

Closed 3-stop Center and Periphery in Graphs

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Abstract A delivery person must leave the central location of the business, deliver packages at a number of addresses, and then return. Naturally, he/she wishes to reduce costs by finding the most efficient route. This motivates the following:

Given a set of k distinct vertices $\mathcal{S} = \{x_1, x_2, \dots, x_k\}$ in a simple graph G , the closed k -stop-distance of set \mathcal{S} is defined to be

$$d_k(\mathcal{S}) = \min_{\theta \in \mathcal{P}(\mathcal{S})} (d(\theta(x_1), \theta(x_2)) + d(\theta(x_2), \theta(x_3)) + \dots + d(\theta(x_k), \theta(x_1))),$$

where $\mathcal{P}(\mathcal{S})$ is the set of all permutations of \mathcal{S} . That is the same as saying that $d_k(\mathcal{S})$ is the length of a shortest closed walk through the vertices $\{x_1, \dots, x_k\}$.

The closed 2-stop distance is twice the standard distance between two vertices. We study the closed k -stop center and closed k -stop periphery of a graph, for $k = 3$.

Keywords Central appendage number, peripheral appendage number, Steiner distance

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1 Definitions and Introduction

In this paper, all graphs are simple (i.e., no loops or multiple edges). For vertices u and v of a graph G , let $d(u, v)$ denote the standard distance from u to v (i.e., the length of a shortest path from u to v). Let $G = (V(G), E(G))$ be a graph of order n ($|V(G)| = n$) and size m ($|E(G)| = m$). Let $x \in V(G)$. Recall that the eccentricity $e(x)$ of a vertex x is $\max_{v \in V(G), v \neq x} d(x, v)$.

Let G and H be two graphs. The join of G and H , namely $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : \forall u \in V(G), \forall v \in V(H)\}$.

The disjoint union of G and H , namely $G \cup H$, is the graph whose $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

The *closed k -stop distance* of a set with k vertices $\mathcal{S} = \{x_1, x_2, x_3, \dots, x_k\}$ ($k \geq 2$) in a connected graph G is defined to be

$$d_k(\mathcal{S}) = \min_{\theta \in \mathcal{P}(\mathcal{S})} (d(\theta(x_1), \theta(x_2)) + d(\theta(x_2), \theta(x_3)) + \dots + d(\theta(x_k), \theta(x_1))),$$

where $\mathcal{P}(\mathcal{S})$ is the set of all permutations of \mathcal{S} and $x_i \neq x_j, 1 \leq i, j \leq k$. This concept was introduced in [1] and expanded in [2]. That is the same as saying that $d_k(x_1, x_2, \dots, x_k)$ is the length of a shortest closed walk through the vertices x_1, x_2, \dots, x_k . The *closed k -stop eccentricity* $e_k(x)$ of a vertex x in a connected graph G is $\max \{d_k(\mathcal{S}) | x \in \mathcal{S}, \mathcal{S} \subseteq V(G), |\mathcal{S}| = k\}$. For a connected graph G , the minimum closed k -stop eccentricity among the vertices of G is the *closed k -stop radius*, that is, $rad_k(G) = \min_{x \in V(G)} e_k(x)$. The maximum closed k -stop eccentricity among the vertices of G is the *closed k -stop diameter*, that is, $diam_k(G) = \max_{x \in V(G)} e_k(x)$. Equivalently, $diam_k(G) = \max \{d_k(\mathcal{S}) | \mathcal{S} \subseteq V(G), |\mathcal{S}| = k\}$. For our purposes, the definition based on the k -stop eccentricities is more useful.

Note that if $k = 2$, then $d_2(\{x_1, x_2\}) = 2d(x_1, x_2)$. We thus consider $k \geq 3$. In particular, the closed 3-stop distance of x, y and z ($x \neq y, x \neq z, y \neq z$) is

$$d_3(\{x, y, z\}) = d(x, y) + d(y, z) + d(z, x).$$

For simplicity, we will write $d_3(x, y, z)$ instead of $d_3(\{x, y, z\})$.

The *closed 3-stop eccentricity* $e_3(x)$ of a vertex x in a connected graph G is the maximum closed 3-stop distance of a set of three vertices containing x , that is,

$$e_3(x) = \max_{y, z \in V(G)} (d(x, y) + d(y, z) + d(z, x)).$$

The minimum closed 3-stop eccentricity among the vertices of G is the *closed 3-stop radius*, that is, $rad_3(G) = \min_{x \in V(G)} e_3(x)$. The maximum closed 3-stop eccentricity among the vertices of G is the *closed 3-stop diameter*, that is, $diam_3(G) = \max_{x \in V(G)} e_3(x)$.

The center $C(G)$ of G is the subgraph induced by those vertices of G having minimum eccentricity. For more on standard center of a graph we refer the reader to [3] and [4]. The *closed 3-stop center* $C_3(G)$ of G is the subgraph induced by those vertices of G having minimum closed 3-stop eccentricity [2]. For a given graph G , if there exists a graph H such that $C_3(H) \cong G$, we define the *closed 3-stop central appendage number of a graph G* , $AC_3(G)$, to be the minimum difference $|V(H)| - |V(G)|$ over all graphs H such that $C_3(H) \cong G$. For more on standard central appendage number of a graph we refer the reader to [5].

The periphery $P(G)$ of a graph G is the subgraph induced by the vertices having maximum eccentricity. For more on standard periphery of a graph we refer the reader to [3, 6] and [7]. The *closed 3-stop periphery* $P_3(G)$ of G is the subgraph induced by the vertices having maximum closed 3-stop eccentricities [2]. For a given graph G , if there exists a graph H such that $P_3(H) \cong G$, we define the *closed 3-stop peripheral appendage number of G* , $AP_3(G)$, to be the minimum difference $|V(H)| - |V(G)|$ over all graphs H such that $P_3(H) \cong G$. For more on standard peripheral appendage number of a graph we refer the reader to [8] and [9].

When more than one graph is discussed, such as G and H , we use the notation $d_3^G(x, y, z)$ and $e_3^G(x)$ to represent the closed 3-stop distance of x, y and z and the closed 3-stop eccentricity of x , respectively, in the graph G , and $d_3^H(x, y, z)$ and $e_3^H(x)$ for the corresponding distance and eccentricity in H .

