On Stratified Domination in Oriented Graphs

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ABSTRACT

An oriented graph is 2-stratified if its vertex set is partitioned into two classes, where the vertices in one class are colored red and those in the other class are colored blue. Let \( H \) be a 2-stratified oriented graph rooted at some blue vertex. An \( H \)-coloring of an oriented graph \( D \) is a red-blue coloring of the vertices of \( D \) in which every blue vertex \( v \) belongs to a copy of \( H \) rooted at \( v \) in \( D \). The \( H \)-domination number \( \gamma_H(D) \) is the minimum number of red vertices in an \( H \)-coloring of \( D \). We investigate \( H \)-colorings in oriented graphs where \( H \) is the red-red-blue directed path of order 3.

Key Words: stratified oriented graph, \( H \)-domination, domination, open domination.

AMS Subject Classification: 05C15, 05C20, 05C69.

1 Introduction

Stratified domination in graphs was first introduced and studied in [?]. As described in [?], the most studied types of domination in graphs can be defined in terms of an appropriately chosen rooted 2-stratified graph, in fact, in terms of an appropriately selected rooted 2-stratified path of order 3. This gives rise to an infinite class of domination parameters, each of which is defined for every graph. The concept of stratified domination was extended to oriented graphs in [?]. We refer to the books [?], [?], [?] for graph theory notation and terminology not described in this paper. In this paper we only consider connected graphs.

If a digraph \( D \) has the property that for each pair \( u, v \) of distinct vertices of \( D \), at most one of \( (u,v) \) and \( (v,u) \) is an arc of \( D \), then \( D \) is an oriented graph. For a vertex \( v \) in an oriented graph, the number of vertices to which a vertex \( v \) is adjacent is the outdegree of \( v \) and is denoted by \( od_v \), and the number of vertices from which \( v \) is adjacent is the indegree of \( v \) and is denoted by \( id_v \). Thus the degree of \( v \) is \( \text{deg} v = od_v + id_v \).

An oriented graph whose vertex set is partitioned into two subsets is called a 2-stratified oriented graph, where the vertices of one subset are considered to be colored red and those in the other subset are colored blue. For an oriented graph \( D \), a red-blue coloring of \( D \) is a coloring in which every vertex is colored red or blue. It is acceptable if all vertices of \( D \) are colored the same. If there is at least one vertex of each color, then the red-blue coloring of \( D \) produces a 2-stratification of \( D \). Let \( F \) be a (connected) 2-stratified oriented graph rooted at some blue vertex. An \( F \)-coloring of an oriented graph \( D \) is a red-blue coloring of the vertices of \( D \) in which every blue vertex \( v \) belongs to a copy \( F' \) of \( F \) rooted at \( v \) in \( D \). In this case, \( v \) is said to be \( F \)-dominated by some red vertex in \( F' \). A red vertex is \( F \)-dominated by itself. The \( F \)-domination number \( \gamma_F(D) \) is the minimum number of red vertices in an \( F \)-coloring of \( D \). The set of red vertices in an \( F \)-coloring \( c \) of \( D \) is also called an \( F \)-dominating set of \( D \) and is denoted by \( R_c \). If \( |R_c| = \gamma_F(D) \), then \( c \) is a minimum \( F \)-coloring of \( D \) and \( R_c \) is a minimum
**F-dominating set** of $D$. The $F$-domination number of every oriented graph $D$ is defined since $V(D)$ is an $F$-dominating set. Therefore, if $F$ has $r$ red vertices, then

$$r \leq \gamma_F(D) \leq n$$

for every oriented graph $D$ of order $n \geq r$. Furthermore, if $D$ has no subdigraph isomorphic to $F$, then $\gamma_F(D) = n$.

