CLOSED $k$-STOP DISTANCE IN GRAPHS

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Abstract

The Traveling Salesman Problem (TSP) is still one of the most researched topics in computational mathematics, and we introduce a variant of it, namely the study of the closed $k$-walks in graphs. We search for a shortest closed route visiting $k$ cities in a non complete graph without weights. This motivates the following definition. Given a set of $k$ distinct vertices $S = \{x_1, x_2, \ldots, x_k\}$ in a simple graph $G$, the closed $k$-stop-distance of set $S$ is defined to be

$$d_k(S) = \min_{\theta \in \mathcal{P}(S)} \left( d(\theta(x_1), \theta(x_2)) + d(\theta(x_2), \theta(x_3)) + \cdots + d(\theta(x_k), \theta(x_1)) \right),$$

where $\mathcal{P}(S)$ is the set of all permutations from $S$ onto $S$. That is the same as saying that $d_k(S)$ is the length of the shortest closed walk through the vertices $\{x_1, \ldots, x_k\}$. Recall that the Steiner distance $sd(S)$ is the number of edges in a minimum connected subgraph containing all of the vertices of $S$. We note some relationships between Steiner distance and closed $k$-stop distance.
The closed 2-stop distance is twice the ordinary distance between two vertices. We conjecture that \( \text{rad}_k(G) \leq \text{diam}_k(G) \leq \frac{k}{k-1} \text{rad}_k(G) \) for any connected graph \( G \) for \( k \geq 2 \). For \( k = 2 \), this formula reduces to the classical result \( \text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G) \). We prove the conjecture in the cases when \( k = 3 \) and \( k = 4 \) for any graph \( G \) and for \( k \geq 3 \) when \( G \) is a tree. We also study the closed \( k \)-stop center and closed \( k \)-stop periphery of a graph, for \( k = 3 \).

**Keywords:** Traveling Salesman, Steiner distance, distance, closed \( k \)-stop distance.

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1. **Definitions and Motivation**

In this paper, all graphs are simple (i.e., no loops or multiple edges). For vertices \( u \) and \( v \) in a connected graph \( G \), let \( d(u, v) \) denote the standard distance from \( u \) to \( v \) (i.e., the length of the shortest path from \( u \) to \( v \)). Recall that the eccentricity \( e(v) \) of a vertex \( v \) is the maximum distance \( d(v, w) \) over all other vertices \( w \in V(G) \). The radius \( \text{rad}(G) \) of \( G \) is the minimum eccentricity \( e(v) \) over all vertices \( v \in V(G) \), and the diameter \( \text{diam}(G) \) is the maximum eccentricity \( e(v) \) taken over all vertices \( v \in V(G) \).

Let \( G = (V(G), E(G)) \) be a graph of order \( n \) (\( |V(G)| = n \)) and size \( m \) (\( |E(G)| = m \)). Let \( S \subseteq V(G) \). Recall ([2, 4, 5, 6, 7]) that a Steiner tree for \( S \) is a connected subgraph of \( G \) of smallest size (number of edges) that contains \( S \). The size of such a subgraph is called the Steiner distance for \( S \) and is denoted by \( sd(S) \). Then, the Steiner \( k \)-eccentricity \( se_k(v) \) of a vertex \( v \) of \( G \) is defined by \( se_k(v) = \max\{sd(S) | S \subseteq V(G), |S| = k, v \in S \} \). Then the Steiner \( k \)-radius and \( k \)-diameter are defined by \( \text{srad}_k(G) = \min\{se_k(v) | v \in V(G) \} \) and \( \text{sdiam}_k(G) = \max\{se_k(v) | v \in V(G) \} \).

