Efficient Employment of Non-Reactive Sensors

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Abstract

We consider two types of non-reactive aerial sensors, which are subject to false-positive and false-negative errors. The sensors search for threat objects such as ballistic missile launchers or improvised explosive devices. The objects are located in a certain area of interest, which is divided into a grid of area-cells. The grid is defined such that each area-cell may contain at most one object. The objective of a sensor is to *determine* if a certain area-cell is likely or unlikely to contain an object. An area-cell is said to be *determined* if the searcher can ascertain with a given high probability these events. Since definitive identification of a threat object, and subsequent handling of that threat, are done by limited number of available ground combat units, the *determination* of an area-cell can help field commanders better allocate and direct these scarce resources. We develop two models, one for each type of sensor, that describe the search process and maximize the expected number of *determined* area-cells.

1 Introduction

Advents in sensing, unmanned aerial vehicles (UAVs), and satellite technologies are expected to increase the military use of aerial or space sensors for detecting threat objects such as improvised explosive devices or missile launchers. These advanced technologies may generate powerful and effective sensors, which necessitate operational concepts in order to facilitate their efficient utilization. In this paper we address operational concepts associated with employing sensors in persistent search missions over an extended search area. Specifically, we consider the problem of efficiently allocating *non-reactive* sensors across a search area of interest. The sensors are non-reactive in the sense that the search plan is set in advance, and it is not updated in real time during the search process following new information (e.g., pre-programmed “send-and-forget” UAVs). The details of the operational search setting are given in Section 2.

The theory of optimal search has a history of principal importance in military operations. The theory has fundamental applications to anti-submarine warfare, counter-mine warfare, and search and rescue operations. The books [6] and [10] are classical references in this area; with [11] a valuable recent reference. Discrete search problems of the type addressed in this paper are not new. Optimal whereabout search, where we seek to maximize the probability of determining which box contains a certain object, is studied in [1] and [5]. Chew [3] considers an optimal search with stopping rule where all search outcomes are independent, conditional on the location of the searched object and the search policy. Wegener [12] investigates a search process where the search time of a cell depends on the number of searches so far. A minimum cost search problem is discussed in [8], where only one search mode is considered and the sensor has perfect specificity. The paper [9] deals with discrete search with multiple sensors in order to maximize the probability of successful search of a single target during a specified time period. Other discrete search problems are studied in [2, 7, 13]. However, all of the aforementioned references assume that the sensor has perfect specificity, that is, there are no false positive detections. Our models, which are based on [4], relax this assumption.

The main contribution of this paper, in addition to the relaxation of the perfect specificity assumption, is the development of two novel sensor models (*smart* and *dummy* sensors; see Section 2), and their application to a variety of scenarios. For the scenarios examined, the results and analysis indicate that,

- The level of initial intelligence regarding the area of interest has a significant effect on the optimal employment of the sensors and on the expected number of *determined* area-cells, and this effect is quantified.
- The optimal employment of a sensor follows a greedy rule where search effort is first invested in area-cells that are more likely to be determined than others.
- The *smart* sensor significantly outperforms the *dummy* sensor in situations of minimum uncertainty regarding the presence or absence of the threat object. In other situations the effect is not significant.
- The effectiveness of a sensor is determined by the relative values of its sensitivity and specificity and not by the absolute values of these parameters, except when either of these two parameters is very small, in which case sensor's effectiveness is very sensitive to the values of the other parameter.

The paper is organized as follows. In Section 2 we describe the operational setting and in Section 3 we formulate the models for the dummy and smart sensors. In Section 4 we analyze the models with respect to various scenarios, and in Section 5 we discuss the conclusions of the paper.

2 Operational Setting

Targets (e.g., missile launchers) are scattered in an area of interest and the objective of the field commander is to detect as many as possible of them. The area of interest is divided into a grid of area-cells such that each area-cell may contain at most one target. A sensor is assigned to search a certain area-cell for a certain time period during which it can make a finite number of discrete observations or *looks*. The result of each look is either a *detect* result or a *no detect* result. The sensor is imperfect – it is subject to false-positive and false-negative errors – and therefore the sensor's cues may be erroneous. The information provided by the sensor is used by the field commander to decide on further tactical or operational actions. Our goal is to help the field commander to determine the best search plan such that the information provided by the search results – his awareness regarding which area-cells are likely to contain targets and which area-cells are likely to be empty – is maximized. This *informational* MOE is described next.

An area-cell is said to be *determined* if it can be ascertained, with a given (high) probability, whether it contains a target or not. Specifically, given two probability thresholds, selected by the commander and reflecting his attitude regarding uncertainty, an area-cell is determined to be empty if the post-search probability that a target is in that cell is lower than the lower threshold. The area-cell is determined as containing a target if that posterior probability is higher than the higher threshold. The objective is to maximize the expected number of area cells that are determined. This type of information – classifying area-cells as being very likely or very unlikely to contain targets – can help field commanders filter a sizable area of interest down to only those area-cells that are likely to contain a target, and therefore better focus their operational effort.

The sensors we consider are *non reactive*; the assignment of looks to area-cells is made in advance and it does not change dynamically following information (*detection* and *no-detection* results) obtained during the search. This situation is applicable in particular to pre-programmed UAVs whose way-points and search pattern cannot be modified during the search mission.

We consider two types of sensors: *dummy* and *smart*. The *dummy* sensor evaluates the detection/no-detection results of a certain area-cell only at the end of the search, after all assigned looks have been exhausted. Based on the resulting posterior probability and the two probability thresholds, the searcher decides at that point if the area-cell is determined or not. This sensor represents a *batch* handling of the sensor data; the searcher examines the sensor's results and decides upon them only after the search process is over. The *smart* sensor examines the detection/no-detection results and computes the probability of a target continuously during the search. If at any point during the search this probability crosses either of the two thresholds, the area-cell is determined before all the looks are exhausted.

3 Models

We start this section by describing the basic framework shared by the two models. Specifically, we assume that one sensor is assigned an area of interest to search, which is partitioned into a grid of I area-cells. We assume that the area of interest can be partitioned in such a way that each area-cell i, for $i = 1, 2, \ldots, I$, contains at most one threat object. The sensor has a finite number of L looks that it can apply to the search. These looks are allocated to the various area-cells prior to the start of the search mission.