As with graphs, the most studied type of domination in digraphs can be defined in terms of an appropriately chosen rooted 2-stratified digraph. Let the filled in vertices be colored red, and the empty vertices be colored blue. If $F$ is a connected 2-stratified oriented graph of order 2, then $F$ is one of the 2-stratifcations of $\vec{P}_2$ in Figure 1. In each case, the $F$-domination number is a well-known domination parameter in digraphs.

$$F_1 : \bullet \to \circ \quad F_2 : \circ \to \bullet$$

Figure 1: Two 2-stratified oriented graphs of $\vec{P}_2$

Let $D$ be an oriented graph. A vertex $v$ is said to dominate (or out-dominate) itself together with all vertices adjacent from $v$. A set $S \subseteq V(D)$ is a dominating set for $D$ if every vertex in $D$ is dominated by some vertex in $S$. The domination number $\gamma(D)$ is the minimum cardinality of a dominating set in $D$. A dominating set of cardinality $\gamma(D)$ is called a minimum dominating set of $D$. The converse $D^*$ of an oriented graph $D$ has the same vertex set as $D$ and the arc $(u,v)$ is in $D^*$ if and only if the arc $(v,u)$ is in $D$. The following was established in [?].

**Proposition 1.1** For the 2-stratified oriented graphs $F_1$ and $F_2$ of $\vec{P}_2$,

$$\gamma(D) = \gamma_{F_1}(D) \text{ and } \gamma_{F_2}(D) = \gamma_{F_1}(D^*)$$

for every oriented graph $D$.

By Proposition 1.1, we need not be concerned with studying the $F$-domination for any oriented connected graph $F$ of order 2 and so we proceed to consider $F$-domination for 2-stratified oriented connected graphs of higher order. This leads us to 2-stratified paths of order 3. There are six of these, as shown in Figure 2.

$$H = H_1 \quad H_2 \quad H_3 \quad H_4 \quad H_5 \quad H_6$$

Figure 2: Six 2-stratifcations of $\vec{P}_3$

As with graphs, each of these 2-stratifed oriented graphs in Figure 2 gives rise to a domination parameter in digraphs. However, we are especially interested in the 2-stratified oriented graph $H_1$. Not only is the $H_1$-domination number a new domination parameter, it also possesses several interesting features. For this reason, $H_1$-domination was studied in [?]. To simplify the notation, we write $H = H_1$. Necessarily, the only blue vertex in $H$ is the root of $H$. Since $H$ has two red vertices, it follows by (??) that if $D$ is a digraph of order $n \geq 2$, then

$$2 \leq \gamma_H(D) \leq n.$$  

(2)
To illustrate $H$-domination, consider the tournament $T$ in Figure 3. Since the red-blue coloring of $T$ shown in Figure 3 is an $H$-coloring of $T$ with three red vertices, $\gamma_H(T) \leq 3$. Therefore, either $\gamma_H(T) = 2$ or $\gamma_H(T) = 3$. In fact, it can be verified that $\gamma_H(T) = 3$.

![Figure 3: An $H$-coloring of an oriented graph](image)

**2 Some Known Results**

In this section, we summarize some elementary and known results on $H$-domination in oriented graphs (see [?]). Some additional definitions will be useful in what follows. Let $c$ be an $H$-coloring of an oriented graph $D$ and let $w$ be a blue vertex of $D$. Necessarily, $w$ is the terminal vertex of a directed red-red-blue path $P_3$, say $P_3 = u, v, w$. In this case, we say that $w$ is $H$-dominated by $v$ or that $v$ $H$-dominates $w$. That is, in this context, every blue vertex $w$ of $D$ is $H$-dominated by some red vertex $v$ and so $w$ is adjacent from $v$, which, in turn, is adjacent from another red vertex. Consequently, if a red vertex $v$ $H$-dominates a blue vertex $w$, then $v$ is adjacent to $w$ and adjacent from another red vertex.

The following three lemmas are useful.

**Lemma 2.1** Let $v$ be a vertex in an oriented graph $D$.

(a) If $\text{id} v = 0$, then $v$ cannot dominate or $H$-dominate any other vertex in $D$.