In this paper, we study an alternate but related method of defining the distance of a set of vertices. The closed \( k \)-stop distance was introduced by Gadzinski, Sanders, and Xiong [3] as \( k \)-stop-return distance. The closed \( k \)-stop-distance of a set of \( k \) vertices \( S = \{x_1, x_2, \ldots, x_k \} \), where \( k \geq 2 \), is defined to be

\[
dl_k(S) = \min_{\theta \in P(S)} \left( d(\theta(x_1), \theta(x_2)) + d(\theta(x_2), \theta(x_3)) + \cdots + d(\theta(x_k), \theta(x_1)) \right),
\]
where $\mathcal{P}(\mathcal{S})$ is the set of all permutations from $\mathcal{S}$ onto $\mathcal{S}$. That is the same as saying that $d_k(\mathcal{S})$ is the length of the shortest closed walk through the vertices $\{x_1, \ldots, x_k\}$. The closed $k$-stop eccentricity $e_k(x)$ of a vertex $x$ in $G$ is $\max\{d_k(S) | x \in S, S \subseteq V(G), |S| = k\}$. The minimum closed $k$-stop eccentricity among the vertices of $G$ is the closed $k$-stop radius, that is, $\text{rad}_k(G) = \min_{x \in V(G)} e_k(x)$. The maximum closed $k$-stop eccentricity among the vertices of $G$ is the closed $k$-stop diameter, that is, $\text{diam}_k(G) = \max_{x \in V(G)} e_k(x)$.

Note that if $k = 2$, then $d_2(\{x_1, x_2\}) = 2d(x_1, x_2)$. We thus consider $k \geq 3$. In particular, the closed 3-stop distance of $x, y$ and $z$ ($x \neq y, x \neq z, y \neq z$) is

$$d_3(\{x, y, z\}) = d(x, y) + d(y, z) + d(z, x).$$

For simplicity, we will write $d_3(x, y, z)$ instead of $d_3(\{x, y, z\})$.

The closed 3-stop eccentricity $e_3(x)$ of a vertex $x$ in a graph $G$ is the maximum closed 3-stop distance of a set of three vertices containing $x$, that is,

$$e_3(x) = \max_{y,z \in V(G)} \left( d(x, y) + d(y, z) + d(z, x) \right).$$

The central vertices of a graph $G$ are the vertices with minimum eccentricity, and the center $C(G)$ of $G$ is the subgraph induced by the central vertices. Similarly, we define the closed $k$-stop central vertices of $G$ to be the vertices with minimum closed $k$-stop eccentricity and the closed $k$-stop center $C_k(G)$ of $G$ to be the subgraph induced by the closed $k$-stop central vertices. A graph is closed $k$-stop self-centered if $C_k(G) = G$.

The peripheral vertices of a graph $G$ are the vertices with maximum eccentricity, and the periphery $P(G)$ of $G$ is the subgraph induced by the peripheral vertices. Similarly, we define the closed $k$-stop peripheral vertices of $G$ to be the vertices with maximum closed $k$-stop eccentricity and the closed $k$-stop periphery $P_k(G)$ of $G$ as the subgraph induced by the closed $k$-stop peripheral vertices. For simplicity in this paper, we will sometimes omit the words “closed” and “stop”, so for instance, we will refer to the closed 3-stop eccentricity as the 3-eccentricity of a vertex.

Notice that for all values of $k \geq 2$, two times the $k$-Steiner distance is an upper bound on the closed $k$-stop distance of a set of vertices in a graph. (Given a Steiner tree for a set of $k$ vertices, one possible closed walk through those vertices would trace each edge of the Steiner tree twice.) The $k$-Steiner distance plus one is always a lower bound for the closed $k$-stop distance, since the edges of a closed walk form a connected subgraph.
Necessarily, in a closed walk, either an edge is repeated or a cycle is formed, so at least one edge could be omitted without disconnecting the subgraph. That is, for a set \( S \) of \( |S| = k \in \{1, 2, \ldots, n - 1, n\} \) vertices, we have that

\[
\begin{align*}
(1) & \quad s_{ek}(v) + 1 \leq e_{k}(v), \forall v \in V(G), \\
(2) & \quad s\text{rad}_{k}(G) + 1 \leq \text{rad}_{k}(G) \leq 2 s\text{rad}_{k}(G), \text{ and} \\
(3) & \quad s\text{diam}_{k}(G) + 1 \leq \text{diam}_{k}(G) \leq 2 s\text{diam}_{k}(G).
\end{align*}
\]

For other graph theory terminology we refer the reader to [1]. In this paper we study the closed \( k \)-stop distance in graphs. Particularly, we present an inequality between the radius and diameter that generalizes the inequality for the standard distance. We also present a conjecture regarding this inequality that is verified to be true in trees. We also study the closed \( k \)-stop center and closed \( k \)-stop periphery of a graph, for \( k = 3 \).