Recall that the Steiner distance of a set \mathcal{S} of vertices is the number of edges in a minimum connected subgraph containing all of the vertices in \mathcal{S} . The closed k -stop distance can be viewed as an alternative method of defining distance for a set of vertices. For references on Steiner distance, see [10–14]. The relationship between Steiner distance and closed k -stop distance was explored in [2].

For other graph theory terminology we refer the reader to [15]. In this paper we study the closed 3-stop central appendage number and the closed 3-stop peripheral appendage number.

We end this section with the following propositions that have appeared in [2] and which will be used in this paper.

Proposition A *Let G be a connected graph of order at least 3. Then*

$$|V(P_3(G))| \geq 3.$$

Proposition B *For any connected graph G , we have*

$$\text{rad}_3(G) \leq \text{diam}_3(G) \leq \frac{3}{2}\text{rad}_3(G).$$

Observation C *If u and v are adjacent vertices in a connected graph, then*

$$|e_3(u) - e_3(v)| \leq 2.$$

For the rest of the paper we consider graphs with at least 3 vertices.

2 The Closed 3-stop Peripheral Appendage Number

We start with the closed 3-stop peripheral appendage number. This number is zero if and only if G is its own closed 3-stop periphery, i.e., G is closed 3-stop self-peripheral. This occurs exactly when every vertex of G has the same closed 3-stop eccentricity, so a closed 3-stop self-peripheral graph is also closed 3-stop self-centered, that is, the graph is its own closed 3-stop center.

Recall that a graph G is *vertex-transitive* if for every pair of vertices $u, v \in V(G)$, there is an automorphism of $V(G)$ which maps u to v .

Observation 2.1 *If G is connected and vertex-transitive, then G is closed 3-stop self-centered and closed 3-stop self-peripheral.*

However, if some vertex v in G has $e_3(v) \leq 4$ and G is not a closed 3-stop self-peripheral graph, then G is not the closed 3-stop periphery of any supergraph H . To see this, first notice that G cannot be a complete graph. Also since G is not complete, for all $x \in V(H) - V(G)$, we will have $e_3^H(x) \geq 4$, while there exists $v \in G$ such that $e_3^H(v) \leq 4$ in the supergraph H . Thus, we may assume that $\text{rad}_3(G) \geq 5$.

The next three propositions characterize the graphs G with $\text{rad}_3(G) \geq 5$ for which the closed 3-stop peripheral appendage number is defined. Recall that for a vertex $u \in V(G)$,

the open neighborhood of u is $N(u) = \{v : uv \in E(G)\}$ and the closed neighborhood of u is $N[u] = N(u) \cup \{u\}$.

Proposition 2.2 *Assume that $\text{rad}_3(G) \geq 5$. Suppose that either (1) for every vertex v in G , the vertices of $V(G) - N[v]$ induce a complete graph, or (2) for every vertex v in G , $V(G) - N[v]$ contains two nonadjacent vertices. Then G is the closed 3-stop periphery of some graph H , with $AP_3(G) \in \{0, 1\}$, both being realizable.*

Proof First, suppose that for every $v \in V(G)$, the subgraph induced by $V(G) - N[v]$ contains two non-adjacent vertices. Let H be the graph obtained from G , $H = K_1 + G$, by adding one extra vertex x that forms the K_1 . Then $e_3^H(x) = 4$, and $e_3^H(y) = 6$, for all $y \in V(G)$.

Suppose now that for every vertex $u \in V(G)$, the vertices in $V(G) - N[u]$ induce a complete graph. Since this is true for every vertex in G , the standard $\text{diam}(G) \leq 3$. Since $\text{rad}_3(G) \geq 5$, it follows that $|V(G) - N[u]| \neq \emptyset$ for all $u \in V(G)$. We can conclude that $\text{diam}(G)$ is either 2 or 3. First, suppose $\text{diam}(G) = 2$. Let u be an arbitrary vertex in $V(G)$. If there is exactly one vertex in $V(G) - N[u]$, it is not adjacent to every vertex of $N(u)$; otherwise, $e_3(u) = 4$. Consider $H = G + K_1$, with a new vertex x corresponding to K_1 . Then $e_3^H(x) = 4$. We now show that $e_3^H(u) = 5$ for every $u \in V(G)$. If $v, w \in N(u)$, then $d_3^H(u, v, w) \leq 4$. If $v, w \in V(G) - N[u]$, then $d_3^H(u, v, w) = 5$. If $v \in N(u)$ and $w \in V(G) - N[u]$, then $d_3^H(u, v, w)$ is either 4 or 5, depending on whether v and w are adjacent. (Note that $d(v, w) \leq \text{diam}(G) = 2$.) If either v or w equals x , then $d_3^H(u, v, w)$ is 3 or 4. Thus, $e_3^H(u) = 5$ for every $u \in V(G)$.

Now, consider the case that for every vertex $u \in V(G)$, the vertices in $V(G) - N[u]$ induce a complete graph, and $\text{diam}(G) = 3$. Let u and v be two vertices of G such that $d_G(u, v) = 3$. Since $N[u] \subseteq V(G) - N[v]$, it follows that $N[u]$ must induce a complete graph. Similarly, $N[v]$ must induce a complete graph, $N[u] \cup N[v] = V(G)$, and $N[u] \cap N[v] = \emptyset$. Let $H = G + K_1$, and label the new vertex x . Then $e_3^H(x) = 4$. If w is any vertex other than u, v and x , then w is adjacent to x and exactly one of u and v , so $d_3^H(u, v, w) = 5$. It follows that $d_3^H(u) = d_3^H(v) = d_3^H(w) = 5$.

A class with $AP_3(G) = 0$ is P_n , with $n \geq 3$, and a class $AP_3(G) = 1$ is an extended star, formed by subdividing each edge of $K_{1,n}$, with $n \geq 3$. Notice that P_n satisfies hypothesis (2) for $n \geq 6$ and the extended star satisfies hypothesis (2) for $n \geq 3$. A class which satisfies hypothesis (1) and has $AP_3(G) = 0$ is formed by starting with a complete graph on at least 5 vertices and removing the edges of a hamiltonian cycle. A class which satisfies (1) and has $AP_3(G) = 1$ is formed by starting with two complete graphs K_r and K_s with $r \geq 2$ and $s \geq 3$ and joining a vertex of K_r with at least one and at most $s - 2$ vertices of K_s . \square

Notice that in a connected graph G if v is a vertex such that $V(G) - N[v]$ induces a complete graph, then the standard eccentricity of v is at most 3.

