Vertex and edge critical total restrained domination in graphs

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Abstract  
A graph $G$ with no isolated vertices is vertex critical with respect to total restrained domination if, for any vertex $v$ of $G$ that is not adjacent to a vertex of degree one, the total restrained domination number of $G - v$ is less than the total restrained domination number.

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of $G$. We call these graphs $\gamma_{tr}$-vertex critical. Similarly, a graph with no isolated vertices is edge critical with respect to total restrained domination if for any non-edge $e$ of $G$, the total restrained domination number of $G + e$ is less than the total restrained domination number of $G$. We call these graphs $\gamma_{tr}$-edge critical. In this paper, we characterize the $\gamma_{tr}$-vertex critical trees, as well as those $\gamma_{tr}(G)$-vertex critical graphs $G$ for which $\gamma_{tr}(G) - \gamma_{tr}(G - v) = n - 2$ for some $v \in V(G)$. Moreover, we also characterize the $\gamma_{tr}$-edge critical trees, as well as those $\gamma_{tr}(G)$-edge critical graphs $G$ for which $\gamma_{tr}(G) - \gamma_{tr}(G + e) = n - 2$ for some $e \notin E(G)$.

**Keywords:** Total restrained domination, vertex critical, edge critical.

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1 Introduction

A vertex in a graph $G$ dominates itself and its neighbors. A set of vertices $S$ in a graph $G$ is a dominating set if each vertex not in $S$ is dominated by some vertex of $S$. The domination number of $G$, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A dominating set $S$ is called a total dominating set if each vertex is dominated by some vertex of $S$, and the total domination number of $G$, denoted $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. A leaf in a graph $G$ is a vertex of degree one, and a remote vertex is a vertex which is adjacent to a leaf. Let $S(G)$ denote the set of remote vertices of $G$.

Note that the removal of a vertex in a graph may decrease the domination number. A graph $G$ is called domination vertex critical if $\gamma(G - v) < \gamma(G)$ for every vertex $v$ in $G$. For references on domination vertex critical graphs see [1, 4, 8].

Goddard et al. [5] studied the concept of vertex criticality for total domination. They defined a connected graph $G$ of order at least two to be total domination vertex critical or just $\gamma_t$-vertex critical if, for every vertex $v \in V(G) - S(G)$, we have $\gamma_t(G - v) < \gamma_t(G)$. Note that if $G$ is $\gamma_t$-vertex critical and $v \in V(G) - S(G)$, then $\gamma_t(G - v) = \gamma_t(G) - 1$.

Chen et al. [2] and Zelinka [10] introduced the study of total restrained domination, which was further studied by Hattingh et al. [6] and Cyman et al. [3]. A set $S \subseteq V(G)$ is a total restrained dominating set, denoted TRDS, if every vertex is adjacent to a vertex in $S$ and every vertex in $V(G) - S$ is also adjacent to a vertex in $V(G) - S$. The total restrained
domination number of $G$, denoted $\gamma_{tr}(G)$, is the minimum cardinality of a total restrained dominating set of $G$. A total TRDS of cardinality $\gamma_{tr}(G)$ is called a $\gamma_{tr}(G)$-set.

Let $G$ be a connected graph of order at least three. We say that $G$ is total restrained domination vertex critical or just $\gamma_{tr}$-vertex critical if, for any vertex $v$ of $V(G) - S(G)$, we have $\gamma_{tr}(G - v) < \gamma_{tr}(G)$. Similarly, we say $G$ is total restrained domination edge critical or just $\gamma_{tr}$-edge critical if for any $e \notin E(G)$, we have $\gamma_{tr}(G + e) < \gamma_{tr}(G)$.

In Section 2, we characterize the $\gamma_{tr}$-vertex critical trees, as well as those $\gamma_{tr}(G)$-vertex critical graphs $G$ for which $\gamma_{tr}(G) - \gamma_{tr}(G - v) = n - 2$ for some $v \in V(G)$. In Section 3, we characterize the $\gamma_{tr}$-edge critical trees, as well as those $\gamma_{tr}(G)$-edge critical graphs $G$ for which $\gamma_{tr}(G) - \gamma_{tr}(G + e) = n - 2$ for some $e \notin E(G)$.