2. Possible Values of Closed 3-stop Eccentricities

It is well-known that the ordinary radius and diameter of a graph \( G \) are related by \( \text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G) \). Furthermore, for every \( k \) such that \( \text{rad}(G) < k \leq \text{diam}(G) \), a graph must have at least two vertices with eccentricity \( k \), and at least one vertex with eccentricity \( \text{rad}(G) \). In the case of closed 3-stop distance, there is at least one vertex with closed 3-stop eccentricity \( \text{rad}_{3}(G) \), and there are at least three vertices with closed 3-stop eccentricity \( \text{diam}_{3}(G) \).

**Proposition 1.** A connected graph \( G \) of order at least 3 has at least three closed 3-stop peripheral vertices.

**Proof.** Let \( x \in V(P_{3}(G)) \). Then there exist vertices \( x_{1} \) and \( x_{2} \in V(G) \) such that \( e_{3}(x) = d(x, x_{1}) + d(x_{1}, x_{2}) + d(x_{2}, x) = e_{3}(x_{1}) = e_{3}(x_{2}) \). Thus \( x, x_{1}, x_{2} \in V(P_{3}(G)) \). \( \square \)

Recall that in a graph \( G \), the following relation holds: \( \text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G) \). We present a similar sharp inequality between the closed 3-stop radius and closed 3-stop diameter.

**Proposition 2.** For a connected graph \( G \), we have

\[ \text{rad}_{3}(G) \leq \text{diam}_{3}(G) \leq \frac{3}{2} \text{rad}_{3}(G). \]
Proof. The first inequality follows by definition. Let \( u \in V(C_3(G)) \), and let \( y \in V(P_3(G)) \). There are vertices \( w \) and \( x \), necessarily also in the closed 3-stop periphery, such that \( e_3(y) = d(y, w) + d(w, x) + d(x, y) = e_3(x) = e_3(w) \). Assume, without loss of generality, that \( d(u, y) + d(y, x) + d(x, u) \leq d(u, w) + d(w, x) + d(x, u) \) and \( d(u, w) + d(w, y) + d(y, u) \leq d(u, w) + d(w, x) + d(x, u) \). This gives \( d(u, y) + d(y, x) \leq d(u, w) + d(w, x) \) and \( d(w, y) + d(y, u) \leq d(w, x) + d(x, u) \).

**Case I.** \( d(w, x) \leq 2d(u, y) \).

Using the inequalities above,

\[
e_3(y) = d(y, w) + d(w, x) + d(x, y) \
\leq d(w, x) + d(x, u) - d(y, u) + d(w, x) + d(u, w) + d(w, x) - d(u, y) \
= d(u, x) + d(x, w) + d(w, u) + 2(d(w, x) - d(u, y)) \
\leq e_3(u) + 2(d(w, x) - d(u, y)).
\]

Now, clearly, \( d(w, x) \leq d(w, u) + d(u, x) \), and from our assumption for this case, \( 2d(w, x) \leq 4d(u, y) \). Thus, \( 4d(w, x) \leq d(w, u) + d(u, x) + d(w, x) + 4d(u, y) \), which simplifies to

\[
2(d(w, x) - d(u, y)) \leq \frac{1}{2} (d(u, w) + d(w, x) + d(x, u)) 
\leq \frac{1}{2} e_3(u).
\]

Thus, \( e_3(y) \leq \frac{3}{2} e_3(x) \).