Proposition 2.3 *Suppose that G is a graph with $\text{diam}_3(G) > \text{rad}_3(G) \geq 5$. Furthermore, suppose that G contains at least one vertex v such that $V(G) - N[v]$ induces a complete graph and $e(v) \leq 2$, and at least one vertex u such that $V(G) - N[u]$ contains a pair of nonadjacent vertices. Then G is not the closed 3-stop periphery of any supergraph H .*

Proof Suppose, to the contrary, that G is the closed 3-stop periphery of some supergraph H .

Claim 1 $e_3^H(v) \leq 5$.

In the graph G , $e_3^G(v) \leq 6$. For $e_3^H(v)$ to be larger than $e_3^G(v)$, there would have to be vertices in $V(H) - V(G)$ with the same closed 3-stop eccentricity as v , which contradicts G being the closed 3-stop periphery of H .

Furthermore, if $e_3^G(v) = 6$, then there must be vertices t and s in $V(G)$ such that $d(v, t) = 2$, $d(v, s) = 1$, and $d(t, s) = 3$. If $e_3^H(v) = 6$, then there must exist vertices t' and s' in $V(G)$ such that $d(v, t') = 2$, $d(v, s') = 1$, and $d(t', s') = 3$ in H as well as in G . Let $x \in V(H) - V(G)$. If x is adjacent to both s' and t' , then $d(s', t')$ is reduced to 2. Otherwise, $d_3(x, s', t') = d(x, s') + d(s', t') + d(t', x) \geq 2d(s', t') \geq 6$, which implies that $e_3(x) \geq 6 = e_3(v)$ and contradicts the fact that $v \in P_3(H)$ and $x \notin P_3(H)$. Thus, we may assume that $e_3(v) \leq 5$ in H , and hence, $e_3^H(x) \leq 4$ for all $x \in V(H) - V(G)$.

Claim 2 Every $x \in V(H) - V(G)$ is adjacent to every $y \in V(G)$.

Suppose there is a vertex $x \in V(H) - V(G)$ and a vertex $y \in V(G)$ such that x and y are not adjacent in H . Since $\text{rad}_3(G) \geq 5$, there must be some vertex z in $V(G)$ which is not adjacent to y . Thus, $d_3^H(x, y, z) = d^H(x, y) + d^H(y, z) + d^H(z, x) \geq 5$. This contradicts $e_3^H(x) \leq 4$. Therefore, we may assume that every vertex $x \in V(H) - V(G)$ is adjacent to every vertex $y \in V(G)$.

Finally, consider the vertex u in $V(G)$ such that $V(G) - N[u]$ contains two vertices q and r which are not adjacent in G . In H , $d_3^H(u, q, r) = 6$. Thus, $e_3^H(u) > e_3^H(v)$, so v cannot be in the closed 3-stop periphery of H . This is a contradiction. \square

We have the following partial result for the remaining cases.

Proposition 2.4 *Suppose that G is a graph with $\text{diam}_3(G) > \text{rad}_3(G) \geq 5$. Furthermore, suppose that G contains at least one vertex u such that $V(G) - N[u]$ contains a pair of non-adjacent vertices and at least one vertex v such that $V(G) - N[v]$ induces a complete graph, but $e(v) = 3$. Let $A = \{w \in V(G) - N[v] \mid d(v, w) = 2\}$, $B = \{w \in V(G) - N[v] \mid d(v, w) = 3\}$, $C = \{c \in N(v) \mid \text{There is some } w \notin N[v] \text{ such that } wc \in E(G)\}$, and $D = N(v) - C$. If there exists $a \in A$ such that $d(a, d) = 2$ for all $d \in D$, then G is not the closed 3-stop periphery of any supergraph H . Otherwise, $AP_3(G) \in \{0, 1, 2\}$.*

Proof Suppose G has vertices u and v as described above and suppose G is $P_3(H)$ for some H , with $G \subseteq H$. Each $y \in V(G)$ must have 3-stop eccentricity $\text{diam}_3(H)$ in H .

Claim 1 First we will show that $\text{diam}_3(H) = 7$ and $\text{rad}_3(H) = 6$ and that G contains a subgraph isomorphic to G_0 , with $V(G_0) = \{a, b, c, d, e\}$ and $E(G_0) = \{ab, bc, ac, cd, de\}$.

Let q and r be non-adjacent vertices in $V(G) - N[u]$. Then, since none of u , q , or r is adjacent in G or in H , $e_3^H(u) \geq 6$. Now, consider v . Define sets A and B as above. Given any two vertices s and t in $N[v] \cup A$, $d_3(v, s, t) \leq 6$. If $s \in A$ and $t \in B$, then $d_3(v, s, t) = 6$. If $s, t \in B$, then $d_3(v, s, t) = 7$. If $s \in N[v]$ and $t \in B$, then $d_3(v, s, t) \leq 8$. Thus, $\text{diam}_3(H) \in \{6, 7, 8\}$.

Let s and t be vertices such that $e_3^H(v) = \text{diam}_3(H) = d_3(v, s, t) \geq 6$. Thus, $d^H(v, s) +$

$d^H(s, t) + d^H(t, v) \geq 6$. These distances cannot all be 2, since if s and t are both in $V(G) - N[v]$, then they would be adjacent to each other. At least one of these three distances must be 3. It follows that, given $x \in V(H) - V(G)$, x cannot be adjacent to every vertex in $V(G)$. If, for example, $d^H(s, t) \geq 3$, then x could be adjacent to at most one of s and t , say s , and $d_3(x, s, t) = d(x, s) + d(s, t) + d(t, x) \geq 1 + 3 + 2 = 6$. The other cases are similar. We have $e_3(x) \geq 6$. Since x is not in $P_3(H)$, $\text{diam}_3(H) \geq 7$.

Furthermore, since $d_3^H(v, s, t) = \text{diam}_3(v, s, t) \geq 7$, either s and t are both in B or without loss of generality $s \in N[v]$ and $t \in B$. Suppose that s and t are both in B . There is a v - s path of length 3 in G , say v, q, r, s . Since r and $t \in V(G) - N[v]$, r and s are both adjacent to t , and $\{v, q, r, s, t\}$ induces a subgraph isomorphic to G_0 . Notice that in this case, $e_3^H(v) = 7 = \text{diam}_3(H)$, so $e_3^H(x) = \text{rad}_3(H)$ must be 6.

