Ch 7: Numerical Linear Algebra

7.4 Matrix Norms and Condition Numbers

This section considers the accuracy of computed solutions to linear systems. At every step, the computer will roundoff, and we would like to know how far the computer answer is from the real answer.

1. Let A be an $m \times n$. Then

   (a) recall the Frobenius norm of an

   $$||A||_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^m a_{ij}^2} = \sqrt{a_{11}^2 + a_{12}^2 + a_{13}^2 + \ldots + a_{1n}^2 + a_{21}^2 + a_{22}^2 + \ldots + a_{2n}^2 + \ldots + a_{mn}^2}$$

   (b) the 1-norm:

   $$||A||_1 = \max_{1 \le j \le n} \sum_{i=1}^m |a_{ij}| =$$

   $$= \max\left(|a_{11}| + |a_{21}| + |a_{31}| + \ldots + |a_{m1}|\right), \left(|a_{12}| + |a_{22}| + \ldots + |a_{m2}|\right), \ldots, \left(|a_{1n}| + |a_{2n}| + \ldots + |a_{mn}|\right),$$

   i.e. it is the maximum among the sums of the absolute values of the elements of each column

   (c) the ∞-norm:

   $$||A||_\infty = \max_{1 \le i \le m} \sum_{j=1}^n |a_{ij}| =$$

   $$= \max\left(|a_{11}| + |a_{12}| + |a_{13}| + \ldots + |a_{1n}|\right), \left(|a_{21}| + |a_{22}| + \ldots + |a_{2n}|\right), \ldots, \left(|a_{m1}| + |a_{m2}| + \ldots + |a_{mn}|\right),$$

   i.e. it is the maximum among the sums of the absolute values of the elements of each row

   (d) the 2-norm:

   $$||A||_2 = \max_{\mathbf{x} \neq \mathbf{0}} \frac{||A\mathbf{x}||_2}{||\mathbf{x}||_2},$$

   where $||x||_2 = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}$ and similarly for $||A\mathbf{x}||_2$.
   or

   $$||A||_2 = \sigma_1,$$

   where σ_1 is the largest singular value in the SVD of $A = U\Sigma V^T$.

   As a consequence

   $$||A^{-1}||_2 = \frac{1}{\sigma_n},$$

   where σ_n is the smallest nonzero singular value in the SVD of $A = U\Sigma V^T$.

2. some properties of the Frobenius norm:

   (a) $||A||_F = \sqrt{\sum_{i=1}^n ||a_j||_2^2}$, i.e. it $||A||_F$ is the square root of the sums of the squares of the standard 2-norms of its columns (similarly it could be computed using rows instead of columns)

(b) $||A\mathbf{x}||_2 \leq ||A||_F||\mathbf{x}||_2$

(c) $||AB||_F \leq ||A||_F||B||_F$

(d) $||A^n||_F \leq ||A||_F^n$ (this is a consequence of the property above)

3. the useful matrix norms need satisfy the triangle inequality $||AB|| \leq ||A||||B||$, so that we can also have $||A^n|| = ||A||^n$ (note that Frobenius norm satisfies it)

4. a matrix norm $\|\cdot\|_M$ and a vector norm $\|\cdot\|_V$ are *compatible* if Cauchy Buniakowsky holds: $\|A\mathbf{x}\|_V \leq \|A\|_M\|x\|_V$. Particularly, $\|\cdot\|_M$ and $\|\cdot\|_V$ are copatible (see $2(b)$ above)

5. for each standard vector norm, we can define a compatible matrix norm, and the matrix norm thus defined is said to be *subordinate to the vector norm.* These norms satisfy the property in 3 above. And so we can define the matrix norms $1(a) - (d)$ above.

6. if A is an $n \times 1$ matrix, i.e. a vector in $\mathbb{R}^n$, then the Frobenius norm is the standard 2-norm used before

7. the *operator norm* $||A||$ is defined as $||A|| = \max_{\mathbf{x}\neq\mathbf{0}} \frac{||A\mathbf{x}||}{||\mathbf{x}||}$

8. a matrix A is *ill-conditioned* if relatively small changes in the input (in the matrix A) can cause large change in the output (the solution of $A\mathbf{x} = \mathbf{b}$), i.e. the solution is not very accurate if input is rounded. Otherwise it is *well-conditioned.*

9. if a matrix is ill-conditioned, then a small roundoff error can have a drastic effect on the output, and so even pivoting techniques such as the one in Example 3 page 423 will not be useful. However, if the matrix is well-conditioned, then the computerized solution is quite accurate. Thus *the accuracy of the solution depends on the conditioning number of the matrix*

10. let $\mathbf{x}$ be the solution of $A\mathbf{x} = \mathbf{b}$, and $\mathbf{x}'$ be the calculated solution of $A\mathbf{x} = \mathbf{b}$, so the

$$\textit{error} \text{ in the solution is } \mathbf{e} = \mathbf{x} - \mathbf{x}', \text{and}$$

$$\textit{relative error} \text{ in the solution is } \frac{\|\mathbf{e}\|}{\|\mathbf{x}\|} = \frac{\|\mathbf{x} - \mathbf{x}'\|}{\|\mathbf{x}\|}.$$

11. Generally, we can't verify the error or relative error (we don't know $\mathbf{x}$), so a possible way to check this is by testing the accuracy back in the system $A\mathbf{x} = \mathbf{b}$:

$$\textit{residual } \mathbf{r} \text{ is the difference } \mathbf{r} = A\mathbf{x} - A\mathbf{x}' = \mathbf{b} - \mathbf{b}', \text{ and}$$

$$\textit{relative residual} = \frac{\|A\mathbf{x} - A\mathbf{x}'\|}{\|A\mathbf{x}\|} = \frac{\|\mathbf{b} - \mathbf{b}'\|}{\|\mathbf{b}\|} = \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}.$$

12. how close is the relative residual to the relative error? We need condition number for that:

$$\textit{condition number: } cond(A) = \|A\|\|A^{-1}\|$$

where the $\|\cdot\|$ above could be any of the norms defined for matrices.

13. so how big can the relative error be? For a matrix A we have

$$\frac{1}{cond(A)}\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \leq \frac{\|\mathbf{e}\|}{\|\mathbf{x}\|} \leq cond(A)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}.$$

    And so if the condition number is close to 1, then the **relative error** $\frac{\|\mathbf{e}\|}{\|\mathbf{x}\|}$, and relative residual $\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}$ will be close. If the condition number is large, then the **relative residual** is much larger than the relative error. The condition number helps find an upper bound of the relative error.

14. If A is nonsingular, and we have the SVD decomposition of A (or at least the eigenvalues of A^TA), we can compute the

    condition number using the 2-norm: $cond_2(A) = \|A\|_2\|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_n}$,

15. Since $\sigma_n(A) = \sqrt{\lambda_n(A^TA)}$ = the smallest nonzero eigenvalue, it follows that the smallest nonzero eigenvalue of A^TA is a measure of how close the matrix is to being singular: if σ_n is small, then $cond_2(A)$ is large. That is, the closer the matrix is to being singular, the more ill-conditioned it is.