**Case II.** \( d(w, x) > 2d(u, y) \).

If we restrict the paths from \( y \) so that they must come and go through \( u \), the resulting paths will be the same length or longer than they would be without the restriction. Thus, \( e_3(y) \leq 2d(y, u) + e_3(u) < d(w, x) + e_3(u) \). Since \( e_3(u) \geq d(u, w) + d(w, x) + d(x, u) \) and \( d(w, x) \leq d(w, u) + d(x, u) \), it follows that \( d(w, x) \leq \frac{1}{2} e_3(u) \). Thus, \( e_3(y) \leq \frac{3}{2} e_3(u) \).

Recall that, for the standard eccentricity, \( |e(u) - e(v)| \leq 1 \) for adjacent vertices \( u \) and \( v \) in a connected graph. Gadzinski, Sanders and Xiong noted a similar relationship for the closed \( k \)-stop eccentricities of adjacent vertices. Suppose \( u \) and \( v \in V(G) \) are adjacent. Let \( x_2, x_3, \ldots, x_k \) be vertices such that \( e_k(u) = d_k(\{u, x_2, x_3, \ldots, x_k\}) \). One possible closed walk through \( \{u, x_2, x_3, \ldots, x_k\} \) would be from \( u \) to \( v \), followed by a shortest closed walk
through \(\{v, x_2, x_3, \ldots, x_k\}\), and then from \(v\) to \(u\). Thus, \(e_k(u) \leq e_k(v) + 2\).
Similarly, \(e_k(v) \leq e_k(u) + 2\).

**Proposition 3** [3]. If \(u\) and \(v\) are adjacent vertices in a connected graph, then \(|e_k(u) - e_k(v)| \leq 2\).

The following example shows that it is possible for every vertex between \(rad_3(G)\) and \(diam_3(G)\) to be realized as the closed 3-stop eccentricity of some vertex, though it is also possible that some values may only be achieved once.

Let \(V(G) = \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k, x_0, x_1, \ldots, x_k\}\) and \(E(G) = \{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1}, x_ix_{i+1} | 1 \leq i \leq k - 1\} \cup \{x_0x_1, x_0u_1, x_0v_1, x_0w_1, u_1v_1, v_1w_1\}\). Then \(rad_3(G) = e_3(x_0) = 4k\), \(e_3(u_i) = e_3(x_i) = e_3(w_i) = 4k + 2i\), and \(e_3(v_i) = 4k + 2i - 1\). Notice that all odd eccentricities larger than \(4k + 2M - 1\) may be skipped by leaving out the vertices \(v_i\) for \(i > M\).

Thus, this construction also shows that not all integers between \(rad_3(G)\) and \(diam_3(G)\) must be realized. Figure 1 shows an example of this construction with \(k = 3\).

![Figure 1. Graph with closed 3-stop eccentricities 12, 13, 14, 15, 16, 17, 18.](image)

In any graph \(G\), there is at least one vertex with closed 3-stop eccentricity \(rad_3(G)\) and at least three vertices with closed 3-stop eccentricity \(diam_3(G)\).

From Proposition 3, we may conclude that, for any two consecutive integers \(k\) and \(k+1\) with \(rad_3(G) \leq k < diam_3(G)\), there must be a vertex with closed
3-stop eccentricity either \( k \) or \( k + 1 \). In fact, for every pair of consecutive numbers between \( \text{rad}_3(G) \) and \( \text{diam}_3(G) \), there must be at least two vertices with closed 3-stop eccentricity equal to one of those numbers.

**Proposition 4.** Let \( G \) be a connected graph and let \( k \) be an integer such that \( \text{rad}_3(G) < k < \text{diam}_3(G) - 1 \). Then there are at least two vertices in \( G \) with closed 3-stop eccentricity either \( k \) or \( k + 1 \).

