Which Graphs Are Divisor Graphs?

Gary Chartrand, Raluca Muntean,
Varaporn Saenpholphat, and Ping Zhang

Department of Mathematics and Statistics
Western Michigan University
Kalamazoo, MI 49008 USA

ABSTRACT

For a finite nonempty set S of positive integers, the divisor graph $G(S)$ of S has vertex set S and two vertices i and j of $G(S)$ are adjacent if i divides j or j divides i, while the divisor digraph $D(S)$ of S has vertex set S and (i, j) is an arc of $D(S)$ if $i \mid j$. A graph G is a divisor graph if there exists a set S of positive integers such that G is isomorphic to $G(S)$. It is shown that a triangle-free graph is a divisor graph if and only if it is bipartite. Also $G \times K_2$ is a divisor graph if and only if G is bipartite. A vertex v in an oriented graph D is a transmitter if $\mathrm{id}\, v = 0$, a receiver if $\mathrm{od}\, v = 0$, and a transitive vertex if $\mathrm{id}\, v, \mathrm{od}\, v > 0$ and for every $u \in N^-(v)$ and $w \in N^+(v)$, $(u, w) \in E(D)$. It is shown that a graph G is a divisor graph if and only if there exists an orientation D of G such that every vertex of D is a transmitter, a receiver, or a transitive vertex.

Keywords: divisor graphs, divisor digraph.

AMS Subject Classification: 05C12, 05C20, 05C78.

1 Introduction

In 1983 Erdös, Freud, and Hegyvári [3] studied, for positive integers n, those sequences $a_1, a_2, \cdots, a_k$ of greatest length, whose terms are distinct elements of the set $\{1, 2, \cdots, n\}$ and such that either $a_i \mid a_{i+1}$ or $a_{i+1} \mid a_i$ for each i with $1 \leq i \leq k-1$. There is another way to formulate this problem. For a positive integer n, the *divisor graph* G_n is that graph whose vertex set is $\{1, 2, \cdots, n\}$ such that vertex i is adjacent to vertex j if and only if $i \mid j$ or $j \mid i$. Thus in G_n a vertex i is adjacent to a vertex j if and only if $lcm(i, j) = \max(i, j)$ (or $gcd(i, j) = \min(i, j)$). Let $f(n)$ be the length of a longest path in G_n. What can be said about $f(n)$? Erdös, Freud, and Hegyvári [3] proved that for sufficiently large positive integers n,

$$f(n) \leq (1 - \log 2)n.$$

Also in 1983, Pollington [5] proved that for every $\epsilon > 0$, there exists a positive integer N such that for every $n > N$,

$$f(n) \geq n \cdot e^{-(2+\epsilon)\sqrt{\log n \cdot \log(\log n)}}.$$

In 1983 as well, Pomerance [6] studied the function f. Moreover, for a positive integer n, Pomerance considered the graph H_n with vertex set $\{1, 2, \cdots, n\}$ where vertex i is adjacent to vertex j if and only if $lcm(i,j) \leq n$. Let $g(n)$ denote the length of a longest path in H_n. Pomerance asked if there is some positive integer n for which $g(n) > f(n)$.

Pomerance [6] also extended the definition of divisor graph to any nonempty set S of positive integers, whose divisor graph $G(S)$ has vertex set S and where two vertices i and j are adjacent in $G(S)$ if and only if $i \mid j$ or $j \mid i$ (that is, $gcd(i,j) = \min(i,j)$). Certainly, $1 \leq gcd(i,j) = \min(i,j)$ for every two distinct positive integers i and j. At the other extreme of the Pomerance definition is that of considering a graph in which there is an edge joining i and j if $gcd(i,j) = 1$.

