Chapter 1
Introduction
-- Formally, a graph $G$ consists of a finite nonempty set $V$ of objects called vertices (the singular is vertex) and a set $E$ of 2-element subsets of $V$ called edges
-- The sets $V$ and $E$ are the vertex set and edge set of $G$, respectively
-- Vertices are sometimes called points or nodes and edges are sometimes called lines
-- There are some who use the term simple graph for what we call a graph
-- Two graphs $G$ and $H$ are equal if $V(G) = V(H)$ and $E(G) = E(H)$, in which case we write $G = H$
-- If $uv$ is an edge of $G$, then $u$ and $v$ are said to be adjacent in $G$
-- The number of vertices in $G$ is often called the order of $G$, while the number of edges is its size
-- A graph with exactly one vertex is called a trivial graph, implying that the order of a nontrivial graph is at least 2
-- A graph $G$ such as the one of Figure 1.3 (a) that has labels on vertices/edges is a labeled graph and Figure 1.3 (b) represents an unlabeled graph
-- If $e = uv$ is an edge of $G$, then the adjacent vertices $u$ and $v$ are said to be joined by the edge $e$
-- The vertices $u$ and $v$ are referred to as neighbors of each other
-- In this case, the vertex $u$ and the edge $e$ (as well as $v$ and $e$) are said to be incident with each other
-- Distinct edges incident with a common vertex are adjacent edges
-- A graph $H$ is called a subgraph of a graph $G$, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$
-- If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then $H$ is a proper subgraph of $G$
-- If a subgraph of a graph $G$ has the same vertex set as $G$, then it is a spanning subgraph of $G$
-- A subgraph $F$ of a graph $G$ is called an induced subgraph of $G$ if whenever $u$ and $v$ are vertices of $F$ and $uv$ is an edge of $G$, then $uv$ is an edge of $F$ as well
-- If $S$ is a nonempty set of vertices of a graph $G$, then the subgraph of $G$ induced by $S$ is the induced subgraph with vertex set $S$. This induced subgraph is denoted by $G[ S ]$. For a nonempty set $X$ of edges, the subgraph $G[ X ]$ induced by $X$ has edge set $X$ and consists of all vertices that are incident with at least one edge in $X$. This subgraph is called an edge-induced subgraph of $G$
-- A **u - v** walk \( W \) in \( G \) is a sequence of vertices in \( G \), beginning with \( u \) and ending at \( v \) such that consecutive vertices in the sequence are adjacent, that is, we can express \( W \) as where \( k \geq 0 \) and \( v_i \) and \( v_{i+1} \) are adjacent for \( i = 0, 1, 2, \ldots, k-1 \). If \( u = v \), then the walk \( W \) is **closed**; while if \( u \neq v \), then \( W \) is **open**.

-- The number of edges encountered in a walk (including multiple occurrences of an edge) is called the **length** of the walk.

-- A walk of length 0 is a **trivial walk**.

-- Borrowing terminology from the Old West, we define a **u - v trail** in a graph \( G \) to be a **u - v walk** in which no edge is traversed more than once.

-- A **u - v walk** in a graph in which no vertices are repeated is a **u - v path**.

-- A **circuit** in a graph \( G \) is a closed trail of length 3 or more.

-- A circuit that repeats no vertex, except for the first and last, is a **cycle**.

-- A **k-cycle** is a cycle of length \( k \).

-- A 3-cycle is also referred to as a **triangle**.

-- A cycle of odd length is called an **odd cycle**; while, not surprisingly, a cycle of even length is called an **even cycle**.

-- The vertices and edges of a trail, path, circuit or cycle in a graph \( G \) form a subgraph of \( G \), also called a **trail**, **path**, **circuit** or **cycle**.

-- If \( G \) contains a **u - v path**, then \( u \) and \( v \) are said to be **connected** and \( u \) is **connected to** \( v \) (and \( v \) is connected to \( u \)).

-- A graph \( G \) is **connected** if every two vertices of \( G \) are connected, that is, if \( G \) contains a **u - v path** for every pair \( u, v \) of vertices of \( G \).

-- A graph \( G \) that is not connected is called **disconnected** (i.e. there are two vertices \( x \) and \( y \), and no \( x - y \) path).

-- A connected subgraph of \( G \) that is not a proper subgraph of any other connected subgraph of \( G \) is a **component** of \( G \).

-- In this case, \( G \) is the union of the graphs \( G_1, G_2, \ldots, G_k \).