**Proof.** Suppose to the contrary that \( v \in V(G) \) is the only vertex with closed 3-stop eccentricity either \( k \) or \( k + 1 \). Let \( A = \{ u \in V(G) | e_3(u) < k \} \) and \( B = \{ u \in V(G) | e_3(u) > k + 1 \} \). Notice that both \( A \) and \( B \) are non-empty and \( A \cup B \cup \{ v \} = V(G) \). Consider any \( x \in A \) and \( y \in B \). Since \( e_3(x) \leq k - 1 \) and \( e_3(y) \geq k + 2 \), it follows from Proposition 3 that any \( x-y \) path must contain a vertex with eccentricity either \( k \) or \( k + 1 \). However, \( v \) is the only such vertex. Thus, \( v \) is a cut-vertex and \( A \) and \( B \) are not connected in \( G - v \). Let \( w \) and \( y \) be vertices such that \( e_3(v) = d_3(v, w, y) \). Since \( e_3(w) \geq e_3(v) \) and \( e_3(y) \geq e_3(v) \), both \( w \) and \( y \) must be in \( B \). Now, let \( u \in A \). Every path from \( u \) to \( w \) or \( y \) must go through \( v \), so \( e_3(u) \geq d_3(u, w, y) = 2d(u, v) + d_3(v, w, y) = 2d(u, v) + e_3(v) \). But this contradicts the fact that \( e_3(u) < e_3(v) \).

In every example that we have found, there are at least three vertices with closed 3-stop eccentricity either \( k \) or \( k + 1 \) for \( \text{rad}_3(G) < k < \text{diam}_3(G) - 1 \).

**Conjecture 5.** Let \( G \) be a connected graph and let \( k \) be an integer such that

\[ \text{rad}_3(G) < k < \text{diam}_3(G) - 1. \]

Then there are at least three vertices in \( G \) with closed 3-stop eccentricity either \( k \) or \( k + 1 \).

### 3. Closed \( k \)-stop Radius and Closed \( k \)-stop Diameter

In this section we study closed \( k \)-stop eccentricity. Proposition 1 can be generalized for \( k \geq 4 \).

**Proposition 6.** Let \( G \) be a connected graph of order at least \( k \), \( k \in \mathbb{N} \). Then \( G \) has at least \( k \) vertices that are closed \( k \)-stop peripheral.
Proof. Let $x_1 \in V(P_k(G))$. Then there exist vertices $x_2, x_3, \ldots, x_k \in V(G)$ such that $e_k(x_1) = d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_k, x_1) = e_k(x_2) = e_k(x_3) = \cdots = e_k(x_k)$. Thus $x_1, x_2, \ldots, x_k \in V(P_k(G))$. ■

Also, Proposition 2 can be generalized for $k = 4$.

**Proposition 7.** For any connected graph $G$, we have
\[
rad_4(G) \leq diam_4(G) \leq \frac{4}{3}rad_4(G).
\]

**Proof.** Let $G$ be a connected graph. Suppose $u \in V(C_4(G))$ and $v \in V(P_4(G))$. Furthermore, suppose that $e_4(v)$ is attained by visiting $w$, $x$, and $y$, not necessarily in that order. We must have $w, x$, and $y \in V(P_4(G))$, and $e_4(v) = e_4(w) = e_4(x) = e_4(y) = d_4(\{v, w, x, y\})$.

Without loss of generality, we may assume that the minimum distance among $d(v, w)$, $d(v, x)$, $d(v, y)$, $d(w, x)$, $d(w, y)$, and $d(w, y)$ is $d(v, w)$. If we now distinguish $v$ and $w$ from $x$ and $y$, we may assume, without loss of generality, that the distance from $\{v, w\}$ to $\{x, y\}$, that is, the minimum distance among $d(v, x)$, $d(v, y)$, $d(w, x)$, and $d(w, y)$, is $d(v, y)$. Thus, $v$ is the vertex in common in these two distances. Now,

\[
\begin{align*}
4) \quad rad_4(G) &= e_4(u) \\
5) \quad &\geq d_4(u, w, x, y) \\
6) \quad &= \min(d(u, w) + d(w, x) + d(x, y) + d(y, u), d(u, x) + d(x, w) + d(w, y) + d(y, u)) \\
7) \quad &+ d(w, y) + d(y, u), d(u, w) + d(w, y) + d(y, x) + d(x, u)) \\
8) \quad &\geq d(w, y) + d(w, x) + d(x, y).
\end{align*}
\]

The last inequality follows by applying the triangle inequality to each of terms in the minimum. Thus, $4rad_4(G) \geq 4d(w, y) + 4d(w, x) + 4d(x, y)$.