2 $\gamma_{tr}$-vertex critical graphs

In contrast to total domination, the removal of a vertex may decrease the total restrained domination number by more than one. In fact, if $G$ is a $\gamma_{tr}$-vertex critical graph, then $\gamma_{tr}(G) - \gamma_{tr}(G - v) \leq n - 2$ for all $v \in V(G)$. In this section, we characterize $\gamma_{tr}(G)$-vertex critical graphs $G$ for which $\gamma_{tr}(G) - \gamma_{tr}(G - v) = n - 2$ for some $v \in V(G)$. Goddard et. al. [5] have shown that there are no $\gamma_{t}$-vertex critical trees. We will also determine which trees are $\gamma_{tr}$-vertex critical.

Let $A$ be the family of connected graphs $G$ such that $G$ belongs to $A$ if and only if every edge is incident with a remote vertex or a leaf or $G$ is a cycle on three vertices.

The following result is due to Cyman and Raczek, [3].

**Theorem 1** Let $G$ be a connected graph of order $n \geq 2$. Then $\gamma_{tr}(G) = n$ if and only if $G$ belongs to $A$.

Let $P_4$ be a path with consecutive vertices $v_1, v_2, v_3, v_4$. Let $m \geq 0$ be an integer and let $G(m)$ be the graph obtained from $P_4$ by adding $m$ new vertices $u_1, \ldots, u_m$ and joining $u_i$, $i = 1, \ldots, m$, to each of the vertices $v_2$ and $v_3$.

**Proposition 1** Suppose $G$ is a connected graph of order $n \geq 3$. Then $G$ is
a \( \gamma_{tr} \)-vertex critical graph for which \( \gamma_{tr}(G) - \gamma_{tr}(G - v) = n - 2 \) for some \( v \in V(G) \) if and only if \( G \in \{ C_3, K_{1,2}, G(n-4) \} \).

**Proof.** Let \( v \in \gamma \) and so \( \gamma \) is a \( \gamma \)-set of \( G \). For the converse, suppose \( v \) is a vertex may be chosen for \( \gamma \), without loss of generality, that \( u \) is a remote vertex of \( K \), while for \( K_{1,2} \) a leaf may be chosen for \( v \), while for \( K_{1,3} \) a leaf must be chosen for \( v \).

For the converse, suppose \( G \) is a \( \gamma_{tr} \)-vertex critical graph for which \( \gamma_{tr}(G) - \gamma_{tr}(G - v) = n - 2 \) for some \( v \in V(G) \). Then \( \gamma_{tr}(G) = n \), while \( \gamma_{tr}(G - v) = 2 \).

Suppose \( n = 3 \). By Theorem 1, either \( G \) is \( C_3 \) or each vertex is incident with a remote vertex. In the latter case, \( G = K_{1,2} \). We henceforth assume \( n \geq 4 \). By Theorem 1, we may assume that each edge of \( G \) is incident with a remote vertex of \( G \).

If each remote vertex \( u \) of \( G \) is adjacent to at least two leaves or \( \deg(u) = 2 \), then, for every \( v \in V(G) \), each edge of \( G - v \) is still incident with a remote vertex, and so, by Theorem 1, \( \gamma_{tr}(G - v) = n - 1 \geq 3 \), which is a contradiction.

Thus, there exists a remote vertex \( u \) of \( G \) such that \( \deg(u) \geq 3 \) and \( u \) is adjacent to exactly one leaf \( \ell \) of \( G \). Let \( S_v \) be a \( \gamma_{tr} \)-set of \( G - v \).

**Case 1.** \( v \neq \ell \).

As \( \ell \) is also a leaf of \( G - v \), we have \( S_v = \{ u, \ell \} \), and so each vertex of \( R = V(G) - \{ u, \ell, v \} \) is adjacent to \( u \). Moreover, each vertex of \( R \) is adjacent to another vertex of \( R \). Thus, no vertex in \( R \) is a remote vertex of \( G - v \). However, in \( G \), each edge in \( < R > \) must be incident with a remote vertex of \( G \). Thus, some vertex \( w \) in \( R \) is remote, which implies that \( v \) is the leaf adjacent to \( w \) in \( G \). Note that \( v \) is not adjacent to any of the vertices of \( R - \{ w \} \), and so each vertex of \( R - \{ w \} \) is adjacent to only \( w \) in \( < R > \). Thus, \( G = G(n-4) \).