-- The **distance** between \( u \) and \( v \) is the smallest length of any \( u - v \) path in \( G \) and is denoted by \( d_G(u, v) \) or simply \( d(u, v) \) if the graph \( G \) under consideration is clear.

-- A **u - v path** of length \( d(u, v) \) is called a **u - v geodesic**.

-- If the vertices of a graph \( G \) of order \( n \) can be labeled (or relabeled) \( v_1, v_2, \ldots, v_n \) so that its edges are \( v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n \), then \( G \) is called a **path**; while if the vertices of a graph \( G \) of order \( n \geq 3 \) can be labeled (or relabeled) \( v_1, v_2, \ldots, v_n \) so that its edges are \( v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n \) and \( v_1 v_n \), then \( G \) is called a **cycle**.

-- A graph \( G \) is **complete** if every two distinct vertices of \( G \) are adjacent.
-- The complement of a graph $G$ is that graph whose vertex set is also $V(G)$ and such that for each pair $u, v$ of distinct vertices of $G$, $uv$ is an edge of if and only if $uv$ is not an edge of $G$

-- The graph then has $n$ vertices and no edges; it is called the empty graph of order $n$

-- A graph $G$ is a bipartite graph if $V(G)$ can be partitioned into two subsets $U$ and $W$, called partite sets, such that every edge of $G$ joins a vertex of $U$ and a vertex of $W$. If this does happen, however, then we call $G$ a complete bipartite graph

-- If either $s = 1$ or $t = 1$, then $K_s, t$ is a star

-- A graph $G$ is a $k$-partite graph if $V(G)$ can be partitioned into $k$ subsets $V_1, V_2, ..., V_k$ (once again called partite sets) such that if $uv$ is an edge of $G$, then $u$ and $v$ belong to different partite sets. If, in addition, every two vertices in different partite sets are joined by an edge, then $G$ is a complete $k$-partite graph

-- The complete $k$-partite graphs are also referred to as complete multipartite graphs

-- The join $G + H$ consists of $G \cup H$ and all edges joining a vertex of $G$ and a vertex of $H$

-- For two graphs $G$ and $H$, the Cartesian product $G \times H$ has vertex set $V(G \times H) = V(G) \times V(H)$, that is, every vertex of $G \times H$ is an ordered pair $(u, v)$, where $u \in V(G)$ and $v \in V(H)$

-- The graph $C_4 \times K_2$ is often denoted by $Q_3$ and is called the 3-cube

-- The graphs $Q_n$ are then called $n$-cubes or hypercubes

-- The $n$-cube can also be defined as that graph whose vertex set is the set of ordered $n$-tuples of $0$s and $1$s (commonly called $n$-bit strings) and where two vertices are adjacent if their ordered $n$-tuples differ in exactly $1$ position

-- A multigraph $M$ consists of a finite nonempty set $V$ of vertices and a set $E$ of edges, where every two vertices of $M$ are joined by a finite number of edges (possibly zero)

-- If two or more edges join the same pair of (distinct) vertices, then these edges are called parallel edges

-- In a pseudograph, not only are parallel edges permitted but an edge is also permitted to join a vertex to itself

-- Such an edge is called a loop

-- A digraph (or directed graph) $D$ is a finite nonempty set $V$ of objects called vertices together with a set $E$ of ordered pairs of distinct vertices

-- The elements of $E$ are called directed edges or arcs
Then u is said to be adjacent to v and v is adjacent from u. If, in the definition of digraph, for each pair u, v of distinct vertices, at most one of (u, v) and (v, u) is a directed edge, then the resulting digraph is an oriented graph. The digraph D is also called an orientation of G.

Chapter 2
Degrees

The degree of a vertex v in a graph G is the number of edges incident with v and is denoted by deg G v or simply by deg v if the graph G is clear from the context. The set N (v) of neighbors of a vertex v is called the neighborhood of v. Thus deg v = | N (v)|. A vertex of degree 0 is referred to as an isolated vertex and a vertex of degree 1 is an end-vertex (or a leaf). The minimum degree of G is the minimum degree among the vertices of G and is denoted by δ(G); the maximum degree of G is denoted by Δ (G). Since every edge of G joins a vertex of U and a vertex of W, it follows that adding the degrees of the vertices in U (or in W) gives the number of edges in G, that is, A vertex of even degree is called an even vertex, while a vertex of odd degree is an odd vertex.