(b) If $\text{id} v = 0$, then $v$ can neither $H$-dominate nor be $H$-dominated by any other vertex in $D$.

**Lemma 2.2** Let $v$ be a vertex in an oriented graph $D$ with $\text{id} v = 1$ and let $c$ be an $H$-coloring of $D$.

(a) If $(u, v)$ is an arc of $D$, then at least one of $u$ and $v$ must be colored red by $c$.

(b) If $(u, v)$ is an arc of $D$, $\text{id} u = 1$, and $(w, u)$ is an arc of $D$, then at least two of $u$, $v$, and $w$ must be colored red by $c$.

**Lemma 2.3** Let $D$ be a connected oriented graph of order $n$ and let $\Delta$ be the maximum outdegree among all vertices of $D$ with positive indegree. Then $\gamma_H(D) \leq n - \Delta$.

The following is an immediate consequence of Lemma 2.3.

**Corollary 2.4** Let $I$ be the set of all vertices of an oriented graph $D$ with indegree 0. Then
(a) \( I \) belongs to every dominating set of \( D \), and

(b) \( I \subseteq R_c \) for every \( H \)-coloring \( c \) of \( D \).

Recall that if \( D \) is a nontrivial oriented graph of order \( n \) and \( \gamma_H(D) = k \), then \( 2 \leq k \leq n \). It was shown in [?] that that every pair \( k, n \) of integers with \( 2 \leq k \leq n \) is realizable as the \( H \)-domination number and the order of some connected oriented graph, respectively.

**Theorem 2.5** For each pair \( k, n \) of integers with \( 2 \leq k \leq n \), there exists a connected oriented graph \( D \) of order \( n \) with \( \gamma_H(D) = k \).

A vertex \( v \) is said to openly dominate (or openly out-dominate) all vertices adjacent from \( v \). A set \( S \subseteq V(D) \) is an open dominating set for \( D \) if every vertex in \( D \) is openly dominated by some vertex in \( S \). The open domination number \( \gamma_o(D) \) is the minimum cardinality of an open dominating set of \( D \). Then the open domination number \( \gamma_o(D) \) is defined for an oriented graph \( D \) if and only if \( \text{id} x \geq 1 \) for every vertex \( x \) in \( D \). The relationships among \( H \)-domination number, domination number, and open domination number of an oriented graph were established in [?].

**Theorem 2.6** Let \( D \) be an oriented graph for which \( \gamma_o(D) \) is defined. For the 2-stratification \( H \) of \( P_3 \),

\[
\gamma(D) \leq \gamma_H(D) \leq \gamma_o(D) \leq \left\lfloor \frac{3\gamma_H(D)}{2} \right\rfloor.
\]

Furthermore, for each integer \( k \geq 3 \), there is an oriented graph \( D \) for which

\[
\gamma(D) = \gamma_H(D) = \gamma_o(D) = k.
\]

By Theorem ??, if \( D \) is a connected oriented graph with \( \gamma(D) = a \) and \( \gamma_H(D) = b \), then \( a \leq b \) and \( b \geq 2 \). It was shown in [?] that every pair \( a, b \) of positive integers with \( a \leq b \) and \( b \geq 2 \) is realizable as the domination number and \( H \)-domination number of some connected oriented graph \( D \), as we state next.

**Theorem 2.7** For every pair \( a, b \) of positive integers with \( a \leq b \) and \( b \geq 2 \), there exists a connected oriented graph \( D \) such that \( \gamma(D) = a \) and \( \gamma_H(D) = b \).

Although every pair \( a, b \) of positive integers with \( a \leq b \) and \( b \geq 2 \) is realizable as the domination number and \( H \)-domination number of some connected oriented graph, this is not the case for the \( H \)-domination number \( \gamma_H \) and open domination number \( \gamma_o \). The following characterization of those pairs \( b, c \) of integers with \( 2 \leq b \leq c \) that are realizable as the \( H \)-domination number and open domination number of some connected oriented graph was established in [?].