**Case 2.** \( v = \ell \).

If \( u \in S_v \), then \( S_v \cup \{ \ell \} \) is a TRDS of \( G \), and so \( \gamma_{tr}(G) \leq 3 \), which is a contradiction. We assume \( u \notin S_v \). Let \( S_v = \{ x, y \} \) and suppose, without loss of generality, that \( u \) is adjacent to \( x \). Note that each vertex of \( R = V(G) - \{ x, y, \ell \} \) is adjacent to another vertex of \( R \), and so \( R \) does not contain any leaves. Since the edge \( xy \) is incident with a remote vertex of \( G \), either \( x \) or \( y \) is a remote vertex. But \( y \) cannot be a remote vertex,
and so $x$ is remote, while $y$ is a leaf of $G$. Since $y$ is also a leaf of $G - v$, each vertex of $R$ is adjacent to $x$. However, in $G$, each edge in $<R>$ must be incident with a remote vertex of $G$. The only remote vertex in $R$ is the vertex $u$, and so each vertex of $R - \{u\}$ is adjacent to only $u$ in $<R>$. Thus, $G = G(n - 4)$, as required. $\square$

We next characterize $\gamma_{tr}$-vertex critical trees, and then determine which paths are $\gamma_{tr}$-vertex critical.

Let $P$ be a diametrical path of $T$, and suppose $r$ and $r'$ are the leaves of $T$ which form the two endpoints of $P$. Root $T$ at $r'$, and consider a nonleaf vertex $u$ on a path from $r'$ to a leaf of $T$. A path $u = u_0, u_1, \ldots, u_t$ from $u$ to a leaf $u_t$ is called a maximal reference path if every path $u = u_0, u_1, u'_2, \ldots, u'_s$ has the property $s \leq t$. Let $R_{t,u}$ be the set of all maximal reference paths of length $t$ originating from $u$ which do not contain the parent of $u$. An element of $R_{t,u}$ will be called a $u$-RT-path (or just an RT-path if the context is clear), and denoted by $u = u_0^i, \ldots, u_t^i$ for some $i \in \{1, \ldots, |R_{t,u}|\}$.

The set $S$ will denote a $\gamma_{tr}$-set of $T$, while $S'$ will denote a $\gamma_{tr}$-set of $T'$, where $T'$ will be defined later.

**Theorem 2** Let $T$ be a tree of order $n \geq 2$. $T$ is $\gamma_{tr}$-vertex critical if and only if $\gamma_{tr}(T) = n$.

**Proof.** Suppose first that $\gamma_{tr}(T) = n$. Then $\gamma_{tr}(T - v) \leq n - 1$ for every $v \notin S(T)$, and so $T$ is $\gamma_{tr}$-vertex critical. Suppose now that $T$ is $\gamma_{tr}$-vertex critical. We will employ induction on the $n(T)$, the order of $T$, to show that $\gamma_{tr}(T) = n$. If $1 \leq \text{diam}(T) \leq 3$, then $\gamma_{tr}(T) = n$. Thus, the result is true for all trees of order $n \in \{2, 3, 4\}$. Suppose $T$ is a tree of order $n \geq 5$, and suppose that for any $\gamma_{tr}$-vertex-critical tree $T'$ of order $2 \leq n(T') = n' \leq n$ we have that $\gamma_{tr}(T') = n'$. By the above, we may assume that $\text{diam}(T) \geq 4$.

**Claim 1.** Let $t \in \{2, 3\}$, and consider the RT-path $u = u_0, u_1, \ldots, u_t$. If $u \in S(T)$, then $\gamma_{tr}(T) = n$.

**Proof.** Suppose $u \in S(T)$, and let $T' = T - u_t$. Since $u_{t-1}$ is either a leaf or a support vertex of $T'$, we have that $u_{t-1} \in S'$. Thus, $S' \cup \{u_t\}$ is a TRDS of $T$, and so $\gamma_{tr}(T) \leq \gamma_{tr}(T') + 1$.