Finding the sharpness of a bound: find an infinite class (such as the paths on n vertices, or an infinite class that can be described by adding conditions) that achieves the equality of the bound. For a vertex v in a multigraph or pseudograph G, the degree deg v of v in G is the number of edges of G incident with v, where there is a contribution of 2 for each loop at v in a pseudograph. For the pseudograph G of Figure 2.4, Figure 2.4 : Illustrating degrees in a multigraph and a digraph. For a vertex v in a digraph D, the outdegree od v of v is the number of vertices of D to which v is adjacent, while the indegree id v of v is the number of vertices of D from which v is adjacent. If δ (G) = Δ (G), then the vertices of G have the same degree and G is called regular. If deg v = r for every vertex v of G, where 0 ≤ r ≤ n − 1, then G is r-regular or regular of degree r.

A 3-regular graph is also referred to as a cubic graph. The graphs K 4, K 3, 3 and Q 3 are cubic graphs; however, the best known cubic graph may very well be the Petersen graph, shown in Figure 2.6. The graphs H r, n described above are called Harary graphs, named for Frank Harary.
If the degrees of the vertices of a graph $G$ are listed in a sequence $s$, then $s$ is called a **degree sequence** of $G$

A finite sequence of nonnegative integers is called **graphical** if it is a degree sequence of some graph

The adjacency matrix of $G$ is the $n \times n$ matrix $A = [a_{ij}]$, where while the incidence matrix of $G$ is the $n \times m$ matrix $B = [b_{ij}]$, where These matrices are shown for the graph $G$ of Figure 2.17

Two $u - v$ walks are considered **equal** if, as sequences, they are identical, term by term

A graph $G$ of order at least 2 is **irregular** if every two vertices of $G$ have distinct degrees

Recall that we defined a nontrivial graph $G$ to be **irregular** if every two vertices of $G$ have distinct degrees

For a graph $G$ and a vertex $v$ of $G$, define the **$F$-degree** $\text{F deg } v$ of $v$ in $G$ as the number of copies (unlabeled subgraphs, induced or not, having the same structure) of $F$ in $G$ that contain $v$

A graph $G$ is **$F$-regular** if every two vertices of $G$ have the same $F$-degree, while $G$ is **$F$-irregular** if every two vertices of $G$ have distinct $F$-degrees

If $M$ is a multigraph and all parallel edges joining pairs of vertices of $M$ are replaced by a single edge, then the resulting graph $G$ is called the **underlying graph** of $M$

Formally, two (labeled) graphs $G$ and $H$ are **isomorphic** (have the same structure) if there exists a one-to-one correspondence $f$ from $V(G)$ to $V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. In this case, $f$ is called an **isomorphism** from $G$ to $H$. Thus, if $G$ and $H$ are isomorphic graphs, then we say that $G$ is **isomorphic** to $H$ and we write $G \cong H$

If two graphs $G$ and $H$ are not isomorphic, then they are called **non-isomorphic graphs**

A graph $G$ is **self-complementary** if $G \cong \text{complement}(G)$

As expected, two digraphs $D_1$ and $D_2$ are **isomorphic** if there exists a one-to-one correspondence $V(D_1) \rightarrow V(D_2)$ such that $(u, v) \in E(D_1)$ if and only if $(f(u), f(v)) \in E(D_2)$

One of the major consequences of knowing that isomorphism is an equivalence relation on a set of graphs is that this produces a partition of this set into equivalence classes (subsets) which are **isomorphism classes**

For unlabeled graphs $H$ and $G$, we say that $H$ is **isomorphic to a subgraph** of $G$ if for any labeling of the vertices of $H$ and $G$, the labeled graph $H$ is isomorphic to a subgraph of the labeled graph $G$
There are two other isomorphisms from the graph H to itself, namely 3 and 4, defined by and An isomorphism from a graph G to itself is called an automorphism of G
This group is denoted by Aut (G) and is called the automorphism group of G
Consequently, each of the elements of Aut (F) can be expressed in terms of and, namely, Because of this property, and are generators for the group Aut (F)
F and the group table for Aut (F) For a vertex v of a graph G, the set of all vertices into which v can be mapped by some automorphism of G is an orbit of G
Two vertices u and v are similar if they belong to the same orbit
On the other hand, if a graph G contains a single orbit, then every two vertices of G are similar and G is called vertextransitive
An edge e = uv of a connected graph G is called a bridge of G if G − e is disconnected
An edge e is a bridge of a disconnected graph if e is a bridge of some component of G
A tree is an acyclic connected graph
A tree containing exactly two vertices that are not end-vertices (which are necessarily adjacent) is called a double star
Another common class of trees consists of the "caterpillars." A caterpillar is a tree of order 3 or more, the removal of whose endvertices produces a path called the spine of the caterpillar
There are occasions when it is convenient to select a vertex of a tree T under discussion and designate this vertex as the root of T
The tree T then becomes a rooted tree
Acyclic graphs are also referred to as forests
A spanning subgraph H of a connected graph G such that H is a tree is called a spanning tree of G
Let G be a connected graph each of whose edges is assigned a number (called the cost or weight of the edge
For each subgraph H of G, the weight w (H) of H is defined as the sum of the weights of its edges, that is, We seek a spanning tree of G whose weight is minimum among all spanning trees of G
Such a spanning tree is called a minimum spanning tree
The problem of finding a minimum spanning tree in a connected weighted graph is called the Minimum Spanning Tree Problem
Kruskal’s Algorithm: For a connected weighted graph G, a spanning tree T of G is constructed as follows: For the first edge e 1 of T, we select any edge of G
of minimum weight and for the second edge $e_2$ of $T$, we select any remaining edge of $G$ of minimum weight