We suppose that there is some initial intelligence about the presence of threat objects, which is manifested by a prior probability. Let $\theta = (\theta_1, \ldots, \theta_I)$ be the parameter that describes the presence/absence of threat objects; that is, $\theta_i = 1$ if there is a threat object in area-cell i, and $\theta_i = 0$ otherwise. The intelligence is captured by the prior probability mass function of θ_i,

$$\pi_i^{(0)} = P(\theta_i = 1),$$

for $i = 1, \ldots, I$. Following a single look at an area-cell, the sensor returns either a *detection* or a *no-detection* signal. The sensor is characterized by its sensitivity and specificity; for each area-cell i we have

$$p_i = P(\text{sensor indicates detection}|\theta_i = 1),$$

which is called the *sensitivity* of the sensor. The *specificity* of the sensor is $1 - q_i$, where

$$q_i = P(\text{sensor indicates detection}|\theta_i = 0).$$

Although the p_i's and q_i's may depend on the area-cell, we assume that they do not depend on the number of looks. Without loss of generality we take $p_i > q_i$, because we can reverse the cue if $p_i < q_i$. We explicitly assume that $p_i \neq q_i$, for otherwise the sensor does not provide any valuable information.

After the sensor looks at an area-cell, the intelligence regarding the likelihood of a threat object gets updated, and we obtain a *posterior* probability. More specifically,

$$\pi_i^{(1)}(\omega) = \begin{cases} \frac{p_i \pi_i^{(0)}}{p_i \pi_i^{(0)} + q_i(1-\pi_i^{(0)})}, & \text{if } \omega = \text{sensor indicates detection} \\ \frac{(1-p_i)\pi_i^{(0)}}{(1-p_i)\pi_i^{(0)} + (1-q_i)(1-\pi_i^{(0)})}, & \text{if } \omega = \text{sensor indicates no-detection.} \end{cases} \quad (1)$$

In this way, for area-cell i we have a sequence of posteriors $\pi_i^{(1)}, \pi_i^{(2)}, \ldots$, adapted to the sequence of signals generated by the sensor in that area-cell.

We assume that the collection of look results are independent for a given area-cell; this assumption asserts that there is not systematic bias in the sensor. The results for different area-cells may be dependent. As the number of looks for a area-cell i increases, the posterior approaches 1 (if $\theta_i = 1$) or 0 (if $\theta_i = 0$). In reality, one would stop looking when the posterior becomes sufficiently close to 1 or 0. This motivates the introduction of two thresholds, which are subjective measures set by an individual involved in the search mission, such as the watch officer in the tactical operations center, or the field commander in charge of attacking these threat objects. An area-cell is considered to be *determined* if the posterior has crossed either an upper threshold or a lower threshold. If the posterior has crossed the upper threshold β, then the conclusion is that the area-cell is most likely to contain a threat object. Conversely, if the posterior has crossed the lower threshold α, then the area-cell is most likely to be clear. To make the problem non-trivial, we assume throughout the paper that $0 \leq \alpha < \beta \leq 1$.

In most realistic situations, the number of looks available is not large relative to the number of area-cells I and therefore an optimal resource (looks) allocation is needed. Specifically, the decision variables for both the dummy and smart sensor models are the number of looks allocated to area-cell i, denoted by l_i. The measure of effectiveness is the expected number of area-cells determined with at most L looks. Observe that

$$E(\#\text{ area-cells determined with } l_1, \ldots, l_I \text{ looks}) = \sum_{i=1}^{I} P(\text{area-cells } i \text{ determined in } l_i \text{ looks}).$$

It follows that the optimization problem is to choose $l_1, \ldots, l_I$ that

$$\text{maximize} \sum_{i=1}^{I} P(\text{area-cell } i \text{ determined in } l_i \text{ looks}) \tag{2}$$

subject to

$$\sum_{i=1}^{I} l_i \leq L$$

$$l_i \geq 0 \text{ and integer for } i = 1, \ldots, I.$$

In order to solve this problem we need to find $P(\text{area-cell } i \text{ determined in } l_i \text{ looks})$ for the dummy and smart sensors. This is the subject of the next two subsections.

Remark 1 *The objective function in Problem 2 is non-linear, and even not necessarily concave; see Figure 3. Once $P(\text{area-cell } i \text{ determined in } l_i \text{ looks})$ is found for each number of looks $1, 2, \ldots, L$, Problem 2 can be implemented and solved – we used GAMS to illustrate the results in this paper.*

3.1 Dummy Sensor

The dummy sensor is characterized by the fact that in each area-cell the sensor checks its status (i.e. posterior probability) only after the allocated l looks are exhausted. If at that point the posterior is larger than β or smaller than α then the area-cell is declared *determined*. A smarter sensor would watch the posterior continuously and determine the area-cell as soon as the posterior crosses a threshold. Indeed, this is the characterization of the *smart* sensor discussed in the next subsection.

Let D_i = number of detections in area-cell i. Conditioning on θ_i we have

$$P(D_i = d) = \binom{l}{d} p_i^d (1-p_i)^{l-d} \times \pi_i^{(0)} + \binom{l}{d} q_i^d (1-q_i)^{l-d} \times (1-\pi_i^{(0)}), \tag{3}$$

for $d = 0, 1, \ldots, l$. When $D_i = d$ the dummy posterior $\psi_i^{(l)}(d)$ is given by, after some algebra,

$$\psi_i^{(l)}(d) = \frac{p_i^d (1-p_i)^{l-d} \pi_i^{(0)}}{p_i^d (1-p_i)^{l-d} \pi_i^{(0)} + q_i^d (1-q_i)^{l-d} (1-\pi_i^{(0)})} \tag{4}$$

Next we ask: How many detections will cause the dummy posterior to be outside either threshold? In other words, for what values in the range of D_i do we have $\psi_i^{(l)}(d) \geq \beta$ or $\psi_i^{(l)}(d) \leq \alpha$? Solving for d in Equation (4) we obtain $\alpha < \psi_i^{(l)}(d) < \beta$ if and only if $a_i < d < b_i$, where

$$a_i = \frac{\log\left(\frac{\alpha}{1-\alpha} \frac{1-\pi_i^{(0)}}{\pi_i^{(0)}}\right) + l \log\left(\frac{1-q_i}{1-p_i}\right)}{\log\left(\frac{p_i(1-q_i)}{(1-p_i)q_i}\right)} \tag{5}$$

and

$$b_i = \frac{\log\left(\frac{\beta}{1-\beta}\frac{1-\pi_i^{(0)}}{\pi_i^{(0)}}\right) + l\log\left(\frac{1-q_i}{1-p_i}\right)}{\log\left(\frac{p_i(1-q_i)}{(1-p_i)q_i}\right)} \tag{6}$$

Hence

$$P(\text{area-cell } i \text{ not determined in } l \text{ looks}) = P(a_i < D_i < b_i) \tag{7}$$

where the probability mass function of D_i is computed according to Equation (3). It is beneficial to view the interval (a_i, b_i) as a *no determination* region. Naturally,

$$P(\text{area-cell } i \text{ determined in } l \text{ looks}) = 1 - P(\text{area cell } i \text{ not determined in } l \text{ looks}),$$

can then be employed in the optimization problem.