We first show that $\gamma_{tr}(T') = \gamma_{tr}(T) - 1$: (*)

Since $u$ is a remote vertex of $T$, we have that $u \notin S$. Also, $\{u_{t-1}, u_t\} \subseteq S$. Moreover, if $t = 3$, every vertex in $N(u_1) - \{u\}$ is either a leaf or a remote vertex, and so $N(u_1) \subseteq S$, which implies that $u_1 \in S$. Thus, $S - \{u_t\}$ is a
TRDS of $T'$, and so $\gamma_{tr}(T') \leq |S| - 1 = \gamma_{tr}(T) - 1$.

We next establish the following fact.

**Fact 1.** $T'$ is $\gamma_{tr}$-vertex critical.

**Proof.** Suppose, to the contrary, that there exists $v \notin S(T')$ such that $\gamma_{tr}(T') \leq \gamma_{tr}(T' - v)$. Let $w$ be the leaf adjacent to $u$. We first show that $v \neq w$. For suppose, to the contrary, that $v = w$. Note that $N_T[u_{t-1}] - \{u_{t-2}\} \subseteq S'$, while $\{w, u\} \subseteq S'$. Moreover, if $t = 3$, every vertex in $N(u_1) - \{u\}$ is either a leaf or a remote vertex, and so $N(u_1) \subseteq S'$, which implies that $u_1 \in S$. Thus, $S' - \{w\}$ is a TRDS of $T' - v$, and so $\gamma_{tr}(T' - v) \leq |S'| - 1 = \gamma_{tr}(T' - v) - 1$, which is a contradiction.

Thus, $v \neq w$ and either $v \notin S(T)$ or the only leaf adjacent to $v$ is $u_t$. We eliminate the possibility that the only leaf adjacent to $v$ is $u_t$. For suppose, to the contrary, that the only leaf adjacent to $v$ is $u_t$. Note that $N_T[u_{t-1}] \subseteq S'$, while $\{w, u\} \subseteq S'$. Thus, $S' - \{v\}$ is a TRDS of $T' - v$, and so $\gamma_{tr}(T' - v) \leq |S' - 1 = \gamma_{tr}(T') - 1 \leq \gamma_{tr}(T' - v) - 1$, which is a contradiction.

Thus, $v \neq w$ and $v \notin S(T)$. As $v \notin S(T)$, $\gamma_{tr}(T - v) \leq \gamma_{tr}(T) - 1$. If we can show that $\gamma_{tr}(T' - v) \leq \gamma_{tr}(T' - v) - 1$, then, referring to (*), we have $\gamma_{tr}(T) - 1 = \gamma_{tr}(T') \leq \gamma_{tr}(T' - v) \leq \gamma_{tr}(T - v) - 1 \leq \gamma_{tr}(T) - 2$, which will produce a contradiction, and establish our fact.

Let $U$ be a $\gamma_{tr}(T - v)$-set. Note that $v \notin \{u_{t-1}, u_t\}$. Also, $\{u_{t-1}, u_t\} \subseteq U$.

Suppose $\deg(u_{t-1}) \geq 3$. Suppose $v \in N_T(u_{t-1}) - \{u_{t-2}, u_t\}$. Since $u$ is a remote vertex of $T' - v$, we have that $u \in U$. Moreover, if $t = 3$, every vertex in $N_T(u_{t-1}) - \{u\}$ is either a leaf or a remote vertex, and so $N_T - (u_{t-1}) \subseteq U$, which implies that $u_1 \in U$. Thus, $U - \{u_{t-1}\}$ is a TRDS of $T - v - u_{t-1}$, and so $\gamma_{tr}(T' - v) \leq |U| - 1 = \gamma_{tr}(T - v) - 1$.

If $t \geq 3$ and $v = u_{t-1}$, then, since $v \notin S(T')$, every vertex in $N_T - (v - u_{t-1})$ is a remote vertex, but not a leaf, in $T$, and so $N_T - (v - u_{t-1}) \subseteq U$, which implies that $U - \{u_{t-1}\}$ is a TRDS of $T - v - u_{t-1}$, and so $\gamma_{tr}(T' - v) \leq |U| - 1 = \gamma_{tr}(T - v) - 1$.

Thus, $v \notin N_T - (u_{t-1}) \cup \{u_{t-2}\}$, and so $N_T - (u_{t-1}) \subseteq U$. Thus, $U - \{u_t\}$ is a TRDS of $T - v - u_t$, and so $\gamma_{tr}(T' - v) \leq |U| - 1 = \gamma_{tr}(T - v) - 1$.