--- Prim's Algorithm: For a connected weighted graph $G$, a spanning tree $T$ of $G$ is constructed as follows: For an arbitrary vertex $u$ for $G$, an edge of minimum weight incident with $u$ is selected as the first edge $e_1$ of $T$.

--- The following formula was established in 1889 by Arthur Cayley and is often referred to as the Cayley Tree Formula.

--- By a cofactor of an $n \times n$ matrix $M = [m_{ij}]$, we mean $(-1)^{i+j} \det(M_{ij})$, where $\det(M_{ij})$ indicates the determinant of the $(n-1) \times (n-1)$ submatrix $M_{ij}$ of $M$, obtained by deleting row $i$ and column $j$ of $M$.

--- Chapter 5

--- Connectivity

--- A vertex $v$ in a connected graph $G$ is a cutvertex of $G$ if $G - v$ is disconnected.

--- A nontrivial connected graph with no cut-vertices is called a nonseparable graph.

--- A maximal nonseparable subgraph of a graph $G$ is called a block of $G$.

--- By a vertex-cut in a graph $G$, we mean a set $U$ of vertices of $G$ such that $G - U$ is disconnected.

--- A vertex-cut of minimum cardinality in $G$ is called a minimum vertex-cut.

--- For a graph $G$ that is not complete, the vertex-connectivity (or simply the connectivity) $\kappa(G)$ of $G$ is defined as the cardinality of a minimum vertex-cut of $G$; (The symbol is the Greek letter kappa.)

--- If $G = K_n$ for some positive integer $n$, then $\kappa(G)$ is defined to be $n - 1$.

--- In general then, the connectivity $\kappa(G)$ of a graph $G$ is the minimum value of $|U|$ among all subsets $U$ of $V(G)$ such that $G - U$ is either disconnected or trivial.

--- For a nonnegative integer $k$, a graph $G$ is said to be $k$-connected if $\kappa(G) \geq k$.

--- An edge-cut in a nontrivial graph $G$ is a set $X$ of edges of $G$ such that $G - X$ is disconnected.

--- An edge-cut $X$ of a connected graph $G$ is minimal if no proper subset of $X$ is an edge-cut of $G$.

--- An edge-cut of minimum cardinality is called a minimum edge-cut.

--- The edge-connectivity $\lambda(G)$ of a nontrivial graph $G$ is the cardinality of a minimum edge-cut of $G$, while we define $\lambda(K_1) = 0$.

--- For a nonnegative integer $k$, a graph $G$ is $k$-edge-connected if $\lambda(G) \geq k$.

--- These graphs are referred to as the Harary graphs, named for Frank Harary.
For an integer \( k \) with \( 1 \leq k \leq d \), the \textit{kth power} \( G^k \) of \( G \) is the graph with \( V(G^k) = V(G) \) such that \( uv \) is an edge of \( G^k \) if \( 1 \leq d(G(u, v)) \leq k \).

A set \( S \) of vertices of a graph \( G \) is said to \textit{separate} two vertices \( u \) and \( v \) of \( G \) if \( G - S \) is disconnected and \( u \) and \( v \) belong to different components of \( G - S \).

Such a set \( S \) is called a \textit{\( u - v \) separating set}.

A \( u - v \) separating set of minimum cardinality is called a \textit{minimum \( u - v \) separating set}.

An \textit{internal vertex} of a \( u - v \) path \( P \) is a vertex of \( P \) different from \( u \) and \( v \).

A collection \( \{ P_1, P_2, \ldots, P_k \} \) of \( u - v \) paths is called \textit{internally disjoint} if every two of these paths have no vertices in common other than \( u \) and \( v \).