Let S be a finite nonempty set of positive integers. The *relatively prime graph* $RP(S)$ *of* S has S as its vertex set and two vertices i and j are adjacent if i and j are relatively prime. For graphs F and H, we write $F = H$ if F and H are isomorphic. A graph G is *a relatively prime graph* if $G = RP(S)$ for some finite nonempty set S of positive integers. Hence if G is a relatively prime graph, then there exists a function $f : V(G) \to \mathbf{N}$, called a *relatively prime labeling* of G, such that $G = RP(f(V(G)))$. For $S = \{2, 3, 9, 35, 55, 77\}$, the graph $G = RP(S)$ is shown in Figure 1. Thus G is a relatively prime graph.

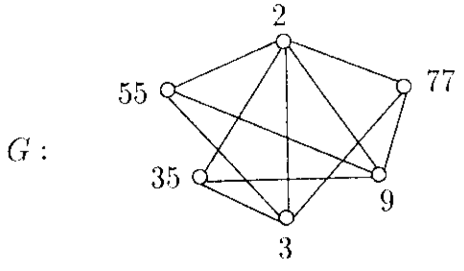


Figure 1: A relatively prime graph G

As we now see, the condition that a graph is a relatively prime graph is not a very demanding one. We write $\overline{G}$ for the complement of G.

Theorem 1.1 *Every graph is a relatively prime graph.*

Proof. Let G be a graph. Label any isolated vertices and the edges of $\overline{G}$ with distinct primes. If v is a vertex of $\overline{G}$ that belongs to a component of $\overline{G}$ of order 3 or more, then label v with the product of the labels of the edges in $\overline{G}$ incident with v. If u and v belong to the same component of order 2 in $\overline{G}$, then label one of u and v with the label p of $e = uv$ and the other with p^2. We claim that this labeling f is a relatively prime labeling of $V(G)$. Let $x, y \in V(G)$. If $xy \notin E(G)$, then $xy \in E(\overline{G})$ and the label assigned to xy is a common factor of $f(x)$ and $f(y)$. Thus $f(x)$ and $f(y)$ are not relatively prime. On the other hand, if $xy \in E(G)$, then $xy \notin E(\overline{G})$. Since no edge is incident with both x and y, it follows that $f(x)$ and $f(y)$ are relatively prime. ■

For $S = \{1, 2, \cdots, n\}$, it was conjectured by R. Entringer that every tree of order n is a subgraph of $PR(S)$. Fu and Huang [4] verified this conjecture for $n \leq 15$.

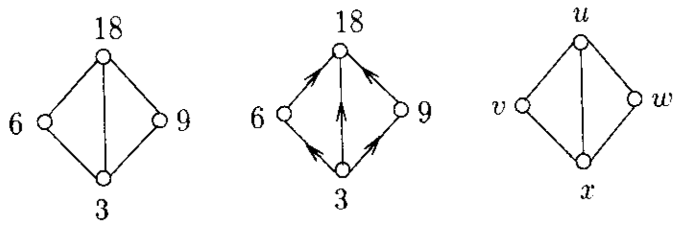
In 2000, Singh and Santhosh [7] defined the concept of divisor graph for finite nonempty sets of integers (rather than positive integers). They defined a graph G to be *a divisor graph* if G is isomorphic to to $G(S)$ for some finite nonempty set S of integers. They showed that every odd cycle of length 5 or more is not a divisor graph while all even cycles and caterpillars (trees, the removal of whose end-vertices produces a path) are divisor graphs.

In this paper, our goal is to study divisor graphs but in terms of nonempty sets of positive integers, as defined in Erdös, Freud, and Hegyvári [3], Pollington [5], and Pomerance [6]. We begin by reviewing the definitions and describing the terminology we will use.

Let S be a finite nonempty set of positive integers. The *divisor graph* $G(S)$ *of* S has S as its vertex set and two vertices i and j are adjacent if either $i \mid j$ or $j \mid i$. The *divisor digraph* $D(S)$ *of* S has vertex set S and (i, j) is an arc of $D(S)$ if $i \mid j$. Thus $G(S)$ is the underlying graph of $D(S)$. For $S = \{3, 6, 9, 18\}$, the divisor graph $G(S)$ and divisor digraph $D(S)$ are shown in Figure 1.

