On Dominator Colorings in Graphs

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ABSTRACT

Given a graph G, the dominator coloring problem seeks a proper coloring of G with the additional property that every vertex in the graph dominates an entire color class. We seek to minimize the number of color classes. We study this problem on several classes of graphs, as well as finding general bounds and characterizations. We also show the relation between dominator chromatic number, chromatic number, and domination number.

Key Words: coloring, domination, dominator coloring.

AMS Subject Classification: 05C15, 05C69.

1 Introduction and motivation

A *dominating set* S is a subset of the vertices in a graph such that every vertex in the graph either belongs to S or has a neighbor in S. The domination number is the order of a minimum dominating set. The topic has long been of interest to researchers [1, 2]. The associated decision problem, DOMINATING SET, has been studied in the computational complexity literature [3], and so has the associated optimization problem, which is to find a dominating set of minimum cardinality. Numerous variants of this problem have been studied [1, 2, 4, 5], and here we study one more that was introduced in [6]. Given a graph G and an integer k, finding a DOMINATING SET of order k is NP-complete on arbitrary graphs [3].

A *proper coloring* of a graph $G = (V(G), E(G))$ is an assignment of *colors* to the vertices of the graph, such that any two adjacent vertices have different colors. The chromatic number is the minimum number of colors needed in a proper coloring of a graph. Graph coloring is used as a model for a vast number of practical problems involving allocation of scarce resources (e.g., scheduling problems), and has played a key role in the development of graph theory and, more generally, discrete mathematics and combinatorial optimization. Graph k-colorability is NP-complete in the general case, although the problem is solvable in polynomial time for many classes [3].

A *dominator coloring* of a graph G is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The *dominator chromatic number* $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of a graph

G. Given a graph G and an integer k, finding a DOMINATING COLORING SET is NP-complete on arbitrary graphs [6].

We start with notation and more formal problem definitions. Let $G = (V(G), E(G))$ be a graph with $n = |V(G)|$ and $m = |E(G)|$. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \mid uv \in E(G)\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. Similarly, for any set $S \subseteq V(G)$, $N(S) = \cup_{v\in S} N(v) - S$ and $N[S] = N(S) \cup S$. A set S is a *dominating set* if $N[S] = V(G)$. The minimum cardinality of a dominating set of G is denoted by $\gamma(G)$. The *distance*, $d(u, v)$, between two vertices u and v in G is the smallest number of edges on a path between u and v in G. The *eccentricity*, $e(v)$, of a vertex v is the largest distance from v to any vertex of G. The *radius*, $rad(G)$, of G is the smallest eccentricity in G. The *diameter*, $diam(G)$, of G is the largest eccentricity in G.

A *graph coloring* is a mapping $f : V(G) \to C$, where C is a set of colors (frequently $C \subseteq \mathbf{Z}^+$). A coloring f is *proper* if, for all $x, y \in V(G)$, $x \in N(y)$ implies $f(x) \neq f(y)$. A *k-coloring* of G is a coloring that uses at most k colors. The *chromatic number* of G is $\chi(G) = \min\{k|G$ has a proper k-coloring$\}$. A coloring of G can also be thought of as a partition of $V(G)$ into color classes $V_1, V_2, ..., V_q$, and a proper coloring of G is then a coloring in which each V_i, $1 \leq i \leq q$ is an independent set of G, *i.e.,* for each i, the subgraph of G induced by V_i contains no edges.

Dominator colorings were introduced in [6] and they were motivated by [7]. The aim in this paper is to study bounds and realization of the dominator chromatic number in terms of chromatic number and domination number. In Section 2 we present the dominator chromatic number for classes of graphs, we then find characterization and realization results in Section 3, and we pose open questions in Section 4. Since disconnected graphs have been studied in [6], we are only concerned with connected graphs.

2 Dominator Coloring for classes of graphs

We begin our study with an example of finding the dominator chromatic number for the graph in Figure 1 in order to show an approach in finding the dominator chromatic number of a graph.

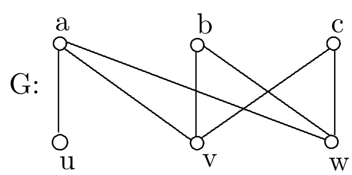

Figure 1: An example

Example 2.1 *The bipartite graph G in Figure 1 has $\chi_d(G) = 3$.*

Proof. Since G is a bipartite graph, at least two colors are needed to obtain a proper coloring of the graph, namely vertices a, b, and c should receive one color, and vertices u, v, and w should receive a different one. Note that this coloring is unique up to isomorphism. However then, the vertex u does not dominate a color class since it is not the only one of its color class (so it cannot dominate its own class), and also it is not adjacent to all vertices in the other color class (thus not dominating the other color class either). Therefore, at least 3 colors are needed for the graph, and so $\chi_d(G) \geq 3$.