More generally, for \( k + 1 \) distinct vertices \( u, v_1, v_2, \ldots, v_k \), a collection \( \{ P_1, P_2, \ldots, P_k \} \) of \( k \) paths, where \( P_i \) is a \( u - v_i \) path (\( 1 \leq i \leq k \)), are \textit{internally disjoint} if every two distinct paths in the collection have only \( u \) in common.

For \( 2k \) distinct vertices \( u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k \), a collection \( \{ P_1, P_2, \ldots, P_k \} \) of \( k \) paths, where \( P_i \) is a \( u_i - v_i \) path (\( 1 \leq i \leq k \)), are \textit{disjoint} if no two distinct paths in the collection have a vertex in common.

Such a graph \( G^k \) is then called a \textit{distance-labeled graph}.

A path \( P \) in \( G^k \) is called \textit{proper} if every two adjacent edges in \( P \) have different labels.

By a \textit{proper edge labeling} of \( G \) we mean a labeling of the edges of \( G \) from the set \( \{ 1, 2, \ldots, k \} \) for some positive integer \( k \) such that no two adjacent edges are labeled the same.

\textbf{Chapter 6}

\textbf{Traversability}

A circuit \( C \) in a graph \( G \) is called an \textit{Eulerian circuit} (pronounced oy-LEER-e-an) if \( C \) contains every edge of \( G \).

A connected graph that contains an Eulerian circuit is called an \textit{Eulerian graph}.

For a connected graph \( G \), we refer to an open trail that contains every edge of \( G \) as an \textit{Eulerian trail}.

This became known as the \textit{Königsberg Bridge Problem}.

A cycle in a graph \( G \) that contains every vertex of \( G \) is called a \textit{Hamiltonian cycle} of \( G \).

A \textit{Hamiltonian graph} is a graph that contains a Hamiltonian cycle.

A path in a graph \( G \) that contains every vertex of \( G \) is called a \textit{Hamiltonian path} in \( G \).

In 1857, Hamilton invented a game called the \textit{Icosian Game}.

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-- The closure \( C(G) \) of a graph \( G \) of order \( n \) is the graph obtained from \( G \) by recursively joining pairs of nonadjacent vertices whose degree sum is at least \( n \) (in the resulting graph at each stage) until no such pair remains

-- A Hamiltonian walk in a connected graph \( G \) is a closed spanning walk of minimum length in \( G \).

**Chapter 7**

**Digraphs**

-- Recall that a **digraph** \( D \) consists of a finite nonempty set \( V \) of objects called **vertices** and a set \( E \) of ordered pairs of distinct vertices.

-- Each element of \( E \) is an **arc** or a **directed edge**.

-- If a digraph \( D \) has the property that for each pair \( u, v \) of distinct vertices of \( D \), at most one of \((u, v)\) and \((v, u)\) is an arc of \( D \), then \( D \) is an **oriented graph**.

-- The digraph \( D \) is then referred to as an **orientation** of \( G \).

-- A digraph \( H \) is called a **subdigraph** of a digraph \( D \) if \( V(H) \) is a subset of \( V(D) \) and \( E(H) \) a subset of \( E(D) \).

-- A digraph \( D \) is **symmetric** if whenever \((u, v)\) is an arc of \( D \), then \((v, u)\) is an arc of \( D \) as well.

-- Also, recall that if \((u, v)\) is an arc of a digraph, then \( u \) is said to be **adjacent to** \( v \) and \( v \) is **adjacent from** \( u \).

-- The vertices \( u \) and \( v \) are also said to be **incident with** the arc \((u, v)\).

-- The number of vertices to which a vertex \( v \) is adjacent is the **outdegree** of \( v \) and is denoted by \( od(v) \).

-- The number of vertices from which \( v \) is adjacent is the **indegree** of \( v \) and is denoted by \( id(v) \).

-- A sequence of vertices of \( D \) such that \( u_i \) is adjacent to \( u_{i+1} \) for all \( i \) \((0 \leq i \leq k-1)\) is called a **(directed) \( u-v \) walk** in \( D \).

-- The number of occurrences of arcs on a walk is the **length** of the walk.

-- A walk in which no arc is repeated is a **(directed) trail**, while a walk in which no vertex is repeated is a **(directed) path**.

-- A \( u-v \) walk is **closed** if \( u = v \) and is **open** if \( u \neq v \).