A graph G is *a divisor graph* if $G = G(S)$ for some finite nonempty set S of positive integers. Hence if G is a divisor graph, then there exists a function $f : V(G) \to \mathbf{N}$, called a *divisor labeling* of G, such that $G = G(f(V(G)))$. Consequently, the graph $G = K_4 - e$ of Figure 1 is a divisor graph, and the function $f : V(G) \to \mathbf{N}$ defined by

$$f(u) = 18,\ f(v) = 6,\ f(w) = 9, \text{ and } f(x) = 3$$



$G(S)$ $D(S)$ G

Figure 2: Divisor graphs and digraphs

is a divisor labeling.

If f is a divisor labeling of a divisor graph G and $k \in \mathbf{N}$, then the function $f' : V(G) \to \mathbf{N}$ defined by $f'(v) = kf(v)$ is also a divisor labeling of G. In particular, every divisor graph G has a divisor labeling all of whose labels are arbitrary large. Therefore, every divisor graph G has a divisor labeling, none of whose labels is 1. Throughout this paper, we will always assume that a given divisor labeling labels no vertex 1.

2 Which Graphs Are Divisor Graphs?

In this section we begin our investigation of graphs that are divisor graphs by making the following observation, which also appeared in Singh and Santhosh [7]. Since our proof is different and shorter, we include it.

Proposition 2.1 *No divisor graph contains an induced odd cycle of length 5 or more.*

Proof. Assume, to the contrary, that there is a divisor graph G containing an induced cycle C_n for some odd $n \geq 5$. Let D be the orientation induced by some divisor labeling f of G. Necessarily, there is a vertex y on the orientation C' of C such that $\text{od}\, y = \text{id}\, y = 1$. Thus there is a directed sub-path x, y, z on C', which implies that $f(x) \mid f(y)$ and $f(y) \mid f(z)$. However, then, $f(x) \mid f(z)$, which implies that xz is an edge of G, an impossibility. ■

The argument given in Proposition 2.1 illustrates the following observations, which we state without proof.

Proposition 2.2 *Every induced subgraph of a divisor graph is a divisor graph.*

Proposition 2.3 *Let G be a divisor graph. Then the orientation of G induced by any divisor labeling of G is transitive.*

Proposition 2.4 *Let G be a divisor graph and f a divisor labeling of G. Then the orientation of G induced by f is acyclic.*

Although divisor graphs cannot contain induced odd cycles of length 5 or more, there are divisor graphs that contains triangles, however. A simple class of divisor graphs are the complete graphs.

Proposition 2.5 *Every complete graph is a divisor graph.*

Proof. Let $n \geq 2$ and let $S = \{2, 2^2, \cdots, 2^n\}$. Then $G(S) = K_n$. ■

Also, consider the graphs G_1, G_2, and G_3 in Figure 2. Divisor labelings are given for G_1 and G_2. Thus G_1 and G_2 are divisor graphs. The graph G_3 is not a divisor graph, however. Assume, to the contrary, that G_3 is a divisor graph. Then there exists a divisor labeling f of G_3. Since f is transitive, we may assume that $f(u) \mid f(v)$, $f(v) \mid f(w)$, and $f(u) \mid f(w)$. Since vy is an edge of G_3, either $f(v) \mid f(y)$ or $f(y) \mid f(v)$. If $f(v) \mid f(y)$, then $f(u) \mid f(y)$, implying that $uy \in E(G_3)$. If $f(y) \mid f(v)$, then $f(y) \mid f(w)$, implying that $yw \in E(G_3)$. This is impossible, though, since G_3 contains neither edge.

In addition to showing that the complete graphs are divisor graphs, Singh and Santhosh [7] also showed that all paths, stars, double stars, and caterpillars are divisor graphs. In fact, all trees are divisor graphs, as we next show.