Suppose that $s \in N[v]$ and $t \in B$. Then $d^H(v, s) = 1$ and $d^H(v, t) = 3$. We must have $d^H(s, t) = 3$ or 4. If $d^H(s, t) = 4$, then $d_3(x, s, t) = d^H(x, s) + d^H(s, t) + d^H(t, x) \geq 2d^H(s, t) = 8$, which is not possible. Thus, $d^H(s, t) = 3$ and $d^H(v, t) = 3$. Again in this case, $\text{diam}_3(H) = 7$ and $\text{rad}_3(H)$ must be 6. Consider a shortest s - t path, s, q, r, t . Notice that q must be in $N[v]$, since otherwise, q would be adjacent to t , and q cannot be v , since $d^H(v, t) = 3$. Thus, both q and s are adjacent to v , and $\{v, s, q, r, t\}$ induces a subgraph isomorphic to G_0 .

Claim 2 Next we show that, for every vertex $x \in V(H) - V(G)$, there is at least one vertex in $V(G)$ not adjacent to x in H . If $V(H) - V(G) = \{x\}$, then there are at least two vertices in $V(G)$ not adjacent to x in H , but any two vertices in $V(G)$ not adjacent to x must be within distance 2 of each other.

Since there must be vertices $s, t \in V(G)$ such that $d^H(s, t) = 3$, it follows that for any vertex $x \in V(H) - V(G)$, there is at least one vertex, say $w \in V(G)$, not adjacent to x . Suppose $V(H) - V(G) = \{x\}$. If x is not adjacent to any vertex in $N(w) \cap V(G)$, then since $e_3^H(x) = 6$, every vertex of H must lie on some x - w geodesic. But then $e_3^H(w) = 6$, which is a contradiction. Thus, x is adjacent to some $z \in V(G)$ such that z is adjacent to w . If x is adjacent to every vertex of $V(G)$ except for w , we would have $e_3^H(z) \leq 6$, which is a contradiction. There must be another vertex $y \in V(G)$ which is not adjacent to x in H . Now, $6 = e_3^H(x) \geq d_3^H(x, y, w) = 2 + d(y, w) + 2$, so $d(y, w) \leq 2$. Thus, y and w are either adjacent or share a common neighbor in G . If they share a common neighbor, such as z , then there must be a third vertex not adjacent to x which is not adjacent to z .

Define sets A, B, C , and D as in the statement of the proposition. Suppose, for all $a \in A$, there exists $d \in D$ such that $d(a, d) = 3$. Notice that $D \neq \emptyset$ and that for every $c \in C$, there must exist $d \in D$ such that $cd \notin E(G)$. Define H by $V(H) = V(G) \cup \{x, x'\}$ and $E(H) = E(G) \cup \{xy | y \in N[v]\} \cup \{x'y | y \in C \cup A \cup B\} \cup \{xx'\}$. Then $e_3^H(x) = e_3^H(x') = 6$. Notice that $d_3^H(b, c, d) = 7$ for every $b \in B, c \in C$ and $d \in D$ with $cd \notin E(G)$, $d_3^H(v, b, d) = 7$ for every $b \in B$ and $d \in D$, and $d_3^H(a, b, d) = 7$ for every $a \in A, b \in B$ and $d \in D$ with $d(a, d) > 2$. We have $e_3^H(y) = 7$ for every $y \in V(G)$.

However, if there exists $a \in A$ with $d(a, d) = 2$ for all $d \in D$, then $e_3^H(a) \leq 6$. This

contradicts the fact that $e_3^H(x) = \text{diam}_3(H) = 7$ for all $x \in V(G)$ (see Claim 1). \square

We have seen examples of classes with $AP_3(G)$ equal to 0 or 1. We now show that $AP_3(G) = 2$ from Proposition 2.4 is realizable. Let G be the graph with $V(G) = \{v, v', w, w', y, y', u\}$ and $E(G) = \{vw', vy', w'y', vu, v'u, v'w, v'y, wy\}$ (see Figure 1). We claim that $AP_3(G) = 2$. First, we calculate $e_3^G(v) = e_3^G(u) = e_3^G(v') = 8$ and $e_3^G(w) = e_3^G(y) = e_3^G(w') = e_3^G(y') = 9$, so $AP_3(G) \geq 1$. Suppose $AP_3(G) = 1$, with $V(H) - V(G) = \{x\}$. From the proof of Proposition 2.4, there must be at least two vertices not adjacent to x , and any two vertices not adjacent to x must be at distance at most two in G . Since by the proof of Proposition 2.4, $e_3^H(v) = 7$, at least one of v, w , and y is not adjacent to x . Similarly, at least one of v', w' , and y' is not adjacent to x . If v and v' are both not adjacent to x , then w, y, w' and y' must all be adjacent to x , since each one is distance 3 from one of v and v' . But then $e_3^H(u) \leq 6$, which is not possible. The only other possibility, without loss of generality, is that v', w , and y are not adjacent to x , while v, w' and y' are all adjacent to x . But then $e_3^H(x) \geq d_3^H(x, y, y') \geq 8$, which is also not possible.

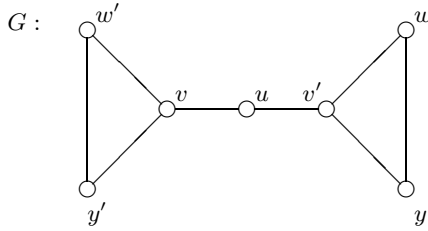


Figure 1 A graph G with $AP_3(G) = 2$

Now, consider the graph H formed by adding two vertices x and x' to G . Add edges $\{xx', xv, xw', xy', xu, x'v', x'w, x'y, x'u\}$. Notice that in H , $e_3(x) = e_3(x') = 6$, while every other vertex has 3-stop eccentricity 7.

We next show that it is possible to have the closed 3-stop periphery and the standard periphery as $P_3(G) \subseteq P(G)$, or $P(G) \subseteq P_3(G)$, or even $P(G) \cap P_3(G) = \emptyset$. For instance, for a path, $P(G) \subseteq P_3(G)$. A C_6 with a pendant edge and vertex added to each of three nonadjacent vertices has $P_3(G) \subseteq P(G)$.