-- A closed trail of length at least 2 is a **(directed) circuit**; a closed walk of length at least 2 in which no vertex is repeated except for the initial and terminal vertices is a **(directed) cycle**.
The underlying graph of a digraph $D$ is obtained by removing all directions from the arcs of $D$ and replacing any resulting pair of parallel edges by a single edge.

A digraph $D$ is connected (sometimes called weakly connected) if the underlying graph of $D$ is connected.

A digraph $D$ is strong (or strongly connected) if $D$ contains both a $u-v$ path and a $v-u$ path for every pair $u, v$ of distinct vertices of $D$.

The directed distance or, more simply, the distance $(u, v)$ from $u$ to $v$ is the length of a shortest $u-v$ path in $D$.

A $u-v$ path of length $(u, v)$ is a $u-v$ geodesic.

An Eulerian circuit in a (strong) digraph $D$ is a circuit containing every arc of $D$.

An Eulerian digraph is a digraph containing an Eulerian circuit.

A tournament is an orientation of a complete graph.

Therefore, a tournament can be defined as a digraph such that for every pair $u, v$ of distinct vertices, exactly one of $(u, v)$ and $(v, u)$ is an arc (Ties are not permitted).

A tournament $T$ is transitive if whenever $(u, v)$ and $(v, w)$ are arcs of $T$, then $(u, w)$ is also an arc of $T$.

As with graphs, a path $P$ in a digraph $D$ is a Hamiltonian path of $D$ if $P$ contains all vertices of $D$.

A cycle $C$ in $D$ is a Hamiltonian cycle if $C$ contains every vertex of $D$.

If $D$ has a Hamiltonian cycle, then $D$ is a Hamiltonian digraph.

Chapter 8

Matchings and Factorization

A set of edges in a graph is independent if no two edges in the set are adjacent.

By a matching in a graph $G$, we mean an independent set of edges in $G$.

We say that $M$ matches the set $\{u_1, u_2, \ldots, u_k\}$ to the set $\{w_1, w_2, \ldots, w_k\}$.

Let $G$ be a bipartite graph with partite sets $U$ and $W$ such that $|U| \leq |W|$.

For a nonempty set $X$ of $U$, the neighborhood $N(X)$ of $X$ is the union of the neighborhoods $N(x)$, where $x$ is in $X$.

The graph $G$ is said to satisfy Hall’s condition if $|N(X)| \geq |X|$ for every nonempty subset $X$ of $U$.

Then this collection of sets has a system of distinct representatives if there exist nondistinct elements $x_1, x_2, \ldots, x_n$ such that $x_i \in S_i$ for $1 \leq i \leq n$.
Such a matching is called a **maximum matching**

If a graph $G$ of order $2k$ has a matching $M$ of cardinality $k$, then this (necessarily maximum) matching $M$ is called a **perfect matching** as $M$ matches every vertex of $G$ to some vertex of $G$

The edge independence number $\alpha'(G)$ of a graph $G$ is the maximum cardinality of an independent set of edges

A vertex and an incident edge are said to **cover** each other

An edge cover of a graph $G$ without isolated vertices is a set of edges of $G$ that covers all vertices of $G$

The edge covering number $\beta'(G)$ of a graph $G$ is the minimum cardinality of an edge cover of $G$

An edge cover of $G$ of cardinality $\alpha'(G)$ is a **minimum edge cover** of $G$

A set of vertices in a graph is **independent** if no two vertices in the set are adjacent

The vertex independence number (or the independence number) $(G)$ of a graph $G$ is the maximum cardinality of an independent set of vertices in $G$

An independent set in $G$ of cardinality $(G)$ is called a **maximum independent set**

A vertex cover in a graph $G$ is a set of vertices that covers all edges of $G$

The minimum number of vertices in a vertex cover of $G$ is the vertex covering number $(G)$ of $G$

A vertex cover of cardinality $\alpha(G)$ is a **minimum vertex cover** in $G$

A 1-regular spanning subgraph of a graph $G$ is also called a **1-factor** of $G$

A component of a graph is **odd** or **even** according to whether its order is odd or even

A graph $G$ is said to be **1-factorable** if there exist 1-factors $F_1, F_2, \ldots, F_r$ of $G$ such that $\{ E(F_1), E(F_2), \ldots, E(F_r) \}$ is a partition of $E(G)$

We then say that $G$ is **factored** into the 1-factors $F_1, F_2, \ldots, F_r$, which form a **1-factorization** of $G$

For this reason, the 1-factorization described in the proof is called a **cyclic factorization**