Proposition 2.6 *Every tree is a divisor graph.*

Proof. We proceed by induction on the order n of a tree T. The result is certainly true if $n = 1$. Assume that all trees of order k are divisor graphs, and let T be a tree of order $k + 1$. Let v be an end-vertex of T and let u be adjacent to v. Then $T' = T - v$ is a tree of order k and is therefore a divisor graph. Hence there is a divisor labeling f of T'. Let D' be the associated orientation of T'. No vertex of D' can have both positive outdegree and indegree, for otherwise, T' has a cycle. There are two cases.

Case 1. $\text{id}\, u > 0$ in D'. Let p be a prime not occurring as a factor in any label of f and define $f'(x) = pf(x)$ for all x in T'. Let $f'(v) = f(u)$.

Case 2. $\text{od}\, u > 0$ in D'. Let p be a prime not occurring as a factor in any label of f and define $f'(x) = f(x)$ for all x in T' and $f'(v) = pf(u)$. ■

It was shown in [7] that all even cycles are also divisor graphs. This result can be extended as well.

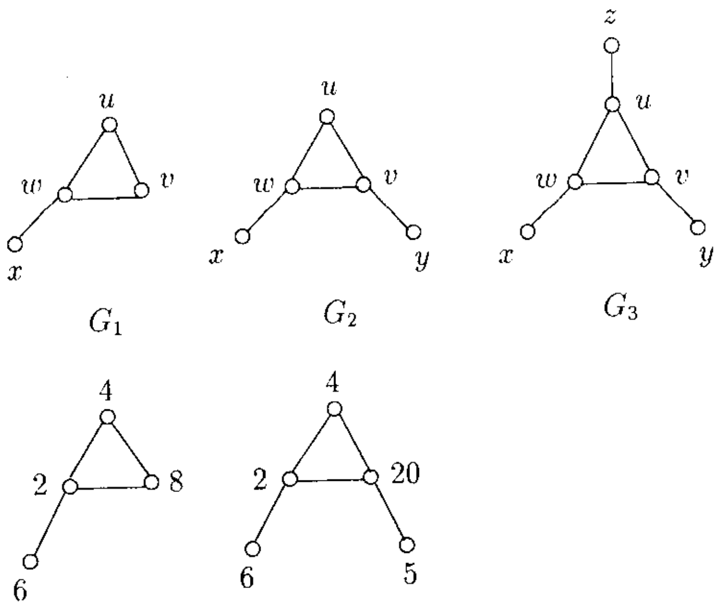


Figure 3: Divisor graphs G_1 and G_2 and a graph G_3 that is not a divisor graph

Theorem 2.7 *Every bipartite graph is a divisor graph.*

Proof. We proceed by induction on the order n of a bipartite graph G. The result is true for $n = 2, 3$. Assume that all bipartite graphs of order $n = k$ are divisor graphs and let G be a bipartite graph of order $n = k + 1$. Let U and W be the partite sets and let u be a vertex in U for which $|U| \geq 2$. Then $G' = G - u$ is a bipartite graph of order k. By induction, G' is a divisor graph. Let f be a divisor labeling of G' with D' as the associated orientation. Necessarily all arcs are directed toward W or away from W. There are two cases.

Case 1. All arcs of D' are directed toward W. Let p be a prime not occurring as a factor in any label of f and define $f'(x) = pf(x)$ if x is a neighbor of u, $f'(x) = f(x)$ if x is not a neighbor of u, and $f'(u) = p$.

Case 2. All arcs of D' are directed away from W. Let p and q be primes not occurring as a factor in any label of f and define $f'(x) = f(x)$ if x is a neighbor of u in G, $f'(x) = pf(x)$ if x is not a neighbor of u in G, and $f'(u) = qa$, where a is the product of the labels of the neighbors of u. ∎

By Propositions 2.1 and 2.7, we have the following corollary.