As an example of the dummy sensor model, consider Figure 1. For $l = 1$, it is impossible to determine the area cell because the posterior is always inside the thresholds; for $l = 2$, having two detections cause the posterior to be above β and so the area-cell is determined. Observe, however, that the probability of determining the area-cell is less for $l = 3$ (.47) than for $l = 2$ (.48), because getting a no-detection after two detections decreases the posterior and pushes it back to within the thresholds.

3.2 Smart Sensor

The smart sensor monitors the posterior continuously and therefore may determine an area-cell as soon as the posterior crosses a threshold, before the looks allocated to that cell are actually exhausted. The subsequent looks are essentially redundant. Although there are several approaches to compute the probability of detection, a simple approach is to use dynamic programming. Define $V_{i,l}(\pi_i)$ as the probability of determining the presence, or absence, of a threat object after l looks in area-cell i, given the current prior probability is π_i. We have the boundary conditions

$$V_{i,l}(\pi_i) = 1, \text{ if } \pi_i \geq \beta \text{ or } \pi_i \leq \alpha,$$

else if $l = 0$

$$V_{i,0}(\pi_i) = 0.$$

For $l \geq 1$, the recursion is given by

$$\begin{aligned} V_{i,l}(\pi_i) &= (p_i\pi_i + q_i(1-\pi_i))V_{i,l-1}\left(\frac{p_i\pi_i}{p_i\pi_i + q_i(1-\pi_i)}\right) \\ &\quad + ((1-p_i)\pi_i + (1-q_i)(1-\pi_i))V_{i,l-1}\left(\frac{(1-p_i)\pi_i}{(1-p_i)\pi_i + (1-q_i)(1-\pi_i)}\right). \end{aligned}$$

Given a prior $\pi_i^{(0)}$ and l looks, we start the above recursion with $V_{i,l}(\pi_i^{(0)})$.

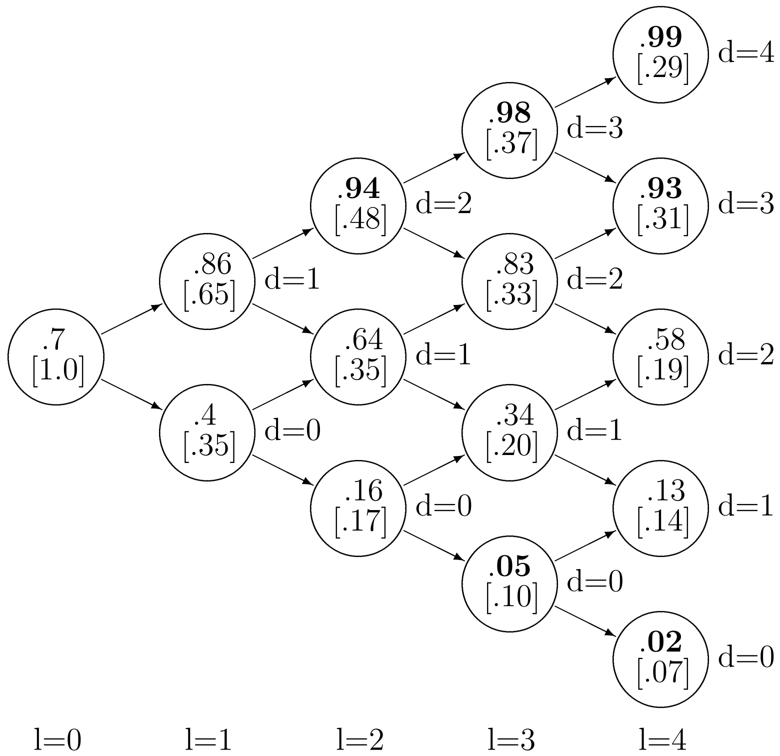

Figure 1: Probability transitions for the dummy sensor model, for $\pi^{(0)} = 0.7$, $p = 0.8$, $q = 0.3$, $\alpha = 0.1$, and $\beta = 0.9$. Inside each node, the top number is the posterior, and the number in brackets is the probability of arriving at the node; nodes with **bold** posteriors occur when the area-cell is *determined*.

Another (computationally faster) approach to compute $V_{i,l}(\pi_i^{(0)})$ is to notice that $\Pi_i = (\pi_i^{(k)} : k \geq 0)$ is a Markov chain defined on $[0,1]$. Let $\tau = \inf\{k \geq 0 : \pi_i^{(k)} \notin (\alpha, \beta)\}$ be the first look at which the posterior crosses either threshold. We have

$$V_{i,l}(\pi_i^{(0)}) = P(\tau \leq l). \tag{8}$$

Let $B = (B_{xy} : \alpha < x, y < \beta)$ be the restriction of the transition kernel of Π_i to (α, β). Then

$$P(\tau > l) = \sum_{\alpha < y < \beta} B^l_{\pi_i^{(0)} y},$$

which together with Eq. (8) leads to $V_{i,l}(\pi_i^{(0)})$.

An example of the smart sensor search process is shown in Figure 2. In this example $B^1_{.7,.86} = .65$, $B^1_{.7,.4} = .35$ and $B^1_{.7,y} = 0$ for all other values of y. Also $P(\tau > 2) = .52$ and $P(\tau > 3) = .42$. The difference between the dummy and smart sensors is that the smart sensor determines the area-cell when arriving at a node whose posterior is outside the thresholds. So while the probability that a dummy sensor determines an area-cell after 3 looks is .37 (see Figure 1) the smart sensor determines it with probability $.48 + .10 = .58$.