A **2-factor** in a graph $G$ is a spanning 2-regular subgraph of $G$

A graph $G$ is said to be **2-factorable** if there exist 2-factors $F_1, F_2, \ldots, F_k$ such that $\{ E(F_1), E(F_2), \ldots, E(F_k) \}$ is a partition of $E(G)$

A Hamiltonian factorization of a graph $G$ is a 2-factorization of $G$ in which each 2-factor is a Hamiltonian cycle

A graph $G$ is **Hamiltonian-factorable** if there exists a Hamiltonian factorization of $G$
-- Hamiltonian factorization of $K_9$ More generally, a spanning subgraph $F$ of a graph $G$ is called a factor of $G$
-- The graph $G$ is said to be factorable into the factors $F_1, F_2, \ldots, F_k$ if \{$E(F_1), E(F_2), \ldots, E(F_k)$\} is a partition of $E(G)$
-- If each factor $F_i$ is isomorphic to some graph $F$, then $G$ is $F$-factorable
-- A graph $G$ is said to be decomposable into the subgraphs $H_1, H_2, \ldots, H_k$ if \{$E(H_1), E(H_2), \ldots, E(H_k)$\} is a partition of $E(G)$
-- Such a partition produces a decomposition of $G$
-- If each $H_i$ is isomorphic to some graph $H$, then the graph $G$ is $H$-decomposable and the decomposition is an $H$-decomposable
-- A Steiner triple system of order $n$ is a set $S$ of cardinality $n$ and a collection $T$ of 3-element subsets, called triples, such that every two distinct elements of $S$ belong to a unique triple in $T$
-- From the $K_3$-decomposition of $K_7$, we have now produced a Steiner triple system of order 7 from the set \{1, 2, \ldots, 7\}, namely: The $K_3$-decomposition is a cyclic decomposition of $K_7$
-- A one-to-one function $f : V(G) \rightarrow \{0, 1, 2, \ldots, m\}$ is called a graceful labeling of $G$ if the induced edge labeling $f' : E(G) \rightarrow \{1, 2, \ldots, m\}$ defined by is also one-to-one
-- The length of a smallest cycle in a graph is referred to as its girth
-- For an integer $g \geq 3$, a $g$-cage is a 3-regular graph of minimum order that has girth $g$

Chapter 9
Planarity
-- A graph $G$ is called a planar graph if $G$ can be drawn in the plane so that no two of its edges cross each other
-- A graph that is not planar is called nonplanar
-- A graph $G$ is called a plane graph if it is drawn in the plane so that no two edges of $G$ cross
-- This problem is referred to as the Three Houses and Three Utilities Problem
-- A plane graph divides the plane into connected pieces called regions
-- This is the exterior region
-- The subgraph of a plane graph whose vertices and edges are Figure 9.5: A planar graph and a plane graph incident with a given region $R$ is the boundary of $R$
-- This is referred to as the Euler Identity
A graph $G$ is **maximal planar** if $G$ is planar but the addition of an edge between any two nonadjacent vertices of $G$ results in a nonplanar graph.

More formally, a graph $G'$ is called a **subdivision** of a graph $G$ if $G' \neq G$ or one or more vertices of degree 2 are inserted into one or more edges of $G$.

A "drawing" in plane is also called an **embedding** of $G$ in the plane. In addition, we say that $G$ can be **embedded** in the plane.

A common surface is the **torus**, a doughnut-shaped surface (see Figure 9.19 (a)). In Figure 9.19 (b), we see that the graph $K_4$ can be embedded on the torus.

The surface $S_k$ is also called a **surface of genus** $k$.

What we have just observed then is that every graph can be **embedded** on some surface.

The smallest nonnegative integer $k$ such that a graph $G$ can be embedded on $S_k$ is called the **genus** of $G$ and is denoted by $\gamma(G)$.

A region is called a **2-cell** if any closed curve that is drawn in that region can be continuously contracted (or shrunk) in that region to a single point.

An embedding, every region of which is a 2-cell, is called a **2-cell embedding**.

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**Chapter 10**

**Coloring**

The conjecture that every map can be colored with four or fewer colors became known as the **Four Color Conjecture** (proof based on a set that was later referred to as an **unavoidable set of reducible configurations**).

A **reducible configuration** is any arrangement of regions that cannot occur in a map.

By a **proper coloring** (or, more simply, a **coloring**) of a graph $G$, we mean an assignment of colors (elements of some set) to the vertices of $G$, one color to each vertex, such that adjacent vertices are colored differently.

