added to $\overline{G}$. 

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**Theorem.** (Van der Merwe, Mynhardt, and H.) A graph $G$ is $4_t$-supercritical if and only if $G$ is the disjoint union of two complete graphs.

NOTE: The complement of a $4_t$-supercritical graph is a complete bipartite graph. The number of edges is minimized when the partite sets are equal in size, and so the Murty-Simon Conjecture holds for this case and a subset of the complements of $4_t$-supercritical graphs yield the extremal graphs of the conjecture.
Equivalent Conjecture

**Conjecture 2:** If $G$ is a $3_t$-critical graph with order $n$ and size $m$, then $m > \binom{n}{2} - n^2/4 = n(n - 2)/4$. 
Van der Merwe’s dissertation provides the following nice bounds on the diameter of $3_t$-critical graphs.

**Theorem** (Van der Merwe, Mynhardt, and H.) If $G$ is a $3_t$-critical graph, then $2 \leq \text{diam}(G) \leq 3$. 
**Diameter 3**

**Theorem** (Hanson-Wang Theorem) If $G$ is a $3_t$-critical graph of diameter three and of order $n$ and size $m$, then $m \geq n(n - 2)/4$.

**NOTE:** In order to prove Conjecture 2, we need to show strict inequality in the Hanson-Wang Theorem. Hence an additional edge was necessary to prove the conjecture in this case.
Indeed a surprising amount of work was required to find this one additional edge.  

**Theorem** (H., Henning, van der Merwe, Yeo, 2009) Conjecture 2 is true for graphs with diameter 3.
**Theorem** (H., Henning, and Yeo, 2010) Conjecture 2 is true for claw-free graphs.

Idea behind proof: Partition into pseudo-cliques.
**The Two Conjectures Involving Diameter and Total Domination in Graphs**

**Theorem** (H., Henning, and Yeo, 2010) Conjecture 2 is true for graphs having connectivity at most 3.

This result is surprising in a sense because intuitively one would think it would be easier to establish a larger edge count for graphs having large connectivity; whereas, here we have proven a lower bound on the number of edges of $3_t$-critical graphs with small connectivity.
Progress Made on the Murty-Simon Conjecture

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- Nice observation by Hanson and Wang yields an equivalent conjecture, providing a new way to approach the problem from the perspective of total domination.

- Using this approach, we have verified the Murty-Simon Conjecture for a number of infinite families of graphs, namely, graphs whose complements have diameter three, are claw-free, or have connectivity at most three.
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- The fact that it remains is a testimony to its difficulty and that there is still much work to do.
- On the other hand, our preliminary work coming at it through the “backdoor with total domination” is evidence that this new approach is promising as it has shed light on the subject and allowed us to obtain some significant partial results.
We now turn our attention to the second conjecture of the talk, an upper bound on the total domination number of a diameter 2 graph. We note that the best known upper bound on the total domination number of any connected graph $G$ with order $n$ and no isolated vertices is $2n/3$. Yeo improved the bound to strictly less than $3n/7$ for graphs with diameter 2 and minimum degree at least 4.
During Desormeaux’s work on his Ph.D. dissertation in 2011, he proved several results that support the following conjecture:

**Conjecture** (Desormeaux, H., Henning 2011) If $G$ is a diameter-2 graph of order $n$, then $\gamma_t(G) \leq \sqrt{n} + 1$. 
In Desormeaux’s dissertation, it is shown that any graph $G$ providing a counterexample to this conjecture would have to have the following properties:

- order $n \geq 12$,
- both $G$ and $\overline{G}$ have diameter 2,
- girth $g(G) \in \{3, 4\}$ and if $g(G) = 4$, then $G$ has an induced 5-cycle,
- neither $G$ nor $\overline{G}$ is planar,
- minimum degree of $G$ is between $\sqrt{n}$ and $\ln(n) \sqrt{n}$.
Note that we can prove the conjecture for a graph having minimum degree $\delta(G) \leq \sqrt{n}$ or having $\delta(G) \geq \sqrt{n} \ln n$. So we were becoming more and more convinced of its validity.

**UPDATE:**
While this talk was being prepared, Desormeaux, H., Henning, and Yeo found a counterexample to the original conjecture and have been able to prove a modified version of the conjecture as follows:

**THEOREM** (we think; proof in process): If $G$ is a diameter-2 graph, then $\gamma_t(G) \leq 1 + \sqrt{n \ln(n)}$. 


Ya-Chen Chen and Z. Füredi, Minimum vertex-diameter-2-critical graphs. *J. Graph Theory* 50 (2005), no. 4, 293–315.


Z. Füredi, The maximum number of edges in a minimal graph of diameter 2. *J. Graph Theory* **16** (1992), 81–98.


T. W. Haynes, M. A. Henning, and A. Yeo, A proof of a conjecture on diameter two critical graphs whose complements are claw-free, Submitted January 2010.