Now, define a coloring where the vertices a, b, and c receive color 1, vertex u receives color 2, and vertices v and w receive color 3. Since each of the vertices a, b, and c dominate the color class 3, the vertex u dominates the color class 2, and each of the vertices v and w dominate the color class 1, it follows that 3 colors will suffice, and so $\chi_d(G) \leq 3$. Therefore $\chi_d(G) = 3$. □

Note that in the above graph every color class is dominated by some vertex. This does not always happen, as we will see below in paths or multistars for example.

We now proceed with classes of graphs. The following proposition appeared in [6].

Proposition A: The dominator chromatic number for some classes of graphs.

(1) The star $K_{1,n}$ has $\chi_d(K_{1,n}) = 2$.

(2) The path P_n of order $n \geq 3$ has

$$\chi_d(P_n) = \begin{cases} 1 + \lceil \frac{n}{3} \rceil & \text{if } n = 2, 3, 4, 5, 7 \\ 2 + \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

(3) The complete graph K_n has and $\chi_d(K_n) = n$.

We now investigate other classes of graphs that have not been studied. The star can be generalized to the *multistar graph* $K_m(a_1, a_2, \ldots, a_m)$, which is formed by joining $a_i \geq 1$ $(1 \leq i \leq m)$ end-vertices to each vertex x_i of a complete graph K_m, with $V(K_m) = \{x_1, x_2, \ldots, x_m\}$. A 2-star and a 3-star graphs are shown in Figure 2.

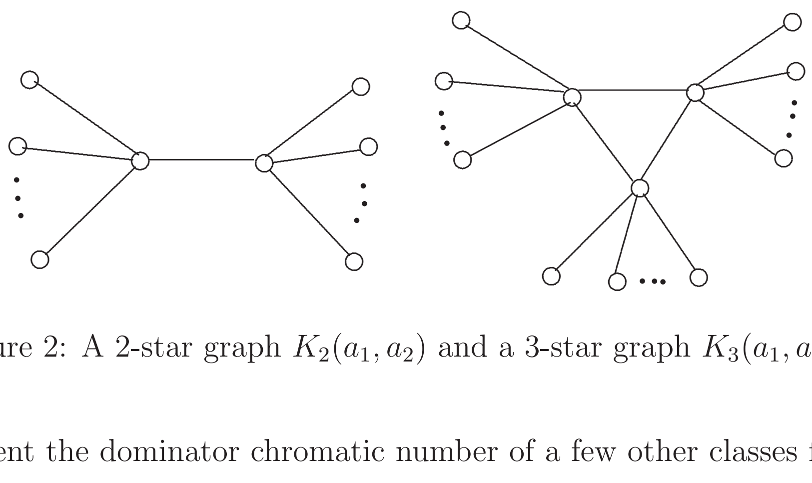

Figure 2: A 2-star graph $K_2(a_1, a_2)$ and a 3-star graph $K_3(a_1, a_2, a_3)$

We present the dominator chromatic number of a few other classes for completeness.

Proposition 2.2 (1) *The cycle C_n has*

$$\chi_d(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } n = 4 \\ \lceil \frac{n}{3} \rceil + 1 & \text{if } n = 5 \\ \lceil \frac{n}{3} \rceil + 2 & \text{otherwise.} \end{cases}$$

(2) *The multistar $K_n(a_1, a_2, \ldots, a_n)$ has $\chi_d(K_n(a_1, a_2, \ldots, a_n)) = n + 1$*

(3) *The wheel $W_{1,n}$ has*

$$\chi_d(W_{1,n}) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd.} \end{cases}$$

(4) *The complete k-partite graph $K_{a_1,a_2,\ldots,a_k}$ has $\chi_d(K_{a_1,a_2,\ldots,a_k}) = k$*

Proof.