We henceforth assume that $\deg(u_{t-1}) = 2$. Note that if $t = 3$, then, since $v \notin S(T')$, $v \neq u_{t-1}$. Moreover, every vertex in $N_T - (u_{t-1}) - \{u\}$ is either a leaf or a remote vertex, and so $N(u_{t-1}) \subseteq U$, which implies that $u_1 \in U$. Thus, $U - \{u_{t-1}\}$ is a TRDS of $T - v - u_{t-1}$, and so $\gamma_{tr}(T' - v) \leq |U| - 1 = \gamma_{tr}(T - v) - 1$. 

6
By the induction assumption and Fact 1, $\gamma_{tr}(T') = n - 1$, and, since $\gamma_{tr}(T') = \gamma_{tr}(T) - 1$, we have $\gamma_{tr}(T) = n$. ♦

Since $\text{diam}(T) \geq 4$, let $r' = v_k, \ldots, v_1, u = u_0, u_1, u_2, u_3 = r$ be a diametrical path. By our Claim, $u$ is not a remote vertex. Consider the tree $T' = T - u$. Hence, by the criticality of $T$, it follows that $|S'| \leq |S| - 1$. For $i = 1, \ldots, m$, let $u, u_1', u_2', u_3'$ be the R3-paths originating from $u$. By our Claim, $u_1'$ is not a remote vertex for $i = 1, \ldots, m$. Thus, all the vertices of the subtree of $T'$ induced by $u_1'$ and its descendants must be contained in $S'$. Hence, $N(u) - \{v_1\} \subseteq S'$. If $v_1 \in S'$, then $S'' = S' - \bigcup_{i=1}^{m}\{u_i'\}$ is a TRDS of $T$, and so $\gamma_{tr}(T) \leq |S''| = |S'| - m \leq |S| - m - 1 \leq \gamma_{tr}(T) - 2$, which is a contradiction. Thus, $v_1 \notin S'$, and $S'$ is a TRDS of $T$ of size at most $\gamma_{tr}(T) - 1$, which is a contradiction. □

As an immediate consequence (cf. Theorem 1), we obtain:

**Corollary 1** Let $T$ be a tree of order $n \geq 2$. Then $T$ is $\gamma_{tr}$-vertex critical if and only if $T$ belongs to $A - \{C_3\}$.

**Corollary 2** The path $P_n$ of order $n \geq 3$ is $\gamma_{tr}$-vertex critical if and only if $n \in \{3, 4, 5\}$.

**Proof.** The only paths in which every edge is incident with a remote vertex or a leaf, are $P_3$, $P_4$ and $P_5$. Thus, $(A - \{C_3\}) \cap \{P_n | n \geq 1\} = \{P_3, P_4, P_5\}$, and so $P_3, P_4$ and $P_5$ are the only $\gamma_{tr}$-vertex critical paths. □

A **caterpillar** is a tree with the property that the removal of its leaves results in a path $v_1, \ldots, v_s$ as the **spine** of the caterpillar. A caterpillar $T$ is uniquely determined by the sequence of nonnegative integers $(t_1, \ldots, t_s)$, where $t_i$ is the number of leaves adjacent to $v_i$, for $s \geq 2$, and $t_1 \geq 1$ and $t_s \geq 1$. For example, the sequence $(1, 0, 0, 1)$ determines the caterpillar path $P_5$.

Let $W$ be a caterpillar with sequence $(a_1, a_2, \ldots, a_n)$ such that whenever $a_i = 0$ for some $2 \leq i \leq n - 1$, then $a_{i-1} \geq 1$ and $a_{i+1} \geq 1$. Then $\text{diam}(W) = n + 1$, and, by Corollary 1, $W$ is a $\gamma_{tr}$-vertex critical tree. Hence, $\gamma_{tr}(W) - \text{diam}(W) = (\sum_{i=1}^{n} a_i) + n - (n + 1) = (\sum_{i=1}^{n} a_i) - 1$, and so there exists a $\gamma_{tr}$-vertex critical tree $W$ such that the difference $\gamma_{tr}(W) - \text{diam}(W)$ can be made arbitrarily large.
Note if $G$ is a $\gamma_{tr}$-edge critical graph, then $\gamma_{tr}(G) - \gamma_{tr}(G + e) \leq n - 2$ for all $e \notin E(G)$. In this section, we characterize those $\gamma_{tr}(G)$-edge critical graphs $G$ for which $\gamma_{tr}(G) - \gamma_{tr}(G + e) = n - 2$ for some $e \notin E(G)$. We also determine which trees are $\gamma_{tr}$-edge critical.