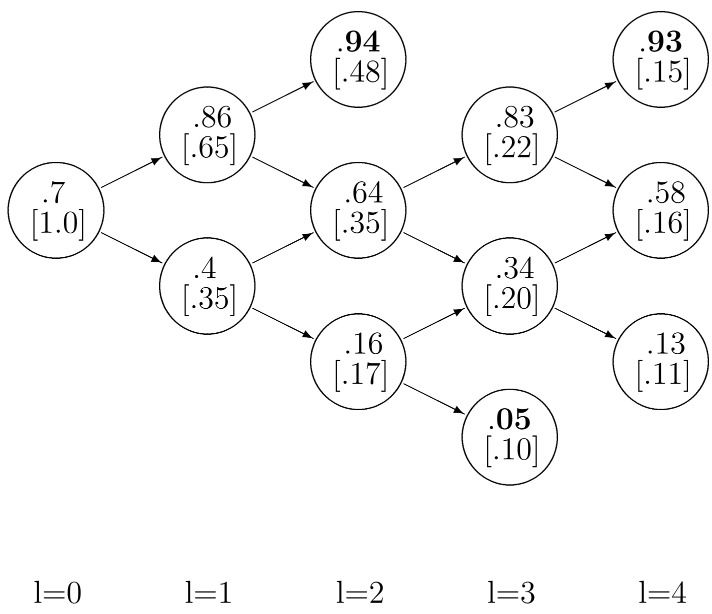

Figure 2: Probability transitions for the smart sensor model, for $\pi^{(0)} = 0.7$, $p = 0.8$, $q = 0.3$, $\alpha = 0.1$, and $\beta = 0.9$. Inside each node, the top number is the posterior, and the number in brackets is the probability of arriving at the node; nodes with **bold** posteriors occur when the area-cell is *determined*.

4 Results and Analysis

In this section we present the results and their analysis for both the dummy and smart sensor models. In the first subsection we discuss the single area-cell scenario, while in the second subsection we optimize sensor employment in multiple area-cells.

4.1 Single area-cell Scenario

We analyze the effect of the model parameters on the probability of determining a single area-cell, so the optimization problem (2) does not come into play. To simplify notation, we drop the subindex i in the discussion that follows in this subsection.

First, we consider the dummy sensor. From the definition of a and b, it is easy to see that they are linear functions of l with positive slope since $p > q$, and that the difference $b - a$ is constant in l. So, we have the following result (see the Appendix for all proofs).

Proposition 1 *The probability of determining an area-cell approaches 1 as the number of looks grows to infinity.*

Observe that the number of integers that lie in the open interval (a, b) is not necessarily constant as a function of the number of looks l. That is, on the sample paths where the dummy posterior is outside the thresholds, a sensor signal may push the dummy posterior back into (α, β), thus increasing the probability of not determining the area-cell. Looking at Equation (7), this suggests that $P(\text{area-cell } i \text{ determined in } l \text{ looks})$ may not be monotonic in the number of looks l. Indeed, for certain parameter settings, increasing the number of looks actually lowers the probability of determining an area-cell. Figure 3 illustrates this situation: When the number of looks goes from 3 to 4, the probability of determining the area-cell decreases; the same happens when going from 5 to 6 looks, 7 to 8 looks, etc. This phenomenon is demonstrated also in Figure 1, as discussed above. From the definition of a in (5) and of b in (6), it follows that the cardinality of the open interval (a, b) is generally not continuous in the model parameters. Ultimately, this causes Figure 3 – Figure 8 to be jagged for the dummy sensor.

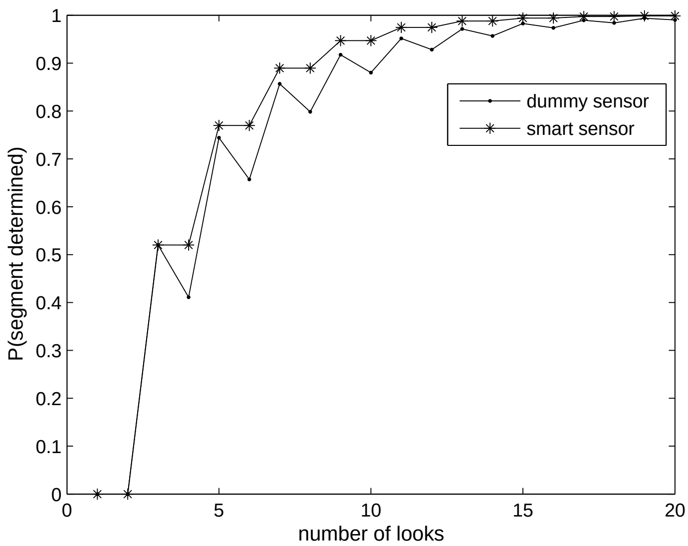

Figure 3: Probability of determining an area-cell as a function of number of looks, for $\pi^{(0)} = 0.5$, $p = 0.8$, $q = 0.2$, $\beta = 0.95$, and $\alpha = 0.05$.

An important issue is the effect of the prior intelligence on the probability of determining an area-cell. Note that the prior $\pi^{(0)}$ is the mixture parameter of the Binomial mixture in Equation (3), and it appears in the definition of a and b. When the number of looks is large, the area-cell is determined with very high probability, regardless of the prior. Hence, for the purpose of our analysis we assume that l is not too large and consider three ranges of $\pi^{(0)}$:

- $\pi^{(0)}$ is close to the lower threshold α. In this case $D \overset{\mathcal{D}}{\approx} Bin(l, q)$ (where $\overset{\mathcal{D}}{\approx}$ means approximately distributed, and $Bin(l, q)$ is a binomial distribution with l looks and probability of detection q). Also, a and b are in the highest part of their range; that is, we determine the area-cell for small values of D. But this is precisely what happens when $D \overset{\mathcal{D}}{\approx} Bin(l, q)$ and q is not too large: D is most likely to be a small number. Hence, when $\pi^{(0)} \approx \alpha$ and q not too large, the probability of determining the area-cell is large.

- $\pi^{(0)}$ is close to the upper threshold. In this case $D \overset{\mathcal{D}}{\approx} Bin(l, p)$ so that D puts most of its mass in the higher end of its range if p is large. Also, a and b are in the low part of their range, so that we determine the area-cell for large values of D. Putting the last two observations together, we conclude that the probability of determining an area-cell is large when $\pi^{(0)}$ is close to β and p is large.

- $\pi^{(0)}$ is not close to either threshold, p and q are mid range. In this situation $\pi^{(0)}$ is a mixture of binomials, and since p and q are mid range, D is most likely to take values in the middle of its range. Also, a and b are in the middle part of their range. Putting these two arguments together, we conclude that the probability of determining an area-cell will be small under these circumstances.

Figure 4 illustrates the above analysis. Other than confirming our explanation of the effect of the prior, Figure 4 is rather striking because of its jumps; these are due to the change in the number of integers that lie in the interval (a, b) as we change the prior, a phenomenon observed and discussed earlier.