Proposition 2.5 *Let F be a graph with at least two components and let G be a graph with at least three components. Then for every integer $k \geq 3$, there exists a connected graph H such that $P(H) \cong F$, $P_3(H) \cong G$, and $d(P(H), P_3(H)) = k + 3$.*

Proof Let $V(G) = \{x_0, x_1, x_2, x'_0, x'_1, x'_2, u, v, w\} \cup \{u_i, u'_i, v_i, v'_i, w_i, w'_i \mid 1 \leq i \leq k - 1\} \cup \{u_k, v_k, w_k\}$ and let $E(G) = \{x_0x_1, x_1x_2, x'_0x'_1, x'_1x'_2, x_2u_1, x_2v_1, x_2w_1, x'_2u'_1, x'_2v'_1, x'_2w'_1, u_ku, v_kv, w_kw\} \cup \{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1}, u'_iu'_{i+1}, v'_iv'_{i+1}, w'_iw'_{i+1} \mid 1 \leq i \leq k - 1\}$, where $u'_k = u_k, v'_k = v_k$ and $w'_k = w_k$. Then $P(G) = \{x_0, x'_0\}$, with $e(x_0) = \text{diam}(G) = 2k + 4$, while $P_3(G) = \{u, v, w\}$ with $e_3(v) = 6k + 6$. Each of these vertices could be replaced with one or more components of the appropriate graph. Figure 2 shows an example with $k = 3$. In the example, $e(x_0) = e(x'_0) = 10$, $e(x_1) = e(x'_1) = 9$, $e(x_2) = e(x'_2) = e(u) = e(v) = e(w) = 8$,

$e(u_1) = e(v_1) = e(w_1) = e(u'_1) = e(v'_1) = e(w'_1) = e(u_3) = e(v_3) = e(w_3) = 7$, and $e(u_2) = e(v_2) = e(w_2) = e(u'_2) = e(v'_2) = e(w'_2) = 6$. The closed 3-stop eccentricities are $e_3(x_0) = e_3(x'_0) = e_3(u_3) = e_3(v_3) = e_3(w_3) = 22$, $e_3(u) = e_3(v) = e_3(w) = 24$, and the closed 3-stop eccentricity of each of the remaining vertices is 20. \square

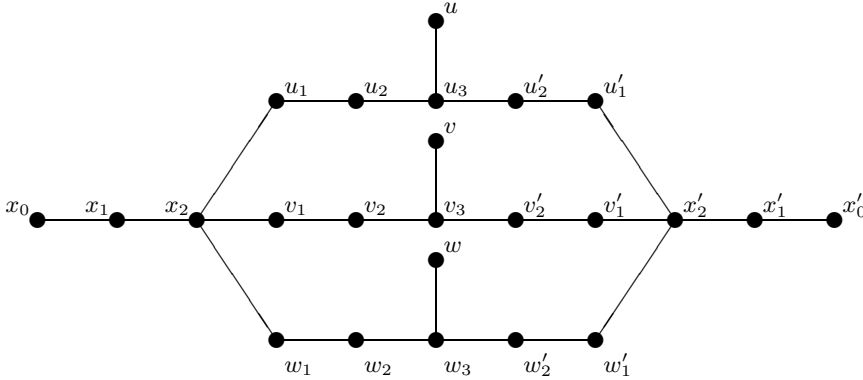


Figure 2 A graph with $P(G) = \{x_0, x'_0\}$ and $P_3(G) = \{u, v, w\}$ at distance 6

3 The Closed 3-stop Central Appendage Number

We now turn our attention to the center of a graph. We first show that every graph can be the closed 3-stop center of some graph, and the closed 3-stop central appendage number is at most 5.

Proposition 3.1 *Let G be any graph. Then there is a supergraph H of G such that $C_3(H) = G$. In general, $|V(H)| - |V(G)| \leq 5$.*

Proof Let G be a graph. We obtain H by adding three new vertices x , y , and z and joining each of them to every vertex in G . Then add vertices u and v and edges uv , ux , and vy . It is straightforward to check that for every $w \in V(G)$, $e_3^H(w) = 6$ in the new graph, while $e_3^H(x) = e_3^H(y) = e_3^H(z) = 7$ and $e_3^H(u) = e_3^H(v) = 7$. \square

As a quick corollary of Proposition 3.1 and Proposition A, we have the following:

Corollary 3.2 *The closed 3-stop central appendage number of a graph G is $AC_3(G) \in \{0, 3, 4, 5\}$.*

A class of graphs with $AC_3(G) = 0$ is the class of paths of order at least 3. We say that a graph is *closed 3-stop self-centered* if every vertex has the same closed 3-stop eccentricity. For any closed 3-stop self-centered graph G , we have $AC_3(G) = 0$. We study the closed 3-stop self-centered graphs, and first we make a few observations.

Observation 3.3 *If G has the property that for every vertex $v \in V(G)$, the vertices in $V(G) - N[v]$ induce a graph with at least two non-adjacent vertices, then $G + \overline{K_n}$ is closed 3-stop self-centered for every integer $n \geq 3$.*

It is straightforward to check that for every vertex $v \in V(G)$, $e_3(v) = 6$. Our next observation illustrates that not every graph is closed 3-stop self-centered (as already observed in

Section 2).

Observation 3.4 *If G has a cut-vertex v such that $G - v$ has at least three components, then G is not closed 3-stop self-centered.*

To see this, suppose $e_3(v) = d(v, y, z)$ and let x be a vertex that is not in the same component of $G - v$ as either y or z . Notice that $e_3(x) \geq d(x, y, z) > d(v, y, z) = e_3(v)$.

Recall that an x - y geodesic is a shortest path between vertex x and vertex y , and the interval $I[x, y]$ is the set of all vertices which lie on some x - y geodesic. That is, $I[x, y] = \{v : v \text{ belongs to some } x\text{-}y \text{ geodesic}\}$.