Let the graph $G(m)$ be defined as before.

**Proposition 2** Suppose $G$ is a connected graph of order $n \geq 3$. Then $G$ is a $\gamma_{tr}$-edge critical graph for which $\gamma_{tr}(G) = n$ and $\gamma_{tr}(G + e) = 2$ for some $e \in E(\overline{G})$ if and only if $G \in \{K_{1,3}, G(n - 4)\}$.

**Proof.** Let $G = G(n - 4)$. By Theorem 1, $\gamma_{tr}(G) = n$, while $2 \leq \gamma_{tr}(G + e) \leq n - 2$ for every $e \notin E(G)$. Thus, $G$ is a $\gamma_{tr}$-edge critical. Moreover, $\gamma_{tr}(G + v_1v_4) = 2$. If $G = K_{1,3}$, then $\gamma_{tr}(G) = n$, while $\gamma_{tr}(G + e) = 2$ for every $e \notin E(G)$, as required.

For the converse, suppose $G$ is a $\gamma_{tr}$-edge critical graph for which $\gamma_{tr}(G) = n$ and $\gamma_{tr}(G + e) = 2$ for some $e \in E(G)$ if and only if $G \in \{K_{1,3}, G(n - 4)\}$.

**Fact 2.** $\deg_G(a) \geq 2$, $\forall a \in R$

We proceed with the following cases.

**Case 1.** $\{u, v\} = \{x, y\}$.

Without loss of generality, assume $u = x$ and $v = y$. It follows from Fact 2 and Theorem 1, that every vertex in $R$ is either a remote vertex or adjacent to a remote vertex. Moreover, also by Fact 2 no vertex of $R$ is a leaf of $G$. Let $w \in R$ be a remote vertex of $G$. Then $w$ is adjacent to a leaf, which must be either $x$ or $y$. Without loss of generality assume it is $x$. Let $w' \in R$ be a vertex which is adjacent to $w$. Then $w'$ must be adjacent to $y$, as $x$ is a leaf. Since at least one of the endpoints of $yw'$ is a remote vertex of $G$, and since $\deg(r) \geq 2$ for every $r \in R$, vertex $y$ is not a remote vertex, whence $w'$ must be remote. But then $y$ is also a leaf of $G$. Hence, $G = P_4 = G(0) = G(n - 4)$.

**Case 2.** $x = u$ and $y \in R$.

Again, every vertex in $R$ is adjacent to a vertex of $R$, whence $\deg(z) \geq 2$.
for every $z \in R - \{y\}$. By Theorem 1, at least one of $u$ or $v$ is a remote vertex.

Suppose $u$ is a remote vertex. Then $v$ is a leaf, and every vertex of $R - \{y\}$ is adjacent to $u$. If $y$ is adjacent to at least two vertices of $R - \{y\}$, then no vertex in $R$ can be remote. Thus, $y$ is a leaf of $G$. Let $w \in R - \{y\}$ be the vertex adjacent to $y$. No vertex in $R - \{y, w\}$ is a remote vertex, and so, by Theorem 1, $R - \{y, w\}$ is an independent set of $G$. Thus, every vertex in $R - \{y, w\}$ is adjacent to $w$. Since $\{u, v\}$ is a minimum TRDS, it follows that $uw \in E(G)$, and so $G = G(n - 4)$.

We may therefore assume that $u$ is a leaf, and every vertex of $R - \{y\}$ is adjacent to $v$. If $y$ is adjacent to at least two vertices of $R - \{y\}$, then no vertex in $R$ can be remote. Thus, $y$ is a leaf of $G$. Let $w \in R - \{y\}$ be the vertex adjacent to $y$. No vertex in $R - \{y, w\}$ is a remote vertex, and so, by Theorem 1, $R - \{y, w\}$ is an independent set of $G$. Thus, every vertex in $R - \{y, w\}$ is adjacent to $w$, and so $G = G(n - 4)$.