The smallest number of colors in any coloring of a graph $G$ is called the **chromatic number** of $G$ and is denoted by $\chi(G)$. (The symbol is the Greek letter "chi"). If it is possible to color (the vertices of) $G$ from a set of $k$ colors, then $G$ is said to be **$k$-colorable**.

A coloring that uses $k$ colors is called a **$k$-coloring**.

If $\chi(G) = k$, then $G$ is said to be **$k$-chromatic** and every $k$-coloring of $G$ is a **minimum coloring** of $G$.

A **clique** in a graph $G$ is a complete subgraph of $G$.

The order of the largest clique in a graph $G$ is its **clique number**, which is denoted by $\omega(G)$.

A coloring of a graph $G$ can also be thought of as a function $c$ from $V(G)$ to the set $\mathbb{N}$ of positive integers (or natural numbers) such that adjacent vertices have...
distinct functional values, that is, a coloring of $G$ is a function $c : V(G) \rightarrow N$ such that $uv \in E(G)$ implies that $c(u) \neq c(v)$

-- This graph is triangle-free (it has no triangles) but has chromatic number 4
-- A graph $G$ is called perfect if $(H) = \omega(H)$ for every induced subgraph $H$ of $G$
-- An edge coloring of a nonempty graph $G$ is an assignment of colors to the edges of $G$, one color to each edge, such that adjacent edges are assigned different colors
-- The minimum number of colors that can be used to color the edges of $G$ is called the chromatic index (or sometimes the edge chromatic number) and is denoted by $'(G)$
-- An edge coloring that uses $k$ colors is a $k$-edge coloring

Chapter 12
Distance
-- For two vertices $u$ and $v$ in a graph $G$, the distance $d(u,v)$ from $u$ to $v$ is the length of a shortest $u-v$ path in $G$
-- A $u-v$ path of length $d(u,v)$ is called a $u-v$ geodesic
-- $d(u,v) = d(v,u)$ for all $u,v \in V(G)$ [the symmetric property]. 4
-- $d(u,w) \leq d(u,v)+d(v,w)$ for all $u,v,w \in V(G)$ [the triangle inequality]. That a connected graph satisfies all four of these properties should be clear, with the possible exception of property 4 (the triangle inequality), which we now verify
-- The fact that the distance $d$ satisfies properties 1 – 4 means that $d$ is a metric and $(V(G),d)$ is a metric space
-- For a vertex $v$ in a connected graph $G$, the eccentricity $e(v)$ of $v$ is the distance between $v$ and a vertex farthest from $v$ in $G$
-- The minimum eccentricity among the vertices of $G$ is its radius and the maximum eccentricity is its diameter, which are denoted by $\text{rad}(G)$ and $\text{diam}(G)$, respectively
-- A vertex $v$ in $G$ is a central vertex if $e(v) = \text{rad}(G)$ and the subgraph induced by the central vertices of $G$ is the center $\text{Cen}(G)$ of $G$
-- If every vertex of $G$ is a central vertex, then $\text{Cen}(G) = G$ and $G$ is called self-centered
-- A vertex $v$ in a connected graph $G$ is called a peripheral vertex if $e(v)=\text{diam}(G)$
-- The subgraph of $G$ induced by its peripheral vertices is the periphery $\text{Per}(G)$ of $G$
-- Such a vertex $v$ is called an eccentric vertex of $u$
A vertex v is an eccentric vertex of the graph G if v is an eccentric vertex of some vertex of G

Chapter 13
Domination

-- The neighborhood (or open neighborhood) N(v) of v is the set of neighbors of v
-- The closed neighborhood \( N[v] \) is defined as \( N[v] = N(v) \cup \{v\} \)
-- A vertex v in a graph G is said to dominate itself and each of its neighbors, that is, v dominates the vertices in its closed neighborhood \( N[v] \). Therefore, v dominates \( 1 + \deg v \) vertices of G
-- A set S of vertices of G is a dominating set of G if every vertex of G is dominated by some vertex in S
-- A minimum dominating set in a graph G is a dominating set of minimum cardinality
-- The cardinality of a minimum dominating set in G is called the domination number of G and is denoted by \( \gamma(G) \)
-- If S is a dominating set of a graph G and no proper subset of S is a dominating set of G, then S is called a minimal dominating set. (In this context, a vertex does not dominate itself.) This type of domination is called total domination
-- A set S of vertices in a graph G is a total dominating set of G if every vertex of G is adjacent to at least one vertex of S
-- The minimum cardinality of a total dominating set is the total domination number \( \gamma_t(G) \) of G
-- A total dominating set of cardinality \( \gamma_t(G) \) is a minimum total dominating set for G