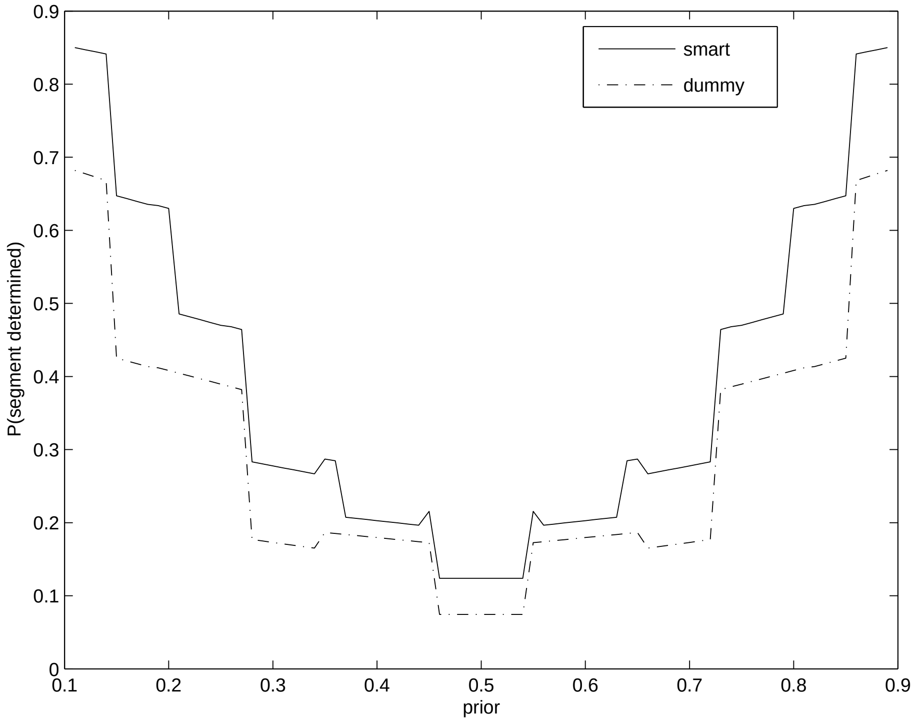

Figure 4: Probability of determining an area-cell as a function of the initial prior, for $l = 11$, $p = 0.6$, $q = 0.4$, $\beta = 0.9$, and $\alpha = 0.1$.

Now we address the effect of sensitivity (p) and specificity ($1 - q$). The basic question regarding the parameters p and q is: What is a good dummy sensor with respect to these parameters? Naturally, $p = 1$ and $q = 0$ is *the* perfect sensor, but this situation is unattainable in practice. As Figure 5 suggests,

- it is the difference $p - q$ that makes a sensor better or worse, regardless of the absolute values of the parameters; the probability of determining an area-cell increases with $p - q$.

- for low values of p, the probability of determining an area-cell is very sensitive to q; that is, a small increase in q causes the probability of determining an area-cell to decrease significantly. The reason for this behavior is that (a, b) expands to include all the integers in $[0, l]$ as q gets closer to p small.
- for high values of q, the probability of determining the area-cell is very sensitive to p. As p moves from q to 1, the no determination region (a, b) moves away from $[0, l]$, thus causing the area-cell to be determined.

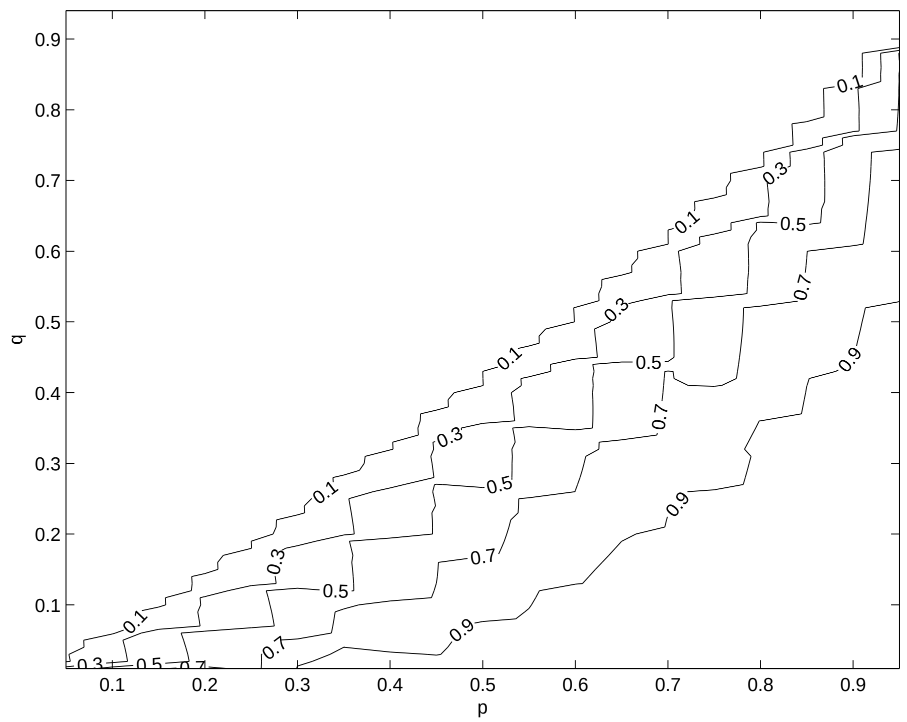

Figure 5: Contour plot of the probability of determining an area-cell for the dummy sensor, as a function of the sensitivity and specificity, for $l = 11$, $\pi^{(0)} = .8$, $\beta = 0.9$, and $\alpha = 0.1$.

Regarding the smart sensor, the following proposition summarizes some of its properties.

Proposition 2 *The smart sensor has a determination probability that is non-decreasing, approaches one as the number of looks increases, and is not smaller than the determination probability of the dummy sensor for the same number of looks.*

Figure 3 illustrates the last proposition. We explain the observation that the probability of determining an area-cell remains constant when going from 3 to 4 looks, from 5 to 6 looks, etc, by the fact that for the parameter settings of Figure 3, on any sample path where the posterior is within (α, β) prior to looking at the area-cell, the posterior remains within (α, β) regardless of the sensor signal (detection or no-detection). Like in the dummy sensor case, a determination probability that is not everywhere differentiable causes Figure 3 – Figure 8 to be non-smooth for the smart sensor as well.

The effect of prior intelligence on the smart sensor is illustrated in Figure 4, with the following interpretation:

- As $\pi^{(0)}$ gets close to either threshold, the probability of determining the area-cell approaches 1. That is, when $\pi^{(0)} \approx \alpha$ and α is small, according to Equation (1), $\pi^{(1)}$(no-detection) ≈ 0 (so we cross the lower threshold) with probability equal to P(no-detection signal) $\approx 1 - q$, so that $P(\tau = 1) \approx 1 - q$; if we get a detection in the first look, the same analysis shows that $P(\tau = 2) \approx q(1 - q)$. Proceeding in that fashion we see that P(area-cell determined for l small) ≈ 1 when $\pi^{(0)} \approx \alpha$ and $\alpha \approx 0$.