Corollary 2.8 *A triangle-free graph G is a divisor graph if and only if G is bipartite.*

Proof. By Proposition 2.7, every bipartite graph is a divisor graph. Thus it remains to verify the converse. Let G be a triangle-free divisor graph. Assume, to the contrary, that G is not bipartite. Then G contains an induced odd cycle C. Since G is triangle-free, the length of C is 5 or more, which contradicts Proposition 2.1. ■

Proposition 2.9 *If G and H are divisor graphs, then $G+H$ is a divisor graph.*

Proof. Let $V(G) = \{u_1, u_2, \cdots, u_n\}$ and $V(H) = \{v_1, v_2, \cdots, v_k\}$. Since G and H are divisor graphs, there exist divisor labelings g and h of G and H, respectively. Suppose that $g(u_i) = a_i$ and $h(v_j) = b_j$ for $1 \leq i \leq n$ and $1 \leq j \leq k$. Let a be the least common multiple of $a_1, a_2, \cdots, a_n$. Define a labeling

$$f : V(G+H) \rightarrow \mathbf{N} \text{ by } f(u_i) = a_i \text{ and } f(v_j) = ab_j$$

for $1 \leq i \leq n$ and $1 \leq j \leq k$. Since $ab_r \mid ab_s$ for $1 \leq r \neq s \leq k$ if and only if $b_r \mid b_s$ and since $a_i \mid ab_j$ for all i and j with $1 \leq i \leq n$ and $1 \leq j \leq k$, it follows that f is a divisor labeling of $G+H$. ■

Corollary 2.10 *Every complete multi-partite graph is a divisor graph.*

Proof. We have seen that every bipartite graph is a divisor graph, so K_{n_1,n_2} is a divisor graph for positive integers n_1 and n_2. Thus for $t \geq 3$ and positive integers $n_1, n_2, \cdots, n_t$, the graph $K_{n_1,n_2,\cdots,n_t} = K_{n_1,n_2}+K_{n_3,n_4,\cdots,n_t}$ is a divisor graph by Proposition 2.9. ■

3 A Characterization of Divisor Graphs

A *transmitter* is a vertex having indegree 0, while a *receiver* is a vertex having outdegree 0. Thus an isolated vertex is both transmitter and receiver. For a vertex u of D, let

$$N^+(u) = \{x \mid (u,x) \in E(D)\} \quad \text{and} \quad N^-(u) = \{x \mid (x,u) \in E(D)\}.$$

So if u is a transmitter, then $N^-(u) = \emptyset$, while if v is a receiver, then $N^+(u) = \emptyset$. A vertex of D is a *transitive vertex* if (1) od $u > 0$ and id $u > 0$, and (2) for every $x \in N^-(u)$ and $y \in N^+(u)$, $(x,y) \in E(D)$. We can now present a characterization of graphs that are divisor graphs.

Theorem 3.1 *Let G be a graph. Then G is a divisor graph if and only if there exists an orientation D of G such that every vertex of D is a transmitter, a receiver, or a transitive vertex.*

Proof. Assume first that G is a divisor graph. Let f be a divisor labeling of G and D be the orientation of G induced by f. We show that every vertex of D is a transmitter, a receiver, or a transitive vertex. Let u be a vertex of D. If $\operatorname{od} u = 0$ or $\operatorname{id} u = 0$, then u is a transmitter or a receiver in D. So we may assume that $\operatorname{od} u > 0$ and $\operatorname{id} u > 0$. If $v \in N^-(u)$ and $w \in N^+(u)$, then $f(v) \mid f(u)$ and $f(u) \mid f(w)$, implying that $f(v) \mid f(w)$ and so $(v, w) \in E(D)$. Thus u is a transitive vertex in D. Therefore, D has the desired property.