Proposition 3.5 *If a graph G has an end-vertex x' and G is closed 3-stop self-centered, then there must exist a vertex $y \in V(G)$ such that $d(x', y) = \text{diam}(G)$ and the interval $I[x', y] = V(G)$.*

Proof Suppose that G has an end-vertex x' adjacent to a vertex x , and suppose that G is closed 3-stop self-centered. Let w and z be vertices such that $e_3(x) = d_3(x, w, z)$. If neither w nor z is equal to x' , then $d_3(x', w, z) = d_3(x, w, z) + 2$, which is a contradiction. Thus, $e_3(x) = d_3(x, w, x') = 2d(x, w) + 2$ for some vertex w .

Let y be a vertex furthest from x' , so necessarily $e(x') = d(x', y)$ and $e(x) = d(x, y)$. If there is a vertex $z \notin I[x', y]$, then $d_3(x, y, z) \geq 2d(x, y) + 1 \geq 2d(x, w) + 1 = e_3(x) - 1$. However, then $d_3(x', y, z) = d_3(x, y, z) + 2 \geq e_3(x) + 1$, which is a contradiction. Thus, there is no vertex $z \notin I[x', y]$. \square

By Proposition 3.5, a graph with at least 3 pendant edges cannot be closed 3-stop self-centered. We concentrate next on graphs with one or two pendant edges.

The converse of Proposition 3.5 is not true. The graph in Figure 3 has an end-vertex x' and a vertex y such that $d(x', y) = \text{diam}(G)$ and $I[x', y] = V(G)$, yet $e_3(z) \geq d(z, w, v) = 12$, while $e_3(x') = e_3(x) = 10$.

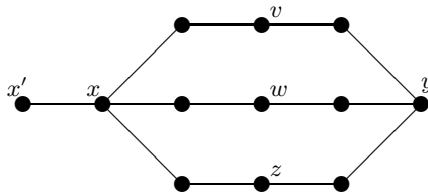


Figure 3 Counterexample to the converse of Proposition 3.5

Corollary 3.6 *If G has two end-vertices x and y and G is closed 3-stop self-centered, then $I[x, y] = V(G)$.*

Corollary 3.7 *If G has two end-vertices x and y and $\text{diam}(G) > d(x, y)$, then G is not closed 3-stop self-centered.*

And so, we now consider graphs that are not closed 3-stop self-centered.

Remark 3.8 For every positive integer $n \geq 3$, $AC_3(K_{1,n}) = 3$.

To see this, we obtain a connected graph H from G by adding 3 vertices x, y, z , so that every

pendant of G is either adjacent to x and y , or to z , such that H is a connected graph. And so the degree of each pendant of the star becomes 2 or 3 in H . The closed 3-stop eccentricities in H are 10 for the vertices x, y , and z , and 8 for the vertices in $V(G)$.

Proposition 3.9 *If G is a graph with no isolated vertices, then $AC_3(G) \leq 3$.*

Proof Consider any spanning forest F of G and let A and B be the partite sets of a bipartition of F . Notice that since G (and F) has no isolated vertices, every vertex of A has at least one neighbor in B and every vertex in B has at least one neighbor in A , and both A and B are nonempty.

Now, add three new vertices x, y and z to G . Join x to every vertex in A , join y to every vertex in B , and join z to every vertex in $V(G)$. Notice that $d_3^H(x, y, z) = 7$, $d_3^H(a, x, y) = 6$, and $d_3^H(b, x, y) = 6$ for every $a \in A$ and $b \in B$. We claim that these distances produce the eccentricities. We can check that $d_3^H(x, b, b') \leq 6$, $d_3^H(x, a, a') \leq 4$, $d_3^H(x, a, b) \leq 6$, $d_3^H(x, a, z) \leq 4$, and $d_3^H(x, z, b) \leq 5$ for every $a, a' \in A$ with $a \neq a'$ and every $b, b' \in B$ with $b \neq b'$. Similarly, every closed 3-stop distance involving y is at most 6 except for $d_3^H(x, y, z)$. Every closed 3-stop distance involving z is at most 6 except for $d_3^H(x, y, z)$. Finally, $d_3^H(a, a', b) \leq 6$ and $d_3^H(a, b, b') \leq 6$ for all $a, a' \in A$ and $b, b' \in B$ using vertex z . \square

Thus, if G has no isolated vertices, then $AC_3(G)$ is either 0 or 3. The graph \overline{K}_2 has closed 3-stop central appendage number 3, since the closed 3-stop center of $K_{2,3}$ consists of the smaller partite set. Similarly, $AC_3(K_1 \cup K_m) = 3$ for any positive integer m , since the graph formed by adding three new vertices x, y and z and joining each of them to every vertex of $K_1 \cup K_m$ has closed 3-stop center $K_1 \cup K_m$. The next result shows that there exist graphs with $AC_3(G) > 3$.

Proposition 3.10 $AC_3(\overline{K}_3) = 5$.

Proof By Proposition 3.1, we have $AC_3(\overline{K}_3) \leq 5$. Since \overline{K}_3 is not connected, $C_3(\overline{K}_3)$ is undefined, and \overline{K}_3 cannot be closed 3-stop self-centered. Thus, by Corollary 3.2, $AC_3(\overline{K}_3) \geq 3$.

Case I Suppose $AC_3(\overline{K}_3) = 3$.

Let H be a (connected) graph of order 6 with $C_3(H) = \overline{K}_3$. Let u, v and w be the vertices of \overline{K}_3 and let x, y , and z be the vertices of $V(H) - V(\overline{K}_3)$. Since u, v , and w form an independent set in H , $d_3^H(u, v, w) \geq 6$. Thus, $d_3^H(x, y, z) = \text{diam}_3(H) \geq 7$. One of $d(x, y)$, $d(y, z)$, and $d(x, z)$ must be at least 3, say without loss of generality $d(y, z) \geq 3$. Furthermore, if any two of x, y , and z are adjacent, say $d(x, y) = 1$, then since H is connected, z must be distance 2 from one of x or y , and $d_3(x, y, z) \leq 6$. Thus, we may assume that no two of x, y , or z are adjacent.

Case IA x is adjacent to u, v , and w .

Notice that y and z cannot be adjacent and cannot have a common neighbor. Since H is a connected graph, each of y and z must be adjacent to at least one of x, u, v , and w , and at most one can be adjacent to each of x, u, v , or w . Thus, we may assume, without loss of generality, that y is adjacent to u and that z is not adjacent to u .

Case IAi $e_3^H(u) = \text{rad}_3(H) \leq 6$.