On the other hand, $3diam_4(G) = 3e_4(v) = 3\min(d(v, w) + d(w, x) + d(x, y) + d(y, v), d(v, w) + d(w, y) + d(y, x) + d(x, v), d(v, x) + d(x, w) + d(w, y) + d(y, v)) \leq 3d(v, w) + 3d(w, x) + 3d(x, y) + 3d(y, v)$.

From our initial assumptions, $3d(v, w) \leq d(x, y) + 2d(w, y)$ and $3d(y, v) \leq d(w, x) + 2d(w, y)$. Thus, we have $3diam_4(G) \leq 3d(v, w) + 3d(w, x) + 3d(x, y) + 3d(y, v) \leq 4d(x, y) + 4d(w, x) + 4d(w, y) \leq 4rad_4(G)$. ■

**Conjecture 8.** For any integer $k \geq 2$ and any connected graph $G$, we have
\[
rad_k(G) \leq diam_k(G) \leq \frac{k}{k-1}rad_k(G).
\]
Notice that for \( k = 2 \), this conjecture reduces to the classical result for ordinary distance that \( \text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G) \). We have shown that the conjecture is true for \( k = 3 \) and \( k = 4 \). However, for higher values of \( k \), the proof would have to take into account the order of the eccentric vertices \( w, x, \) and \( y \) of the peripheral vertex \( v \) in the last step of equation 8. Suppose, for instance, that the vertices \( v_1, v_2, \ldots, v_k \) are arranged so that the length of a closed walk is minimized, that is, \( d(v_1, v_2) + d(v_2, v_3) + \cdots + d(v_{k-1}, v_k) + d(v_k, v_1) \) is as small as possible. If another vertex \( v \) is included, we may wonder if the minimum length closed walk for \( \{v_1, v_2, \ldots, v_k, v\} \) can always be achieved by inserting \( v \) in some location in the list \( v_1, v_2, \ldots, v_k \) or if the original vertices may also have to be rearranged. If \( k \leq 3 \), the minimum can always be achieved by simply inserting \( v \). However, consider the example in Figure 2 for \( k = 4 \). A minimum closed walk containing \( \{v_1, v_2, v_3, v_4\} \) has length 8 and visits these four vertices in order \( v_1, v_2, v_3, v_4, v_1 \) or in reverse order \( v_1, v_4, v_3, v_2, v_1 \). However, a minimum closed walk containing \( \{v_1, v_2, v_3, v_4, v\} \) has length 11 and visits the vertices in one of the following orders: \( v_1, v_3, v_2, v, v_4, v_1, v_1, v_3, v_4, v, v_2, v_1, v_1, v_2, v, v_4, v_3, v_1, v_1, v_4, v, v_2, v_3, v_1 \), or \( v_1, v_4, v, v_2, v_3, v_1 \).

4. Closed \( k \)-stop Distance in Trees

In this section we study the closed \( k \)-stop distance in trees. We start with some observations and illustrations concerning closed \( k \)-stop distance.

Figure 2. The shortest closed walk including \( v_1, v_2, v_3, v_4, v \) cannot be formed by inserting \( v \) into the shortest closed walk including \( v_1, v_2, v_3, v_4 \).
Proposition 9. If $G$ is a graph, and $T$ is a spanning tree of $G$, then for any vertices $x_1, x_2, \ldots, x_k \in V(G)$, $d_k(\{x_1, x_2, \ldots, x_k\})$ in $G$ is at most $d_k(\{x_1, x_2, \ldots, x_k\})$ in $T$.

As a result of Proposition 9 we have that $\operatorname{rad}_k(G) \leq \operatorname{rad}_k(T)$ and $\operatorname{diam}_k(G) \leq \operatorname{diam}_k(T)$. For this reason we study trees next.