To verify the converse, we proceed by induction on the order of a graph G. The result is certainly true for all graphs of some order 3 or less. Assume that the result is true for all graphs of order $k \geq 3$. Let G be a graph of order $k+1$ having an orientation D each of whose vertices is a transmitter, a receiver, or transitive.

Let $u \in V(G)$. We show that every vertex of the orientation $D - u$ of $G - u$ is also a transmitter, a receiver, or a transitive vertex. Let w be a vertex of $D - u$. If w is not a neighbor of u, then certainly w is a transmitter, receiver, or transitive vertex. So we may assume that w is a neighbor of u. We consider three cases.

Case 1. *u is a transmitter in D.* Then $w \in N^+(u)$. If $\deg_D w = 1$, then w is an isolated vertex in $D - u$ and so is a transmitter or a receiver. Hence we may assume that $\deg_D w \geq 2$. Necessarily w is not a transmitter in D. If w is a receiver in D, it is also a receiver in $D - u$. We may assume that w is a transitive vertex in D. Then $N^+(w) \subseteq N^+(u)$. There are two subcases.

Subcase 1.1. $N^-(w) = \{u\}$. Then w is a transmitter in $D - u$.

Subcase 1.2. $N^-(w) - \{u\} \neq \emptyset$. Let $y \in N^-(w)$ and $y \neq u$. For every $x \in N^+(w)$, we have $(y, x) \in E(D)$. So w is a transitive vertex in $D - u$.

Case 2. *u is a receiver in D.* The proof is similar to the proof of Case 1 and is therefore omitted.

Case 3. *u is a transitive vertex in D.* Let $w \in N^-(u)$. Then $\operatorname{od}_D w \geq 1$. If $\operatorname{id}_D w = 0$, then w is a transitive vertex in D and in $D - u$. Thus we may assume that $\operatorname{id}_D w \geq 1$. We show that w is a transitive vertex in $D - u$. Let $x \in N^-(w)$. Then $x \in N^-(u)$. Let $y \in N^+(w)$. If $y \neq u$, then $(x, y) \in E(D)$ because w is a transitive vertex in D. If $y = u$, then $(x, u) \in E(D)$. This implies that w is a transitive vertex in $D - u$. Similarly, if $w \in N^+(u)$, then w is a receiver or a transitive vertex in $D - u$.

Therefore, every vertex of the orientation $D-u$ of $G-u$ is a transmitter, a receiver, or a transitive vertex, as claimed. By the induction hypothesis, then, $G-u$ is a divisor graph. Let f be a divisor labeling of $G-u$. We consider three cases.

Case 1. *u is a transmitter in D.* Let p be a prime not occurring as a factor of $f(x)$ for all $x \in V(G)$. Define a labeling $f' : V(G) \to \mathbf{N}$ by $f'(x) = pf(x)$ if $x \in N(u)$, $f'(x) = f(x)$ if $x \notin N[u]$, and $f'(u) = p$. Then f' is a divisor labeling of G. The digraph D and the divisor labeling f' are shown in Figure 3.

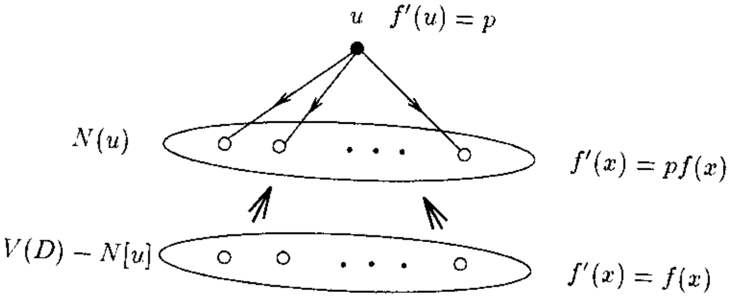


Figure 4: The digraph D and the divisor labeling f' in Case 1

Case 2. *u is a receiver in D.* Let m be the least common multiple of the integers in the set $\{f(x) : x \in N(u)\}$. Moreover, let p and q be distinct primes, neither of which is a factor of $f(x)$ for all vertices x of G. Define a labeling $f' : V(G) \to \mathbf{N}$ by $f'(x) = f(x)$ if $x \in N(u)$, $f'(x) = pf(x)$ if $x \notin N[u]$, and $f'(u) = qm$. Then f' is a divisor labeling of G. Then f' is a divisor labeling of G. The digraph D and the divisor labeling f' are shown in Figure 4.