(1) For $3 \leq n \leq 10$, the result follows by inspection. For $n \geq 11$, let $C_n : v_1, v_2, \ldots, v_n, v_1$ be the cycle on n vertices, and consider the subgraph $P_n : v_1, v_2, \ldots, v_n$. Then by Proposition 2 we have that $\chi_d(P_n) = 2 + \lceil \frac{n}{3} \rceil$. So there is a dominator coloring of the path $v_1, v_2, \ldots, v_n$ with $2 + \lceil \frac{n}{3} \rceil$ colors, such that vertices v_1 and v_n are dominated by themselves or by v_2 and v_{n-1}, respectively. We obtain the cycle C_n from the path P_n, by adding the edge $v_1 v_n$, and so each vertex of C_n will be dominated as it was in the path P_n. However, if the colors assigned to v_1 and v_n are the same, say color i $(1 \leq i \leq \chi_d(P_n))$, then color i is one of the repeated colors in a dominator coloring of P_n. If P_n has only one repeated color, then $\chi_d(P_n) \geq \lceil \frac{n}{2} \rceil + 1$. Since $\chi_d(P_n) = 2 + \lceil \frac{n}{3} \rceil < \lceil \frac{n}{2} \rceil + 1$ $(n \geq 11)$, there is at least one other color that repeats, say j, $1 \leq i \neq j \leq \chi_d(P_n)$. Then assign color j to v_1. Now, the color of v_2 is a non-repeated color since v_1 must dominate some color class in P_n, and so the colors of v_1 and v_2 are different. The rest of the vertices have a proper coloring since the assignment of colors was inherited from P_n. The new coloring is a dominator coloring since it was a dominator coloring of P_n and the possible changes of the addition of the extra edge were studied.

On the other hand, if the colors of v_1 and v_n were nonrepeated colors in a coloring of P_n, then the dominator coloring would not be optimal. And so, at most one of the v_1 and v_n will have a nonrepeated color, say v_1. Then necessarily v_2 has a repeated color in an optimal dominator coloring of P_n. Then adding the edge $v_1 v_n$ will preserve the dominator coloring of P_n in C_n. To see that it cannot be improved, note that if the color of v_1 was to be changed to a repeated color, then v_1 will be adjacent to two repeated colors in C_n, which will be a contradiction as v_1 will not dominate a color class.

Moreover, a coloring of C_n cannot use fewer colors than a coloring of P_n for $n \geq 11$, since the addition of a new edge may only affect the color of the two vertices incident with the edge. Thus $\chi_d(C_n) = \lceil \frac{n}{3} \rceil + 2$ for $n \geq 11$.

(2) Define a coloring of the multistar by assigning colors 1 through n to the n vertices of the complete subgraph, and color $n+1$ to the leaves. So $\chi_d(K_n(a_1, a_2, \ldots, a_n)) \leq n+1$. On the other hand, let f be any coloring of the multistar. Since K_n is a subgraph of $K_n(a_1, a_2, \ldots, a_n)$ it follows that at least n different colors are needed to color the complete subgraph. Assume, to the contrary, that some color i $(1 \leq i \leq n)$ gets repeated on a leaf. Let x be a leaf of the multistar whose color is i. Then there is a vertex of the complete subgraph K_n, say y $(y \neq x)$, such that $f(y) = i$. However then, there is at least one leaf that is adjacent to y and it will not dominate the color class i. Thus the colors of the subgraph K_n cannot be repeated, and so $\chi_d(K_n(a_1, a_2, \ldots, a_n)) \geq n+1$.

(3) Since $\chi_d(W_{1,n}) \geq \chi(W_{1,n})$, the result follows.

(4) Let $K_{a_1,a_2,\ldots,a_k}$ be the complete k-partite graph, and let V_i $(1 \leq i \leq k)$ be the k-partite sets. Then there is a complete subgraph K_k that contains one vertex of each partite set. So $\chi_d(K_{a_1,a_2,\ldots,a_k}) \geq \chi(K_{a_1,a_2,\ldots,a_k}) \geq k$. Also, the coloring that assigns color i to each partite set V_i $(1 \leq i \leq k)$ is a dominator coloring. The result follows.

□

One note on the topic is that for a given graph G, and a subgraph H, the dominator chromatic number of the subgraph can be (1) smaller or (2) larger than the dominator chromatic number of the graph G. To see this, for case (1) consider the graph $G = K_n$ and $H = K_{a,b}$, where $\chi_d(G) = n \geq 2 = \chi_d(H)$, and for case (2) consider the graph $G = K_{n,n}$ and $H = P_{2n}$, where $\chi_d(G) = 2 \leq \frac{n}{3} + 2 = \chi_d(H)$. This is not very helpful when searching for algorithms or trying to use induction in order to find the dominator chromatic number of a graph.

3 Bounds and Realization results

Let G be a connected graph of order $n \geq 2$. Then at least two different colors are needed in a dominator coloring since there are at least two vertices adjacent to each other. Moreover, if each vertex receives its unique color, then we also have a dominator coloring. Thus

$$2 \leq \chi_d(G) \leq n, \tag{1}$$

and these bounds are sharp for $K_{a,b}$ and K_n $(a, b, n \geq 2)$. Also every pair (k, n) is realized as the chromatic domination number and order of some graph [6]. We now present characterizations for lower and upper bounds in (1).