- The analysis for the upper threshold β is analogous when $\pi^{(0)} \approx \beta$ and $\beta \approx 1$, and we have P(area-cell determined for l small) ≈ 1.

- A remarkable feature of Figure 4 is that the smart sensor significantly outperforms the dummy sensor when the prior $\pi^{(0)}$ is close to either threshold. The reason for this behavior is that the dummy sensor only checks the value of the posterior when all the looks have been exhausted, by which time it is possible that $\psi^{(l)}$ is within the thresholds. In the next section we discuss how this phenomenon carries over to the multiple area-cell situation.

The effect of sensitivity and specificity with respect to the smart sensor (see Figure 6) is similar to the dummy sensor: The difference $p - q$ is the important measure concerning sensor performance, and the probability of determining an area-cell increases with $p-q$. Like in the dummy sensor model, the probability of determining the area-cell is very sensitive to q when p is small, and very sensitive to p when q is large.

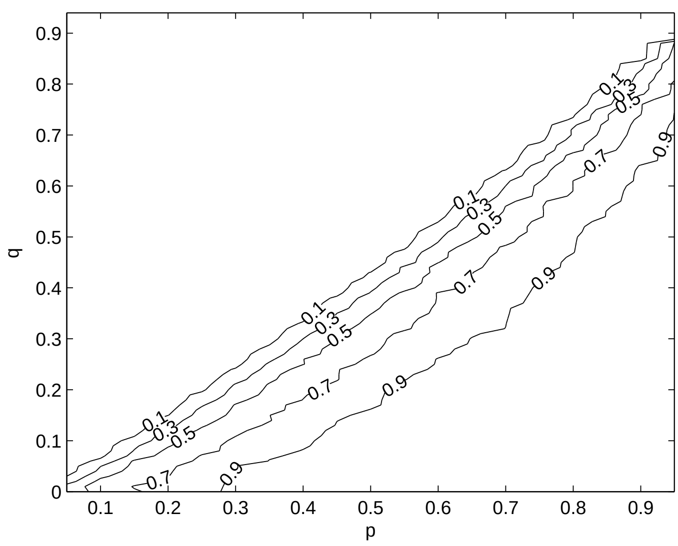

Figure 6: Contour plot of the probability of determining an area-cell for the smart sensor, as a function of the sensitivity and specificity, for $l = 11$, $\pi^{(0)} = .8$, $\beta = 0.9$, and $\alpha = 0.1$.

4.2 Multiple area-cells Scenario

When there is more than one area-cell, we solve Problem (2) to obtain an efficient allocation of looks. Evidently, the effectiveness of the sensors increases with the number of looks, because the uncertainty regarding the presence or absence of threat objects in each area-cell is revealed as we increase the number of looks. In view of Propositions 1 and 2, for any *fractional* allocation $\hat{l}_1 = t_1 L, \hat{l}_2 = t_2 L \ldots, \hat{l}_I = t_I L$ such that all the t's are positive and $t_1 + \cdots + t_I = 1$, we have

$$E(\#\text{ area-cells determined with } \hat{l}_1, \ldots, \hat{l}_I \text{ looks}) \to I,$$

as $L \to \infty$. Since the optimal allocation is no worse than the $\hat{l}_1, \ldots, \hat{l}_I$ allocation, we have

Proposition 3 *Suppose that $l_1^*(L), \ldots, l_I^*(L)$ is an optimal solution to Problem (2) when there are L looks available. Then, for both the dummy and smart sensor models,*

$$E(\#\textit{ area-cells determined with } l_1^*(L), \ldots, l_I^*(L) \textit{ looks}) \to I,$$

as $L \to \infty$.

In words, Proposition 3 states that the expected number of area-cells determined under an optimal allocation of looks approaches the total number of area-cells, as the number of looks available grows. Also, since the smart sensor cannot be worse than the dummy sensor, the expected number of area-cells determined by the smart sensor is never smaller than the expected number of area cells determined by the dummy sensor. This observation and Proposition 3 are demonstrated in Figure 7, where $I = 6$ and all area-cells have the same prior, sensitivity and specificity probabilities. For $L \leq 7$ we have that the dummy and smart sensor models yield the same result, this is because each of the 6 area-cells gets no more than 2 looks, and P(area-cell determined) is the same for both sensors in this situation (cf. Figure 3). As the number of looks available increases, the smart sensor has a larger number of area-cells determined than the dummy sensor, in accordance with Proposition 2.

Next we examine the effect of the prior, sensitivity and specificity probabilities ($\pi^{(0)_i}, p_i$ and $1 - q_i$, respectively) on the the optimal *allocation* of looks. The question is: What area cells get a large (or small) number of looks at optimality? Due to the high dimensionality of the problem, it is impossible to run a full factorial experiment. Therefore, we settle with solving Problem (2) under various representative scenarios that capture the main effects of the above parameters.

Concerning the effect of the prior (see Figure 8), we have the same conclusion as for the single area-cell scenario, namely: The priors close to the thresholds α and β lead to larger expected number of area-cells determined, and the smart sensor significantly improves on the dummy sensor in situations of good prior intelligence. Figure 8 shows these properties for $I = 6$ area-cells when the priors of all area-cells shift together from the lower threshold to the upper threshold. We solved the optimization problem (2) under several other representative configurations, with all the results supporting the above conclusions.

Table 1 summarizes the results of a more detailed analysis. For each one of the two sensors we consider two levels of effectiveness, manifested by the sensitivity and specificity of the sensor, and two configurations of prior probabilities. A relatively *ineffective* sensor

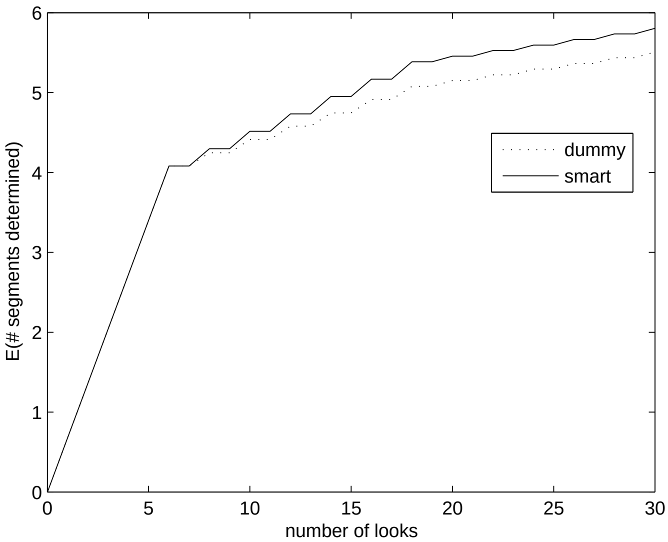

Figure 7: Expected number of area-cells determined as a function of the number of looks, for $I = 6$, $L = 30$, $\pi^{(0)} = 0.8$, $p = 0.8$, $q = 0.2$, $\beta = 0.9$, and $\alpha = 0.1$.