In a tree $T$, the upper inequalities (1), (2), and (3) actually become equalities, so $e_k(v) = 2s_{k}(v)$ for all $v \in V(T)$, $\operatorname{rad}_k(T) = 2s\operatorname{rad}_k(T)$ and $\operatorname{diam}_k(T) = 2s\operatorname{diam}_k(T)$, where the $s\operatorname{rad}_k(T)$ and $s\operatorname{diam}_k(T)$ are the Steiner radius and diameter, respectively. A closed walk containing a set of vertices traces every edge of a Steiner tree for those vertices twice. As a consequence, we have the following observation, also noted independently in [3].

Observation 10. Let $T$ be a tree and let $k \geq 2$ be an integer. Then $e_k(v)$ is even, for all $v \in V(T)$.

For any positive integer $k \geq 2$ and connected graph $G$, the Steiner $k$-center of $G$, $sC_k(G)$, is the subgraph induced by the vertices $v$ such that $s_{k}(v) = s\operatorname{rad}_k(G)$. Notice that since the Steiner distance of two vertices is simply the usual distance, $sC_2(G) = C(G)$. Oellermann and Tian found the following relationship between Steiner $k$-centers of trees.

Theorem 11 [7]. Let $k \geq 3$ be an integer and $T$ a tree of order greater than $k$. Then $sC_{k-1}(T) \subseteq sC_k(T)$.

Similarly, the Steiner $k$-periphery of a graph $G$, $sP_k(G)$, is the subgraph induced by the vertices $v$ such that $s_{k}(v) = s\operatorname{diam}_k(G)$. When $k = 2$, notice that $sP_2(G)$ is the usual periphery $P(G)$. Henning, Oellermann, and Swart found a relationship similar to the one above for the Steiner $k$-peripheries of trees.

Theorem 12 [4]. Let $k \geq 3$ be an integer and $T$ a tree of order greater than $k$. Then $sP_{k-1}(T) \subseteq sP_k(T)$.

Since $\operatorname{rad}_k(T) = 2s\operatorname{rad}_k(T)$ and $\operatorname{diam}_k(T) = 2s\operatorname{diam}_k(T)$ for a tree $T$, we have $sC_k(T) = C_k(T)$ and $sP_k(T) = P_k(T)$. Thus, the results above produce the following corollary.

Corollary 13. Let $T$ be a tree of order $n$. Then $C(T) \subseteq C_3(T)$ and $P(T) \subseteq P_3(T)$. Furthermore, for any $k$ with $3 \leq k \leq n$, we have $C_k(T) \subseteq C_{k+1}(T)$ and $P_k(T) \subseteq P_{k+1}(T)$. 

We next present the only tree that is closed 3-stop self-centered.

**Proposition 14.** Let $T$ be a tree. $T$ is closed 3-stop self-centered if and only if $T \cong P_n$ ($n \geq 3$).

**Proof.** If $T \cong P_n$ ($n \geq 3$), the result follows. For the converse, let $T \not\cong P_n$ be a tree of order $n \geq 3$. Then $T$ has three end-vertices $x, y, z \in V(P_3(T))$ such that $diam_3(T) = d_3(x, y, z)$. Let $x = x_0, x_1, \ldots, x_p = y$ be the geodesic from $x$ to $y$ in $T$. Then $e_3(x) = d(x, y) + d(y, z) + d(z, x)$, and $e_3(x_1) = d(x_1, y) + d(y, z) + d(z, x_1) < e_3(x)$, and so $T$ is not closed 3-stop self-centered.

As a quick corollary of the above proof we have the following result.

**Corollary 15.** Let $T$ be a tree. $T$ is closed 3-stop self-peripheral if and only if $T \cong P_n$ ($n \geq 3$).

As we have seen already, the path $P_n$ has many special properties. The next result shows that $P_n$ is the only tree that has the same closed $k$-stop eccentricity for each vertex and for any $k$ with $1 \leq k \leq n - 1$. This result follows as the path has only two end vertices and a unique path between them.

**Proposition 16.** Let $T$ be a tree of order $n$. Then $e_k(v) = 2n$, for all $v \in V(T)$, and for all $k \in \{1, 2, \ldots, n - 1\}$ if and only if $T = P_n$, the path of order $n$.