**Theorem 2.8** Let \( b \) and \( c \) be integers with \( 2 \leq b \leq c \). Then there exists a connected oriented graph \( D \) such that \( \gamma_H(D) = b \) and \( \gamma_o(D) = c \) if and only if

\[
(b, c) = (2, 3) \quad \text{or} \quad 3 \leq b \leq c \leq \left\lfloor \frac{3b}{2} \right\rfloor.
\]
3 Realizable Triples in $H$-Domination

We have seen that the $H$-domination number of a connected oriented graph is intermediate to its domination number and open domination number. We have also seen that

1. every pair $a, b$ of positive integers with $a \leq b$ and $b \geq 2$ is realizable as the domination number and $H$-domination number of some connected oriented graph and
2. every pair $b, c$ of positive integers with $(b, c) = (2, 3)$ or $3 \leq b \leq c \leq \left\lfloor \frac{3b}{2} \right\rfloor$ is realizable as the $H$-domination number and open domination number of some connected oriented graph.

This suggests the following question.

**Problem 3.1** For which triples $a, b, c$ of positive integers with $a \leq b \leq c$, does there exist a connected oriented graph $D$ such that $\gamma(D) = a$, $\gamma_H(D) = b$, and $\gamma_o(D) = c$?

Since for every connected oriented graph $D$ for which $\gamma_o(D)$ exists,

$$1 \leq \gamma(D) \leq \gamma_H(D) \leq \gamma_o(D) \leq \min \left\{ 2\gamma(D), \left\lfloor \frac{3\gamma_H(D)}{2} \right\rfloor \right\}$$

and $\gamma_o(D) \geq 3$, by a triple we mean an ordered triple $(a, b, c)$ of positive integers with

$$a \leq b \leq c \leq \min \left\{ 2a, \left\lfloor \frac{3b}{2} \right\rfloor \right\}$$

and $c \geq 3$. (3)

Furthermore, we define a triple $(a, b, c)$ to be realizable if there is a connected oriented graph $D$ such that

$$\gamma(D) = a, \gamma_H(D) = b, \text{ and } \gamma_o(D) = c.$$

For example, $(2, 2, 4)$ is nonrealizable by Theorem ???. Certainly, there are infinitely many triples that are not realizable. The following is a consequence of Theorem ???.

**Corollary 3.2** Any triple $(1, b, c)$ is nonrealizable for all positive integers $b$ and $c$.

On the other hand, there are infinitely many realizable triples. The following is a consequence of Theorem ???.

**Corollary 3.3** Every triple $(k, k, k)$ is realizable for each integer $k \geq 3$.

There are infinitely many other realizable triples, as we show next.

**Theorem 3.4** For each triple $a, b, c$ of positive integers with

$$3 \leq a < b < c \leq \min \left\{ 2a, \left\lfloor \frac{b}{2} \right\rfloor + a - 1 \right\},$$

there exists a connected oriented graph $D$ such that

$$\gamma(D) = a, \gamma_H(D) = b, \text{ and } \gamma_o(D) = c.$$ (5)

**Proof.** For a triple $a, b, c$ of positive integers satisfying (5), let

$$k = b - a, \ell = c - b, \text{ and } s = 2a + b - 2c - 1.$$
Thus \( a = k + 2\ell + s + 1, \ b = 2k + 2\ell + s + 1 \) and \( c = 2k + 3\ell + s + 1 \). It then follows by (??) that \( k, \ell, \) and \( s \) are positive integers.