has $p = .6$ and $q = .4$ for all area-cells, while for a relatively *effective* sensor these parameters are .7 and .3, respectively. For each sensor and each level of effectiveness we consider two spatial configurations of the prior probabilities: (1) *Uniform Worse-Case* configuration where $\pi_i^{(0)} = .5, i = 1, ..., 6$, and (2) *Mixed* configuration where the prior is close to the upper threshold for two area-cells, the prior is far from both thresholds for two area-cells, and the prior is close to the lower threshold for two area-cells. For each sensor, level of effectiveness and spatial prior configuration, Table 1 presents the optimal allocation and the maximum expected number of determined area-cells.

Effect of the prior									
p, q	*Spatial Configuration*	$E(\#$ det.$)$	l_1	l_2	l_3	l_4	l_5	l_6	*Sensor*
$p = .6, q = .4$	$\pi^{(0)} = (.5, .5, .5, .5, .5, .5)$	0.7705	16	14	0	0	0	0	smart
		0.6447	16	14	0	0	0	0	dummy
	$\pi^{(0)} = (.8, .8, .5, .5, .2, .2)$	2.4488	8	8	0	0	8	6	smart
		1.9936	10	8	0	0	6	6	dummy
$p = .7, q = .3$	$\pi^{(0)} = (.5, .5, .5, .5, .5, .5)$	3.6186	5	5	5	5	5	5	smart
		3.3540	5	5	5	5	5	5	dummy
	$\pi^{(0)} = (.8, .8, .5, .5, .2, .2)$	4.6857	3	3	9	9	3	3	smart
		4.2253	5	5	9	9	1	1	dummy

Table 1: Effect of the prior for $L = 30$, $I = 6$, $\alpha = 0.1$, and $\beta = 0.9$.

The take-away of Table 1 is:

- When the sensors are relatively ineffective, and and the prior configuration is uniform ($\pi_i^{(0)} = 0.5$, $i = 1, ..., 6$), both the dummy and smart sensors allocate all the looks to

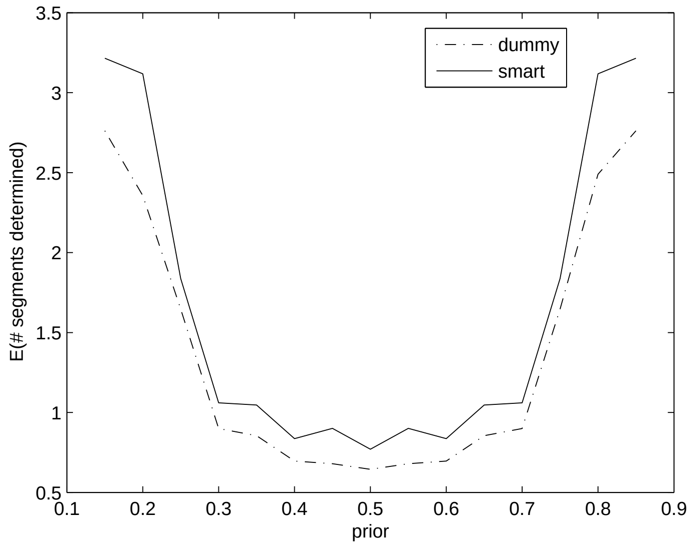

Figure 8: Expected number of area-cells determined as a function of the initial prior, for $I = 6$, $L = 30$, $\pi^{(0)} = 0.8$, $p = 0.8$, $q = 0.2$, $\beta = 0.9$, and $\alpha = 0.1$.

just two area-cells. This happens because in this situation it takes a large number of looks to make the probability of determining an area-cell lift above its zero-value floor. If the prior configuration is mixed, there are four area-cells with prior probabilities close to a threshold. Hence, the model allocates the looks in a greedy fashion so as to determine these four area-cells. In accordance to our single area-cell analysis, the smart sensor significantly outperforms the dummy sensor.

- When the sensors are relatively effective and the prior configuration is uniform, both the smart and dummy sensors uniformly allocate the looks among the 6 area-cells. This occurs because it takes a small number of looks to have the initial shoot up in the probability of determining an area-cell. Observe that the expected number of determined cells shows a remarkable increase from the ineffective-sensor uniform-prior situation; we will have more to say about this issue when we analyze the effect of the p, q configuration.

- When the sensors are effective and the prior has a mixed spatial configuration, it takes a few looks to determine the area-cells whose prior is close to a threshold. Hence, the optimal allocation specifies a large number of looks for the cells whose prior is far from either threshold.

Regarding the sensitivity and specificity of the sensors, in all the scenarios we assume a worse-case prior $\pi^{(0)} = 0.5$ for all area cells $i = 1, ..., 6$, and consider two general cases of sensing situations – relatively effective and relatively ineffective. We consider three types of spatial configurations of these sensing capabilities over the 6 area-cells. Recall that a sensor becomes more effective as $p_i - q_i$ increases (see Figures 5 and 6). Thus, in the *effective* case we assume that the average value of $p_i - q_i$ (denoted as $\bar{p} - \bar{q}$) is 0.7, while in the *ineffective* case this average difference is 0.2. The three spatial configurations are: (1) *Mixed* – Three area-cells with relatively large difference and three with relatively small difference, (e.g., $p_i = .9, q_i = .1, i = 1, 2, 3; p_i = .8, q_i = .8, i = 4, 5, 6$), (2) *Uniform* – All six area-cells have the same difference (e.g., $p_i = .9, q_i = .2, i = 1, ..., 6$), (3) *Monotonic* – the sensitivity is monotonic decreasing, the specificity is monotonic increasing but $p_i - q_i$ remains constant for every $i = 1, ..., 6$.

Table 2 summarizes the results of the optimization models for both sensors. From this table we can draw the following conclusions:

- For an effective sensor ($\bar{p} - \bar{q} = 0.7$) almost all the area-cells are determined. While the smart sensor is obviously better, the difference in the expected number of determined area-cells between the two sensors is 5% or less. Also, the spatial configuration has only a small effect on that measure.