The following is a consequence of the Steiner distance in trees.

**Proposition 17.** Let $T$ be a tree and $k$ an integer with $1 \leq k \leq n$. Then $T$ has at most $k - 1$ end vertices if and only if $T$ is closed $k$-stop self-centered.

**Proof.** Let $T$ be a tree with at most $k - 1$ end vertices, say they form the set $S = \{x_1, x_2, \ldots, x_{k-1}\}$, $k \geq 3$. Then for all $v \in V(G)$,

$$e_k(v) = \min_{\theta \in \mathcal{P}(S)} \left( d(\theta(v), \theta(x_1)) + d(\theta(x_1), \theta(x_2)) + d(\theta(x_2), \theta(x_3)) + \cdots + d(\theta(x_{k-1}), \theta(v)) \right),$$

where $\mathcal{P}(S)$ is the set of all permutations from $\mathcal{P}(S)$ onto $\mathcal{P}(S)$. Since $T$ is a tree with $k - 1$ end vertices, it follows that $e_k(v) = 2m$, $\forall v \in V(G)$. 
For the converse, assume that $T$ is closed $k$-stop self-centered, and assume to the contrary, that $T$ has at least $k$ end vertices, say $y_1, y_2, \ldots, y_t$, for $t \geq k \geq 3$. Let $z_1$ be the support vertex of $y_1$ and let $S = \{y_2, y_3, \ldots, y_{k-1}\}$, $k \geq 3$. Then

$$e_k(z_1) = \min_{\theta \in \mathcal{P}(S)} \left( d(\theta(z_1), \theta(y_2)) + d(\theta(y_2), \theta(y_3)) + d(\theta(y_3), \theta(y_4)) + \cdots + d(\theta(y_{k-1}), \theta(z_1)) \right),$$

where $\mathcal{P}(S)$ is the set of all permutations from $\mathcal{P}(S)$ onto $\mathcal{P}(S)$. However, $e_k(y_1) = 2 + e_k(z_1)$, which is a contradiction to $T$ being closed $k$-stop self-centered.

As a quick corollary of the above proof we have the following result.

**Corollary 18.** Let $T$ be a tree and $k$ an integer with $1 \leq k \leq n$. Then $T$ has at most $k-1$ end vertices if and only if $T$ is closed $k$-stop self-peripheral.

## 5. Further Research

As seen in Section 3, Proposition 2 can be generalized for $k = 4$. The following conjecture was posed in Section 3.

**Conjecture** (Section 3): For any integer $k \geq 2$ and any connected graph $G$, we have

$$\text{rad}_k(G) \leq \text{diam}_k(G) \leq \frac{k}{k-1} \text{rad}_k(G).$$

Chartrand, Oellermann, Tian, and Zou showed a similar result for Steiner radius and diameter for trees.

**Theorem 19** [2]. If $k \geq 2$ is an integer and $T$ is a tree of order at least $k$, then

$$\text{srad}_k(T) \leq \text{sdiam}_k(T) \leq \frac{k}{k-1} \text{srad}_k(T).$$

Since $e_k(v) = 2se_k(v)$ for any vertex $v$ in a tree, we have the corollary.

**Corollary 20.** If $k \geq 2$ is an integer and $T$ is a tree of order at least $k$, then

$$\text{rad}_k(T) \leq \text{diam}_k(T) \leq \frac{k}{k-1} \text{rad}_k(T).$$
We have also been able to verify this conjecture for \( k = 3 \) and \( k = 4 \) for arbitrary connected graphs. As an interesting side note, Chartrand, Oellermann, Tian and Zou conjectured that \( srad_k(G) \leq sdiam_k(G) \leq \frac{k}{k-1}srad(G) \) for any connected graph \( G \) [2]. This conjecture was disproven in [5], but our conjecture for closed \( k \)-stop distance holds for the class of graphs used as a counterexample to the Steiner conjecture.

We propose the extension of the study of centrality and eccentricity for closed \( k \)-stop distance in graphs for \( k \geq 4 \).

References