If z is not adjacent to x , then $d_3^H(u, y, z) = d(u, y) + d(y, z) + d(z, u) \geq 1 + 3 + 3 = 7$, which

contradicts $e_3^H(u) = 6$. If z is adjacent to x , then x is adjacent to every vertex except y and $e_3^H(x) \leq 6$, which contradicts $e_3^H(x) = \text{diam}_3(H)$.

Case IAii $e_3^H(u) = \text{rad}_3(H) \geq 7$.

Thus, $e_3^H(x) = d_3^H(x, y, z) = \text{diam}_3(H) \geq 8$. We have $d(x, y) \leq 2$, and since z must be adjacent to x , v , or w , $d(x, z) \leq 2$. Thus, we must have $d(y, z) = 4$ and z adjacent to v , w , or both. Thus, $d_3^H(x, y, z) = 8$, and $e_H(u) \geq d_3^H(u, y, z) \geq d(y, u) + d(y, z) + d(u, z) \geq 2d(y, z) = 8$. We have a contradiction.

Case IB x is adjacent to at most two of u , v and w .

In this case, without loss of generality, we may assume that x is not adjacent to w . Since $d(y, z) \geq 3$, at most one of y and z is adjacent to w , say y . Thus, $\deg_H(w) = 1$. It follows that $d_3^H(x, w, z) = d_3^H(x, y, z) + 2 = \text{diam}_3(H) + 2$, which is not possible.

Case II Suppose $AC_3(\overline{K}_3) = 4$.

Let H be a connected graph of order 7 with $C_3(H) = \overline{K}_3$, let u , v , and w be the vertices of \overline{K}_3 and let a , b , c , and d be the remaining vertices of H . Without loss of generality, assume that $d^H(a, b)$ is a maximum among $d^H(a, b)$, $d^H(a, c)$, $d^H(a, d)$, $d^H(b, c)$, $d^H(b, d)$, and $d^H(c, d)$. Set $k = d^H(a, b)$. Notice that u , v , and w form an independent set in H , so $\text{rad}_3(H) = d_3^H(u, v, w) \geq 6$ and $\text{diam}_3(H) \geq 7$. It follows that $k \geq 3$. Notice that $\text{rad}_3(H) = e_3(u) \geq d_3^H(u, a, b) \geq 2d(a, b) \geq 2k$.

Since each shortest a - b path has length at least 3 and since u , v , and w are mutually non-adjacent, each shortest a - b path must contain at least one vertex other than a , b , u , v , and w . If both c and d lie on shortest a - b paths, then $\text{diam}_3(H) = 2k$ and $C_3(H) = H$. This is a contradiction. Thus, without loss of generality, c lies on every shortest a - b path, and d does not lie on any shortest a - b path. It also follows that $k \leq 4$.

Case IIA $\text{rad}_3(H) = 2k$.

In this case, each of u , v , and w must lie on some shortest a - b path, since, for example, $e_3^H(u) \geq d_3^H(u, a, b)$.

We now present a proof by contradiction both in the case of $k = 3$ and $k = 4$. If $k = 3$, then without loss of generality, u , v , and w are each adjacent to both a and c , and c is adjacent to b . If $k = 4$, then without loss of generality, u is adjacent to a and c , while v and w are adjacent to b and c . (Any other possibility involves a different partition of $\{u, v, w\}$.) In either case, if d is adjacent to any of c , u , v , or w , then $e_3^H(c) \leq 2k$, which is not possible since $c \notin C_3(H)$. (In the $k = 3$ case, if d is adjacent to c , then $d(a, c) = 2$ and every other vertex is within distance 1 of c , so $e_3^H(c) \leq 6$. In the $k = 3$ case, if d is adjacent to u , v , or w , then only a and d are distance 2 from c , and $d_3^H(c, a, d) = 6$. In the $k = 4$ case, every vertex is within distance 2 of c , so $e_3^H(c) \leq 8$.) Otherwise, d is adjacent only to a or to b , but not to both. But then $d(d, a)$ or $d(d, b)$ is greater than $d(a, b)$, which contradicts our choice of a and b .

Case IIB $\text{rad}_3(H) \geq 2k + 1$.

Thus, $\text{diam}_3(H) \geq 2k + 2$. If $k = 3$, then without loss of generality, a , u , c , b is a shortest

a - b path. Since $e_3^H(c) \geq 8$ and $d^H(c, a) = 2$, $d^H(c, b) = 1$, and each pair of a , b , c , and d is at distance at most 3, we must have $d^H(c, d) = 3$ and $d^H(d, a) = 3$. Now, d cannot be adjacent to a , b , or u , so d is adjacent only to w or v . Say d is adjacent to w ; then w cannot be adjacent to a or c , so w must be adjacent to b . Notice that w cannot be adjacent to anything other than b and d , and d cannot be adjacent to any vertex other than w and possibly v . If d is adjacent to v , then v is also adjacent to only b and d . Now, since $d^H(a, d) = 3$, and b is the only vertex distance 2 from d , we must have a adjacent to b . However, this contradicts our choice of a and b so that $d^H(a, b)$ is a maximum among $d^H(a, b)$, $d^H(a, c)$, $d^H(a, d)$, $d^H(b, c)$, $d^H(b, d)$ and $d^H(a, d)$.

If $k = 4$, then without loss of generality, a shortest a - b path is a, u, c, v, b . We need $e_3^H(c) \geq 10$, but $d^H(a, c) = d^H(b, c) = 2$. We must have $d^H(c, d) \geq 3$, so d is not adjacent to c , u , or v . Now, d can be adjacent to at most one of a or b , and if d is not adjacent to any vertex other than a or b , then $d^H(a, d)$ or $d^H(b, d)$ is greater than $d^H(a, b)$, contradicting our choice of a and b . Thus, d must be adjacent to w , and w is not adjacent to c . If w is adjacent to both a and b , then $d^H(a, b) < 4$. Otherwise, w is adjacent to one of a and b , say b . Since $d^H(a, d) \leq 4$, we must have d adjacent to a . There is now a Hamiltonian cycle in the graph, so $e_3^H(u) \leq 7$. This is not possible. \square

Regarding the standard center and the closed 3-stop center of a graph G , it is also possible that $C_3(G) \subseteq C(G)$, or $C(G) \subseteq C_3(G)$, or even $C(G) \cap C_3(G) = \emptyset$. For instance, $C(P_n) \subseteq C_3(P_n)$ and $C_3(K_{2,n}) \subseteq C(K_{2,n})$ for $n \geq 3$, while $C(G) \cap C_3(G) = \emptyset$ for the graph in Figure 4.