- For the ineffective sensor ($\bar{p} - \bar{q} = 0.2$) the performance of the sensors is quite poor (one or two determined area cells) and it depends on the spatial configuration. Both sensors perform best when the spatial configuration is mixed, and worst when it is uniform. The difference in the expected number of determined area-cells between these two spatial configurations is about 100% for both sensors.

- The optimal allocation of looks depends both on the effectiveness of the sensor and the spatial configuration of its effectiveness across the area-cells, but not on the type of sensor (smart or dummy). When the sensor is effective, looks are spread out more or less evenly across the area-cells, unless the sensitivity and specificity are high (e.g., .9 each), in which case one look will suffice. When the sensor is ineffective ($\bar{p} - \bar{q} = 0.2$), then the search effort is concentrated in a few area-cells, which are most likely to become determined after a considerable number of looks (e.g., 11 looks in the mixed case). The other area cells are ignored.

5 Conclusions

In this paper we developed two bayesian-oriented models that describe the performance of two types of imperfect sensors – *dummy* and *smart* – and presented optimal employment schemes for these sensors in a variety of scenarios. We have shown and quantified the advantage of he smart sensor over the dummy one, which underscores the importance of continuous monitoring of sensor data, in particular in the presence of prior intelligence. We have demonstrated the importance of this prior intelligence on the effectiveness of the search; on the optimal employment of the sensors and on the expected number of *determined* area-cells. The optimal employment of sensors is *greedy* in the sense that search efforts must be allocated to area-cells where they can produce definitive information in the form of determined area-cells. Finally, we demonstrated that for realistic sensing capabilities, the effectiveness of a sensor is determined by the relation between its sensitivity and specificity, rather than the absolute values of these parameters.

The models developed in this paper may be extended to other types of sensors – in particular *reactive* sensors that may facilitate dynamic employment during the search mission.

Effect of sensitivity and specificity									
$\bar{p}-\bar{q}$	*Spatial Configuration*	$E(\#$ det.$)$	l_1	l_2	l_3	l_4	l_5	l_6	*Sensor*
0.7	$p = (.9, .9, .9, .8, .8, .8)$	5.9757	1	1	1	9	8	10	smart
	$q = (.1, .1, .1, .2, .2, .2)$	5.8818	1	1	1	10	8	8	dummy
	$p = (.9, .9, .9, .9, .9, .9)$	5.9255	5	5	5	5	5	5	smart
	$q = (.2, .2, .2, .2, .2, .2)$	5.6277	5	5	5	5	5	5	dummy
	$p = (.96, .91, .86, .81, .76, .71)$	5.9598	5	5	8	5	5	2	smart
	$q = (.26, .21, .16, .11, .06.01)$	5.7693	6	5	6	5	6	2	dummy
0.2	$p = (.8, .8, .8, .5, .5, .5)$	2.0485	11	11	8	0	0	0	smart
	$q = (.5, .5, .5, .4, .4, .4)$	1.6179	11	11	8	0	0	0	dummy
	$p = (.8, .8, .8, .8, .8, .8)$	0.9845	15	15	0	0	0	0	smart
	$q = (.6, .6, .6, .6, .6, .6)$	0.8334	15	15	0	0	0	0	dummy
	$p = (.96, .86, .76, .66, .56, .46)$	1.3654	10	13	0	0	0	7	smart
	$q = (.76, .66, .56, .46, .36, .26)$	1.2320	10	13	0	0	0	7	dummy

Table 2: Sensitivity and specificity effects for $L = 30$, $I = 6$, $\pi^{(0)} = 0.5$, $\alpha = 0.1$, and $\beta = 0.9$.

Appendix

Proof of Proposition 1. Since we focus on just one area-cell, we drop the subindex i from the notation. Consider a collection of random variables that describes the number of detections, indexed by the number of looks: $D_1, \ldots, D_l$. We wish to show that

$$P(D_l \in (a(l), b(l))) \to 0$$

as $l \to \infty$. Since

$$P(D_l \in (a(l), b(l))) = E[P(D_l \in (a(l), b(l))|\theta)],$$

it suffices to show that both $P(D_l \in (a(l), b(l))|\theta = 1)$ and $P(D_l \in (a(l), b(l))|\theta = 0)$ converge to 0 as $l \to \infty$. Observe that, by the independence of looks assumption, the Central Limit Theorem implies

$$P\left(\frac{D_l - lp}{\sqrt{lp(1-p)}} \in (u, v)|\theta = 1\right) \to P(Z \in (u, v))$$

as $l \to \infty$, where Z is a normally distributed random variable with mean 0 and variance 1. Hence

$$P(a(l) < D_l < b(l)|\theta = 1) = P\left(\frac{a(l) - lp}{\sqrt{lp(1-p)}} < \frac{D_l - lp}{\sqrt{lp(1-p)}} < \frac{b(l) - lp}{\sqrt{lp(1-p)}}|\theta = 1\right) \to 0,$$

as $l \to \infty$, since $l^{-1/2}b(l) - l^{-1/2}a(l) \to 0$ as $l \to \infty$. Analogously, it is possible to show that

$$P(a(l) < D_l < b(l)|\theta = 0) \to 0,$$

as $l \to \infty$. Hence we conclude that

$$P(D_l \notin (a(l), b(l))) \to 1 \tag{9}$$

as $l \to \infty$. $\otimes$

Proof of Proposition 2. We now argue that the smart sensor cannot be worse than the dummy sensor. Because $P(\tau \leq l)$ is non-decreasing in l, Equation (8) shows that $V_{i,l}(\pi_i^{(0)})$ is non-decreasing in l too. Moreover,

$$\begin{aligned}
& P(\text{area-cell } i \text{ determined in } l \text{ looks by dummy sensor}) \\
&= \sum_{k=0}^{\infty} P(\text{area-cell } i \text{ determined in } l \text{ looks by dummy sensor} | \tau = k) P(\tau = k) \\
&= \sum_{k=0}^{l} P(\text{area-cell } i \text{ determined in } l \text{ looks by dummy sensor} | \tau = k) P(\tau = k) \\
&\leq \sum_{k=0}^{l} P(\tau = k) \\
&= V_{i,l}(\pi_i^{(0)}).
\end{aligned}$$

The above shows that the smart sensor cannot do worse than the dummy sensor. This, together with Equation (9) shows that

$$P(\text{area-cell } i \text{ determined in } l \text{ looks by smart sensor}) \to 1$$

as $l \to \infty$. $\otimes$

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