For each \( i \) with \( 1 \leq i \leq k \), let \( G_i \) be a directed 4-cycle \( x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}, x_{i,1} \). For each \( j \) with \( 1 \leq j \leq \ell \), let \( H_j \) be the oriented graph obtained from the directed 3-cycle: \( v_{j,1}, v_{j,2}, v_{j,3}, v_{j,1} \) by adding a new vertex \( v_{j,0} \) and a new arc \((v_{j,1}, v_{j,0})\). For each \( t \) with \( 1 \leq t \leq s \), let \( I_t \) be the directed path \( y_{t,0}, y_{t,1}, y_{t,2} \) of order 3. Then \( D \) is the oriented graph obtained from the \( k \) copies of \( G_i \), the \( \ell \) copies of \( H_j \), and the \( s \) copies of \( I_t \) by identifying all the vertices \( x_{i,1} (1 \leq i \leq k), v_{j,0} (1 \leq j \leq \ell) \), and \( y_{t,0} (1 \leq t \leq s) \) and labeling the identified vertex \( v \).

We show that \( D \) has the desired properties described in (??). Let

\[
X_2 = \{x_{i,2} : 1 \leq i \leq k\}, \\
X_3 = \{x_{i,3} : 1 \leq i \leq k\}, \\
V = \{v_{j,1}, v_{j,3} : 1 \leq j \leq \ell\}, \\
V_2 = \{v_{j,2} : 1 \leq j \leq \ell\}, \\
T = \{y_{t,1} : 1 \leq t \leq s\}.
\]

We first show that \( \gamma(D) = a \). Since the set \( V \cup X_3 \cup T \cup \{v\} \) is a dominating set,

\[
\gamma(D) \leq |V \cup X_3 \cup T \cup \{v\}| = 2\ell + k + s + 1 = a.
\]

To show that \( \gamma(D) \geq a \), let \( S \) be a minimum dominating set of \( D \).

For each \( i \) (\( 1 \leq i \leq k \)), since

(i) each vertex \( x_{i,2} \) is only dominated by \( v = x_{i,1} \) or \( x_{i,2} \)

(ii) each vertex \( x_{i,3} \) is only dominated by \( x_{i,2} \) or \( x_{i,3} \), and

(iii) each vertex \( x_{i,4} \) is only dominated by \( x_{i,3} \) or \( x_{i,4} \),

it follows that no single vertex can dominate the vertices in the set \( V(G_i) \). Thus at least two vertices in the set \( V(G_i) \) must belong to \( S \) for \( 1 \leq i \leq k \).

For each \( j \) with \( 1 \leq j \leq \ell \), since

(1) every vertex in \( V(H_j) - \{v_{j,0}\} \) is only dominated by a vertex in \( V(H_j) - \{v_{j,0}\} \), and

(2) no single vertex in \( V(H_j) - \{v_{j,0}\} \) can dominate all vertices in \( V(H_j) - \{v_{j,0}\} \),

it follows that \( S \) must contain at least two vertices in the set \( V(H_j) - \{v_{j,0}\} \) for each \( j \) with \( 1 \leq j \leq \ell \).

For each \( t \) with \( 1 \leq t \leq s \), since each vertex \( y_{t,2} \) can only be dominated by \( y_{t,1} \) or by \( y_{t,2} \), it follows that at least one vertex in \( V(I_t) - \{y_{t,0}\} \) must belong to \( S \) for \( 1 \leq t \leq s \).

Therefore,

\[
\gamma(D) = |S| \geq 2k + 2\ell + s \geq 2\ell + k + 1 + s = a,
\]

and so \( \gamma(D) = 2\ell + k + 1 + s = a \).

Next, we show that \( \gamma_H(D) = b \). Observe that the set \( V \cup X_2 \cup X_3 \cup T \cup \{v\} \) is an \( H \)-dominating set and so

\[
\gamma_H(D) \leq |V \cup X_2 \cup X_3 \cup T \cup \{v\}| = 2\ell + 2k + s + 1 = b.
\]
To show that $\gamma_H(D) \geq b$, let $c$ be a minimum $H$-coloring of $D$.

For each integer $i$ ($1 \leq i \leq k$), we show that at least two vertices in $V(G_i) - \{x_{i,1}\}$ are colored red by $c$. For $1 \leq i \leq k$, since $x_{i,4}$ is only $H$-dominated by $x_{i,3}$ or by itself, we consider these two cases.