Case 3. $\{x, y\} \subseteq R$.

Suppose $x$ is adjacent to both $u$ and $v$. Then, by Theorem 1, either $x$ or $v$ is a remote vertex. If $x$ is a remote vertex, then $x$ is adjacent to a leaf in $R - \{y\}$, which is impossible, since $\deg(z) \geq 2$ for every $z \in R$. Thus, $x$ must be adjacent to a vertex not in a TRDS, whence $v$ is a remote vertex of $G$. Since $\deg(z) \geq 2$ for every $z \in V(G) - \{y\}$, it follows that $y$ must be a leaf of $G$. Now, considering the edge $ux$, vertex $u$ must be adjacent to a leaf in $R - \{y\}$ since $ux$ must be incident to a remote vertex. This produces a contradiction.

Thus, $x$ (and $y$, respectively) is adjacent to exactly one of the vertices in the set $\{u, v\}$.

Suppose $u$ is adjacent to both $x$ and $y$.

Suppose $v$ is adjacent to a vertex in $w \in R$. Then, by the above, $w \in R - \{x, y\}$. As before, either $v$ or $w$ is a remote vertex of $G$. But $v$ cannot be remote, since then a leaf exists in $R - \{x, y\}$, which is a contradiction. Thus, $w$ must be adjacent to a leaf in $R$, which is a contradiction. Hence, $v$ is a leaf of $G$. Since no vertex in $R$ is a remote vertex of $G$, Theorem 1 implies $R = \{x, y\}$. Thus, $G = K_{1,3}$.

We may therefore, without loss of generality, assume that $u$ is adjacent to
only \( x \) in \( \{x, y\} \), while \( v \) is adjacent to only \( y \) in \( \{x, y\} \). Moreover, since no vertex in \( R \) is a remote vertex of \( G \), we must have that \( R = \{x, y\} \). Thus, \( G = P_4 = G(0) = G(n - 4) \). □

**Proposition 3** Suppose \( G \) is a \( \gamma_{tr} \)-edge critical graph. If \( R \) is the set of remote vertices, then \( \langle R \rangle \) is complete.

**Proof.** Let \( \{u, v\} \subseteq R \) such that \( uv \in \bar{G} \). Let \( S \) be a \( \gamma_{tr}(G+uv) \)-set. Then \( \{u, v\} \subseteq S \), and so \( S \) is also a TRDS of \( G \), whence \( \gamma_{tr}(G) \leq \gamma_{tr}(G+uv) \), which is a contradiction. □

**Proposition 4** Suppose \( G \) is a \( \gamma_{tr} \)-edge critical graph. Let \( \{r_1, \ldots, r_\ell\} \) be the remote vertices of \( G \), and let \( L_i \) be the leaves adjacent to \( r_i \) for \( i = 1, \ldots, \ell \). If \( \ell \geq 2 \), then \( |L_i| = 1 \) for \( i = 1, \ldots, \ell \).

**Proof.** Suppose \( \ell \geq 2 \) and, without loss of generality, that \( \{u, v\} \subseteq L_1 \). Moreover, let \( w \in L_2 \). Let \( e = r_2v \), and let \( S \) be a \( \gamma_{tr} \)-set of \( G + e \). Then \( \{u, r_1, r_2, w\} \subseteq S \), whence \( v \in S \), and so \( S \) is a TRDS of \( G \), whence \( \gamma_{tr}(G) \leq \gamma_{tr}(G+uv) \leq \gamma_{tr}(G) - 1 \), which is a contradiction. Thus, \( |L_i| = 1 \) for \( i = 1, \ldots, \ell \), as required. □

**Proposition 5** The only \( \gamma_{tr} \)-edge critical tree \( T \) is \( P_4 \).

**Proof.** Note that \( \text{diam}(T) \leq 3 \), since otherwise (cf. Proposition 3) the two remote vertices on a diametrical path are adjacent, implying that \( T \) has a cycle. If \( \text{diam}(T) = 3 \), then, by Proposition 4, both support vertices on a diametrical path has degree two, implying that \( T \) is isomorphic to \( P_4 \). Lastly, \( P_3 \) is not \( \gamma_{tr} \)-edge critical. □

**References**