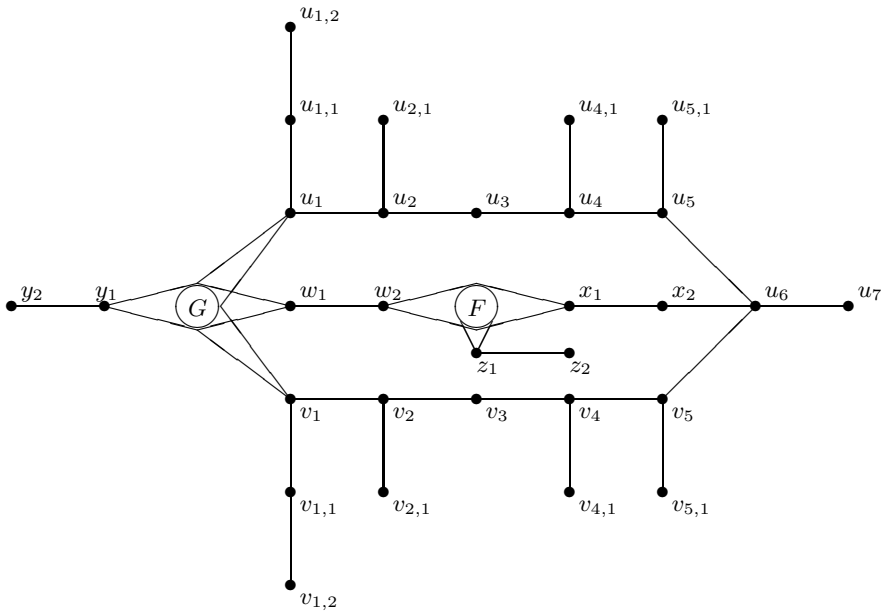


Figure 4 Example of a graph H with $C(H) \cong F$, $C_3(H) \cong G$, and $d(C(H), C_3(H)) = 3$

Proposition 3.11 *For any graphs G and F and any integer $k \geq 3$, there exists a connected graph H such that $C(H) \cong F$, $C_3(H) \cong G$ and $d(C(H), C_3(H)) = k$.*

Proof For $k \geq 3$, we define the graph H as $V(H) = V(G) \cup V(F) \cup \{u_i \mid 1 \leq i \leq 2k + 1\} \cup \{v_i \mid 1 \leq i \leq 2k - 1\} \cup \{w_i, x_i \mid 1 \leq i \leq k - 1\} \cup \{y_i, z_i, u_{1,i}, v_{1,i} \mid 1 \leq i \leq 2\} \cup \{u_{i,1}, v_{i,1} \mid 2 \leq i \leq 2k - 1, i \neq k\}$, and $E(H) = E(F) \cup E(G) \cup \{xx_1, xz_1, xw_{k-1} \mid x \in V(F)\} \cup \{xu_1, xv_1, xw_1, xy_1 \mid x \in V(G)\} \cup \{u_i u_{i+1} \mid 1 \leq i \leq 2k\} \cup \{v_i v_{i+1} \mid 1 \leq i \leq 2k - 2\} \cup \{w_i w_{i+1}, x_i x_{i+1} \mid 1 \leq i \leq k - 2\} \cup \{u_i u_{i,1}, v_i v_{i,1} \mid 1 \leq i \leq 2k - 1, i \neq k\} \cup \{u_{1,1} u_{1,2}, v_{1,1} v_{1,2}, y_1 y_2, z_1 z_2, v_{2k-1} u_{2k}\}$. From the construction of graph H , we have the following:

$$\begin{aligned} e(x) &= 2k \text{ for } x \in V(F), \\ e(x) &> 2k \text{ for } x \in V(H) - V(F), \\ e_3(x) &= 4k + 6 \text{ for } x \in V(G), \\ e_3(x) &> 4k + 6 \text{ for } x \in V(H) - V(G). \end{aligned}$$

See Figure 4 for an example with $k = 3$. □

4 Open Questions

Propositions 2.2, 2.3, and 2.4 characterize the graphs G for which the closed 3-stop central appendage number exists and show that $AP_3(G)$ is 0, 1, or 2 when it exists. An open question is to characterize which graphs are closed 3-stop self-peripheral graphs, which graphs have $AP_3(G) = 1$, and which graphs have $AP_3(G) = 2$. By Proposition 2.5, we know that the periphery and closed 3-stop periphery of a graph may be arbitrarily far apart and that the periphery and closed 3-stop periphery may be any graphs provided that the periphery has at least two components and the closed 3-stop periphery has at least 3 components. However, we have no general construction showing how the periphery and closed 3-stop periphery can overlap. Specifically, given a graph J with subgraphs F and G such that $V(F) \cup V(G) = V(J)$, F has at least 2 components, G has at least 3 components, and $V(F) \cap V(G) \neq \emptyset$, does there exist a graph H with $P(H) \cong F$ and $P_3(H) \cong G$?

By Proposition 3.1 and Corollary 3.2, the closed 3-stop central appendage number exists for every graph G and $AC_3(G) \in \{0, 3, 4, 5\}$. Furthermore, by Proposition 3.9, if G has no isolated vertices, then $AC_3(G) \in \{0, 3\}$. Observations 3.3 and 3.4, Proposition 3.5 and Corollaries 3.6 and 3.7 identify some particular classes of graphs as being closed 3-stop self-centered or not closed 3-stop self-centered, though more work is needed to fully characterize the closed 3-stop self-centered graphs. We have examples of graphs G with $AC_3(G) = 0, 3$, and 5, but have not found an example with $AC_3(G) = 4$. Is it possible for a graph G to have $AC_3(G) = 4$? Finally, Proposition 3.11 shows that the center and closed 3-stop center of a graph can be any graphs and can be arbitrarily far apart. However, we have no general construction showing how the center and closed 3-stop center might overlap. Given a graph J with subgraphs F and G such that $V(F) \cup V(G) = V(J)$ and $V(F) \cap V(G) \neq \emptyset$, does there exist a graph H with $C(H) \cong F$ and $C_3(H) \cong G$?

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