Case 1. $x_{i,4}$ is $H$-dominated by $x_{i,3}$. Then $x_{i,2}$ and $x_{i,3}$ must be colored red by $c$, and so at least two vertices in each set $V(G_i) - \{x_{i,1}\}$ are in $R_c$, where $1 \leq i \leq k$.

Case 2. $x_{i,4}$ is $H$-dominated by itself. Then $x_{i,4}$ is colored red by $c$ for all $1 \leq i \leq k$. Moreover, $x_{i,3}$ is only $H$-dominated by itself, or by $x_{i,2}$, thus at least one vertex in each set $\{x_{i,2}, x_{i,3}\}$ ($1 \leq i \leq k$) is colored red by $c$.

Therefore, at least two vertices in $V(G_i) - \{x_{i,1}\}$ ($1 \leq i \leq k$) are colored red by $c$.

For each $j$ with $1 \leq j \leq \ell$, observe that a vertex in $V(H_j) - \{v_{j,0}\}$ is only $H$-dominated by a vertex in $V(H_j) - \{v_{j,0}\}$ and no single vertex in $V(H_j) - \{v_{j,0}\}$ can $H$-dominate all vertices in $V(H_j) - \{v_{j,0}\}$. It follows that $R_c$ must contain at least two vertices in the set $V(H_j) - \{v_{j,0}\}$ for $1 \leq j \leq \ell$.

For each $t$ with $1 \leq t \leq s$, the vertex $y_{t,2}$ is only $H$-dominated by itself, or by $y_{t,1}$. Thus at least one of $y_{t,1}$ and $y_{t,2}$ must be colored red by $c$. If $y_{t,2}$ is colored red, then $y_{t,1}$ is only $H$-dominated by itself or by $v$, and so at least one of $v$ and $y_{t,1}$ must be colored red. If $y_{t,2}$ is colored blue, then $y_{t,1}$ and $v$ must be colored red. Thus in either case, at least two vertices in $\{v, y_{t,1}, y_{t,2}\}$ must be colored red for $1 \leq t \leq s$. This implies that at least $s + 1$ vertices in $\sum_{t=1}^{s} V(I_t)$ must be red.

Therefore,

$$\gamma_H(D) \geq 2k + 2\ell + (s + 1) = b$$

and so $\gamma(D) = b$.

Finally, we show that $\gamma_o(D) = c$. Since the set $V \cup V_2 \cup X_2 \cup X_3 \cup T \cup \{v\}$ is an open dominating set of $D$, it follows that

$$\gamma_o(D) \leq |V \cup V_2 \cup X_2 \cup X_3 \cup T \cup \{v\}| = 3\ell + 2k + s + 1 = c.$$ 

To show that $\gamma_o(D) \geq c$, let $S_o$ be a minimum open dominating set of $D$.

For each $i$ ($1 \leq i \leq k$), since the vertex $x_{i,p}$, where $2 \leq p \leq 4$, is only openly dominated by the vertex $x_{i,p-1}$, it follows that $\{x_{i,1} = v, x_{i,2}, x_{i,3}\} \subseteq S_o$. Therefore, $v$ and at least two vertices in the set $V(G_i) - \{x_{i,1}\}$ must belong to $S_o$ for all $i$ with $1 \leq i \leq k$.

For each $j$ with $1 \leq j \leq \ell$, the vertex $v_{j,p}$ is only openly dominated by $v_{j,p-1}$, where $1 \leq p \leq 3$, and the second subscript of the vertex is expressed as an integer modulo 3. This implies that $\{v_{j,1}, v_{j,2}, v_{j,3}\} \subseteq S_o$ for all $j$ with $1 \leq j \leq \ell$. Hence at least three vertices in $V(H_j) - \{v_{j,0}\}$ must also belong to $S_o$ for $1 \leq j \leq \ell$.