Proposition 3.1 *Let G be a connected graph of order n. Then*

$$\chi_d(G) = 2 \text{ if and only if } G = K_{a,b} \text{ for } a, b \in \mathbf{N}$$

Proof. Let G be a graph such that $\chi_d(G) = 2$, with V_1 and V_2 being the two color classes. If $|V_1| = 1$ or $|V_2| = 1$, then $G = K_{1,n-1}$ since G is connected. Thus $|V_1| \geq 2$ and $|V_2| \geq 2$. Let $v \in V_1$. Since V_1 is independent and $|V_1| \geq 2$, it follows that v cannot dominate the color class V_1, and so v dominates color class V_2. Similarly for an arbitrary vertex of V_2. Thus each vertex of V_1 is adjacent to each vertex of V_2, and V_1 and V_2 are independent. So $G = K_{a,b}$, for $a, b \geq 1$. By Proposition A we obtain the result. □

Proposition 3.2 *Let G be a connected graph of order n. Then*

$$\chi_d(G) = n \text{ if and only if } G = K_n \text{ for } n \in \mathbf{N}.$$

Proof. Let G be a connected graph of order n with $\chi_d(G) = n$. Assume to the contrary, that $G \neq K_n$. Thus, there are at least two nonadjacent vertices, say x and y. Define a coloring of G such that x and y receive the same color, and each of the remaining vertices receive a unique color. This is a dominator coloring, and so $\chi_d(G) \leq n - 1$, contradiction. Thus $G = K_n$. By Proposition A we obtain the result. □

We next look for bounds related to other graph theoretical parameters. We have seen that the dominator chromatic number can vary from 2 to the order of the graph. In particular, the dominator chromatic number is at least as big as both the domination number and the chromatic number, and bounded above by their sum, as we show next.

Theorem 3.3 *Let G be a connected graph. Then*

$$\max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G).$$

The bounds are sharp.

Proof. Since a dominator coloring must be a proper coloring, we have that $\chi(G) \leq \chi_d(G)$. Also, let c be a minimum dominator coloring of G. For each color class of G, let x_i be a vertex in the class i, with $1 \leq i \leq \chi_d(G)$. We show that the set $S = \{x_i : 1 \leq i \leq \chi_d(G)\}$ is a dominating set. Let $v \in V(G)$. Then v dominates a color class i, for some i $(1 \leq i \leq \chi(G))$. Then v is dominated by the color class i, in particular by x_i.

For the upper bound, let c be a proper coloring of G with $\chi(G)$ colors. Now, assign colors $\chi(G)+1, \chi(G)+2, \ldots, \chi(G)+\gamma(G)$ to the vertices of a minimum dominating set of G leaving the rest of the vertices colored as before. This is a dominator coloring of G since it is still a proper coloring, and the dominating set provides the color class that every vertex dominates.

The lower bounds are sharp as it can be seen for the complete bipartite graphs, and the upper bound is sharp for P_n or C_n $(n \geq 8)$. □

We next present the characterization of the ordered triples of integers (a, b, c) $(c \geq a \geq 1, c \geq b \geq 2)$ that can be realized as domination number, chromatic number and dominator chromatic number of some connected graph. To do so, we first prove the following.

Lemma 3.4 *Let G be a connected graph. If $\gamma(G) = 1$, then $\chi(G) = \chi_d(G)$, and every pair $(1, a)$ with $a \geq 1$ is realizable as $\gamma(G) = 1$ and $\chi(G) = \chi_d(G) = a$.*

Proof. Let G be a connected graph of order n with $\gamma(G) = 1$. It then follows that the radius of G is 1, so the star $K_{1,n-1}$ is a spanning subgraph of G. Let v be a vertex of degree $n-1$ in G. Since v is adjacent to all the vertices of $G - v$, it follows that a minimum coloring of G uses $\chi(G - v) + 1$ colors. This coloring is also a minimum dominator coloring of G, where each vertex dominates the color class of v. To see that every pair $(1, a)$ is realizable, construct G from K_a $(2 \leq a \leq n)$ by attaching $n - a$ pendants to one vertex of K_a. Then $\chi_d(G) = \chi(G) = a$, and $\gamma(G) = 1$. □

We now show that all the values in between the lower and upper bound are attainable as a triple, except when $\gamma(G) = 1$ and $\chi_d(G) = \chi(G) + 1$ as shown above.