Case 3. *u is a transitive vertex in D.* Then $N_G(u) = N^+(u) \cup N^-(u)$, where $N^+(u) \neq \emptyset$, and $N^-(u) \neq \emptyset$. Let p and q be distinct primes, neither of which is a factor of $f(x)$ for all vertices x of G. Furthermore, let m be the least common multiple of the integers in the set $\{f(x) : x \in N^-(u)\}$. Since every vertex of $N^-(u)$ is adjacent to every vertex of $N^+(u)$ in $D-u$, it follows that $m \mid f(y)$ for all $y \in N^+(u)$. Define a labeling $f' : V(G) \to \mathbf{N}$ by $f'(x) = f(x)$ if $x \in N^-(u)$, $f'(y) = pqf(y)$ if $y \in N^+(u)$, $f'(z) = pf(z)$ if $z \in V(D) - N[u]$, and $f'(u) = qm$. Then f' is a divisor labeling of G. The digraph D and the divisor labeling f' are shown in Figure 5.

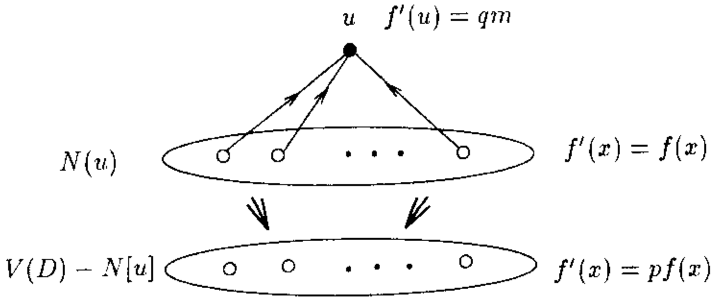


Figure 5: The digraph D and the divisor labeling f' in Case 2

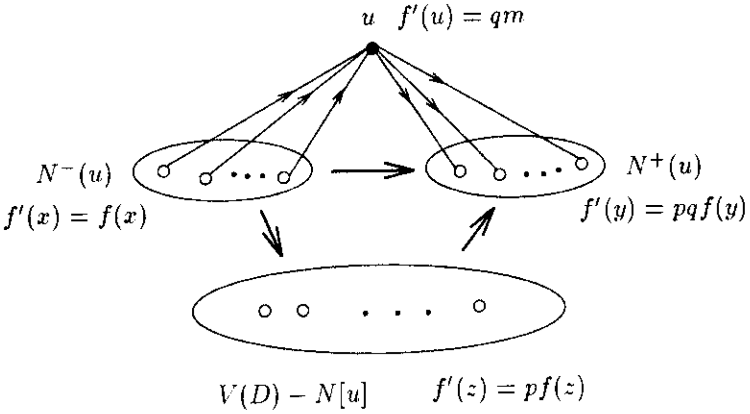


Figure 6: The digraph D and the divisor labeling f' in Case 3

Therefore, G is a divisor graph, as desired. ∎

As an application of Theorem 3.1, we verify the following.