For each $t$ with $1 \leq t \leq s$, each vertex $y_{t,2}$ is only openly dominated by $y_{t,1}$. Hence $y_{t,1}$ belongs to $S_o$ for $1 \leq t \leq s$.

Therefore,

$$\gamma(D) \geq |S_o| = 2k + 3\ell + s + 1 = c,$$

and so $\gamma(D) = 2k + 3\ell + s + 1 = c.$

\[\square\]

**Theorem 3.5** For each triple $(a, a, c)$ of positive integers with $2 \leq a \leq c \leq \left\lfloor \frac{3a}{2} \right\rfloor$, there exists a connected oriented graph $D$ such that $\gamma(D) = \gamma_H(D) = a$ and $\gamma_o(D) = c$. 


Proof. Since $\gamma(D) = \gamma_H(D) = a$, it follows that \[ \min\left\{2\gamma(D), \left\lfloor \frac{3\gamma(D)}{2} \right\rfloor \right\} = \min\left\{2a, \left\lfloor \frac{3a}{2} \right\rfloor \right\} = \left\lfloor \frac{3a}{2} \right\rfloor. \]

By Theorem 3.6, we can assume that $b > a$. We consider two cases.

Case 1. $\gamma(D) = \gamma_H(D) = a$ and $\gamma_0(D) = b = a + 1$. If $a = 2$ then $b = 3$, and the oriented graph $D = \overrightarrow{K}_3$ has the desired properties. If $a = 3$ then $b = 4$. The oriented graph $D$ obtained from a copy of $\overrightarrow{K}_3 : u_1, u_2, u_3, w_1$ and a copy of $u_1v$ by adding one new arc $u_1v$ has the desired properties. Thus $a \geq 4$ and so $b \geq 5$. Let $D_1$ be obtained form a copy the star $\overrightarrow{K}_{1,a-2}$ with the vertices $u, v_1, v_2, \ldots, v_{a-2}$, where $u$ is the central vertex of the star, by subdividing each edge $uv_i$ for $1 \leq i \leq a-3$ so that each edge $uv_i$ is replaced by the oriented path $u, u_i, v_i$. The digraph $D$ is obtained from $D_1$ and a copy of $\overrightarrow{K}_3 : u_1, u_2, u_3, w_1$ by adding a new arc $w_1u$. Then the set \[ U = \{u, u_1, u_2, \ldots, u_{a-3}, w_1, w_3\} \]
is both a minimum dominating set and a minimum $H$-dominating set, and \[ U \cup \{w_2\} \]
is a minimum open dominating set. Thus $\gamma(D) = \gamma_H(D) = a$ and $\gamma_0(D) = b = a + 1$.

Case 2. $\gamma(D) = \gamma_H(D) = a$ and $\gamma_0(D) = b = a + 1$. Since \[ a + 2 \leq b \leq \left\lfloor \frac{3a}{2} \right\rfloor, \]
it follows that $a \geq 4$ and $b \geq 6$. Let $k = b - a - 1 \geq 1$ and $\ell = 3a - 2b + 1 \geq 1$. Let $H_1$ be the digraph obtained from a copy of the oriented star $\overrightarrow{K}_{1,\ell} : u, v_1, v_2, \ldots, v_{\ell}$ by where $u$ is the central vertex of the star, by subdividing the $\ell$ arcs $(u, v_i)$ for $1 \leq i \leq \ell$ so that each arc $(u, v_i)$ is replaced by the oriented path $u, u_i, v_i$. Let $H_2$ be the digraph obtained from $k$ disjoint copies of the complete oriented graph of order three $\overrightarrow{K}_{3, j} : x_j, y_j, z_j, x_j (\ell + 1 \leq j \leq \ell + k)$ by adding $k$ new vertices $v_j$ together with $k$ new arcs $(x_j, y_j)$. Then the digraph $D$ is obtained from $H_1$ and $H_2$ by adding $k$ new arcs $(v_j, u) (\ell + 1 \leq j \leq \ell + k)$. Then the set \[ U = \{u, u_1, u_2, \ldots, u_\ell\} \cup \{x_j, z_j : \ell + 1 \leq j \leq \ell + k\} \]
is both a minimum dominating set and a minimum $H$-dominating set, and \[ U \cup \{y_1, y_2, \ldots, y_k\} \]
is a minimum open dominating set. Thus $\gamma(D) = \gamma_H(D) = a$ and $\gamma_0(D) = b = a + k$ as desired.