Theorem 3.5 *For each ordered triple of integers (a, b, c) $(c > a > 1, c > b \geq 2, c \leq a + b)$ there is a connected graph G with*

$$\gamma(G) = a, \chi(G) = b, \text{ and } \chi_d(G) = c,$$

except (1, b, b+1).

Proof. If $a = 1$, the result follows by Lemma 3.4. If $c = a+b$, then let $G = C_n$ or P_n for $n \geq 8$. Thus assume that $c > a \geq 2, c > b \geq 2$, and $c < a+b$. Let $k = c-b > 0$ be an integer. The graph G is obtained from $K_b : u_1, u_2, \ldots, u_b$ by (1) adding a pendant to each vertex $u_1, u_2, \ldots u_{a-k}$ $(0 < k \leq a)$, and (2) adding $k > 0$ copies of $P_2 : v_i, w_i$ together with k more edges $u_b v_i$ $(0 < i \leq k)$. See graph in Figure 3.

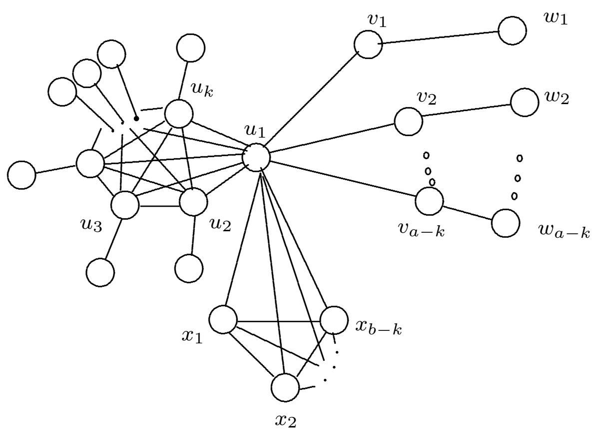

Figure 3: The graph G in Proposition 3.5

The set $\{u_1, u_2, \ldots, u_{a-k}, v_1, v_2, \ldots, v_k\}$ is a minimum dominating set since for each leaf either the leaf or the stem belongs to a dominating set. So $\gamma(G) = a$. Also, in a proper coloring of G, a new color is needed for each vertex in the set $\{u_1, u_2, \ldots, u_b\}$. And since b colors will suffice by repeating two colors on the vertices

v_j and w_j ($1 \leq j \leq k$), we have that $\chi(G) = b$. Finally, in a dominator coloring, a new color is needed for every vertex in the set $\{u_1, u_2, \ldots, u_b, v_1, v_2, \ldots, v_k\}$ since the subgraph K_b is a complete graph that requires b different colors, and for each pendant either the leaf or its stem needs a new color class for the leaf to dominate. Moreover, assigning a new color to each vertex of the complete subgraph K_b, a new color to each vertex v_j, and one repeated color to the vertices w_j ($1 \leq j \leq k$), we obtain $\chi_d(G) = b + k$ where $0 < k \leq a - 1$. □

4 Further Research

One interesting question is to characterize all the graph that have the property that $\chi_d(G) = \chi_d(\bar{G})$. For example, let P be the Petersen graph. Then $\chi_d(P) = 5$ and $\chi_d(\bar{P}) = 5$. In [6] we introduced a new way of finding the dominator chromatic number of a graph whose complement is triangle free, and so we got the above result, which can be used for graphs other than Petersen. For example the graphs C_n and $\bar{C}_n$ ($n \geq 5$) might have this property.

Open Question 1: For what graphs does $\chi_d(G) = \chi_d(\bar{G})$? Or in particular, for what triangle free graphs does $\chi_d(G) = \chi_d(\bar{G})$?

In this paper we found that there is a very nice relation between the chromatic domination, the domination and chromatic number, namely

$$max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \gamma(G) + \chi(G).$$

Is there a a characterization for the bounds of the above equation?

Open Question 2: For what graphs does $\chi_d(G) = \chi(G)$?

Open Question 3: For what graphs does $\chi_d(G) = \gamma(G)$?

Open Question 4: For what graphs does $\chi_d(G) = \gamma(G) + \chi(G)$?

Using the results of this paper, it can be easily seen that $\chi_d(G) = \gamma(G) = \chi(G) = 2$ if and only if $G = K_{a,b}$ for $a, b \geq 2$. This brings up the following question.

Open Question 5: For what graphs does $\chi_d(G) = \gamma(G) = \chi(G)$?

Also, we proved in [6] that dominator chromatic number is NP-complete on arbitrary graphs. That raises the following question.

Open Question 5: What is the complexity of dominator colorings on particular classes of graphs?

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