Proposition 3.2 *The graph $K_3 \times K_2$ is not a divisor graph.*

Proof. Assume, to the contrary, that $K_3 \times K_2$ is a divisor graph and let f be a divisor labeling of $K_3 \times K_2$. Let G_1 and G_2 be two copies of K_3 in $K_3 \times K_2$, where $V(G_1) = \{u_1, u_2, u_3\}$ and $V(G_2) = \{v_1, v_2, v_3\}$ and u_i is adjacent to v_i for $i = 1, 2, 3$. Assume, without loss of generality, that $f(u_1) \mid f(u_2)$, $f(u_2) \mid f(u_3)$, and $f(u_1) \mid f(u_3)$. Since u_2v_2 is an edge in $K_3 \times K_2$, it follows that $f(u_2) \mid f(v_2)$ or $f(v_2) \mid f(u_2)$. In either case, u_2 is not a transmitter, receiver, or transitive vertex, contradicting Theorem 3.1. ∎

By Proposition 2.2, if a graph G is not a divisor graph, then $G \times K_2$ is not a divisor graph since G is an induced subgraph of $G \times K_2$.

Theorem 3.3 *Let G be a graph. Then $G \times K_2$ is a divisor graph if and only if G is a bipartite graph.*

Proof. If G is a bipartite graph, then $G \times K_2$ is a bipartite graph and so $G \times K_2$ is a divisor graph by Proposition 2.7. For the converse, assume, to the contrary, that $G \times K_2$ is a divisor graph but G is not bipartite. Let C be a smallest induced odd cycle of G, say of length ℓ. If $\ell \geq 5$, then G is not a divisor graph by Proposition 2.1 and so $G \times K_2$ is not a divisor graph. Thus $\ell = 3$ and so G contains K_3. However, then $G \times K_2$ contains $K_3 \times K_2$ as an induced subgraph, contradicting Propositions 3.2 and 2.2. ■

4 Another View of Divisor Graphs

We now show that divisor graphs can be considered from a quite different viewpoint. First, we introduce a few definitions. Let D be an oriented graph. The *(directed) distance* $d(u,v)$ from a vertex u to a vertex v in D is the length of a shortest directed $u - v$ path in D. As with graphs, a directed $u - v$ path of length $d(u,v)$ is referred to as a $u - v$ *geodesic*. A vertex w is said to *lie in* a $u - v$ geodesic P if w is an internal vertex of P, that is, w is a vertex of P distinct from u and v. The *closed interval* $I[u,v]$ consists of u and v together with all vertices lying in a $u - v$ geodesic or in a $v - u$ geodesic in D. Hence, if there is no $u - v$ geodesic or $v - u$ geodesic in D, then $I[u,v] = \{u,v\}$.

For a nonempty subset S of $V(D)$, define

$$I[S] = \bigcup_{u,v \in S} I[u,v].$$

Certainly then, $S \subseteq I[S]$. A set S is *convex* if $I[S] = S$. The *convex hull* $[S]$ of S is the smallest convex set containing S. So S is a convex set in D if and only if $[S] = S$. Certainly, the vertex set $V(D)$ is convex. The *convexity number* $\text{con}(D)$ of D is defined in [1] the maximum cardinality of a *proper* convex set of D. A convex set S is defined to be *maximum* if $|S| = \text{con}(D)$. These concepts were introduced for graphs and studied in [2].

Let D be a connected oriented graph of order $n \geq 2$. Since every singleton vertex set is convex,

$$1 \leq \text{con}(D) \leq n - 1.$$

Thus con$(D) = n - 1$ if and only if there exists a vertex v of D such that $V(D - v)$ is convex. The following result was established in [1].

Theorem A *Let D be a connected oriented graph of order $n \geq 2$. Then $V(D - v)$ is convex if and only if v is a transmitter, receiver, or transitive vertex.*

This also provides the following corollary.

Corollary B *Let D be a connected oriented graph of order $n \geq 2$. Then* con$(D) = n - 1$ *if and only if D contains a transmitter, receiver, or transitive vertex.*

An oriented graph D of order at least 2 is called *strongly convex* if $V(D - v)$ is convex *for every* $v \in V(D)$.

Corollary 4.1 *A graph G is a divisor graph if and only if some orientation of G is strongly convex.*

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