**Theorem 3.6** For each triple $(a, b, b)$ of positive integers with $2 \leq a \leq b \leq 2a$, there exists a connected oriented graph $D$ such that $\gamma(D) = a$ and $\gamma_H(D) = \gamma_0(D) = c$.

**Proof.** By Theorem 3.6, we can assume that $b > a$. We consider two cases.

Case 1. $b = a + 1$. If $a = 2$ then $b = 3$ and oriented graph $D$ obtained from the directed cycle $\overrightarrow{C}_3 : u_1, u_2, u_3, u_1$, by adding two new vertices $v_1$ and $v_2$ together with two new edges $(u_1, v_1)$ ($1 \leq i \leq 2$) has the desired properties.

If $a = 3$ then $b = 4$ and the oriented graph $D$ obtained from the directed cycle $\overrightarrow{C}_4 : u_1, u_2, u_3, u_4, u_1$, by adding three new vertices $v_1, v_2$ and $v_3$ together with three new edges $(u_1, v_1)$ ($1 \leq i \leq 3$) has the desired properties.

Thus we may assume that $a \geq 4$. Then the oriented graph $D$ is obtained from the directed 4 cycle $\overrightarrow{C}_4 : u_1, u_2, u_3, u_4, u_1$ by...
1. adding 3 new vertices $v_1, v_2$ and $v_4$ together with 3 new arcs $(u_i, v_i)$ ($i = 1, 2, 4$), and
2. adding $a - 3 \geq 1$ new copies of the directed path of order two $P_{2,j} : x_j, y_j$ together with $k$ new arcs $(u_1, x_j)$ ($1 \leq j \leq a - 3$).

Then the set

$$U = \{u_1, u_2, u_4\} \cup \{x_j : 1 \leq j \leq a - 3\}$$

is a minimum dominating set, and the set

$$U \cup \{u_3\}$$

is both a minimum $H$-dominating set and a minimum open dominating set. Thus $\gamma(D) = a$ and $\gamma_H(D) = \gamma_o(D) = b = a + 1$.

Case 2. $b \geq a + 2$. Let $k = b - a \geq 2$ and $\ell = 2a - b \geq 0$. The oriented graph $D$ is obtained from the directed $2k$ cycle $\vec{C}_{2k} : u_1, v_1, u_2, v_2, \ldots, u_k, v_k, u_1$, by

1. adding $k \geq 2$ new vertices $w_i$ together with $k$ new arcs $(u_i, w_i)$ ($1 \leq i \leq k$), and
2. adding $\ell \geq 0$ new copies of the directed path of order two $\vec{P}_{2,j} : x_j, y_j$ together with $\ell$ new arcs $(u_1, x_j)$ ($1 \leq j \leq \ell$).

Then the set

$$U = \{u_i, x_j : 1 \leq i \leq k, 1 \leq j \leq \ell\}$$

is a minimum dominating set, and the set

$$U \cup \{v_i : 1 \leq i \leq k\}$$

is both and minimum $H$-dominating set and a minimum open dominating set. Thus $\gamma(D) = k + \ell = a$ and $\gamma_H(D) = \gamma_o(D) = 2k + \ell = b$ as desired.

\textbf{Conjecture 3.7} A triple $(a, b, c)$ is realizable if and only if $(a, b, c) \neq (2, 2, c)$ for some integer $c \geq 4$ and $(a, b, c) \neq (1, b, c)$ for $c \geq b \geq 1$.

\textbf{References}


