5 Ch 5: EXISTENCE AND PROOF BY CONTRADICTION

5.1 Counterexamples

Recall: \( \neg (\forall x \in S, P(x)) \iff \exists x \in S, \neg P(x) \)

That is, if we wish to show that \( \forall x \in S, P(x) \) is not true, then it is enough to find a value \( x \in S \) such that \( P(x) \) is false. Such an example is called a counterexample of the statement \( \forall x \in S, P(x) \).

Example: For every real number \( x \), we have that \((x - 1)^2 > 0\).
Solution: The statement is false since the value \( x = 1 \) is a counterexample as \((1 - 1)^2 \neq 0\). \( \diamond \)

Example: For every real number \( x \), \( \tan x = \frac{\sin x}{\cos x} \).
Solution: The statement is false since \( \cos x \) may be zero, and we can’t divide by 0. \( \diamond \)

The above example is not true because the expressions \( \tan x \) and \( \frac{\sin x}{\cos x} \) are not even defined at \( x = 0 \), and so we can’t tell if they are equal. However, the following result is true: For every nonzero real number \( x \), \( \tan x = \frac{\sin x}{\cos x} \).

Example: For each positive integer \( n \), \( 3 | (n^2 - 1) \).
Solution: This is false since \( n = 3 \) is a counterexample. \( \diamond \)

5.2 Proof by Contradiction

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Recall: (1)

(2) \( (P \Rightarrow Q) \iff (\neg (P) \lor Q) \)

(3) \( \neg (P \Rightarrow Q) \iff (P \land (\neg Q)) \)

Our goal: prove a statement \( S : P \Rightarrow Q \) is true. Let \( C \) be a contradiction. If we can show that \( \neg S \Rightarrow C \) is true, then since \( C \) is false we have that \( \neg S \) is false. Thus \( S \) is true.

The method: To prove \( \neg (P \Rightarrow Q) \Rightarrow C \) is true we prove: \( P \land (\neg Q) \Rightarrow C \) is true. That is, assume that \( P \) is true, and \( Q \) is false, and then obtain a contradiction.

Result: There is no smallest positive real number.
(i.e. If \( x \) is a real number, then \( x \) is not the smallest positive real number.)
Proof Assume, to the contrary, that there is a real number \( r \) such that \( r \) is the smallest positive real number. Since \( 0 < \frac{r}{2} < r \), it follows that \( \frac{r}{2} \) is a positive real number that is
smaller than \( r \), which is a contradiction.

**Result:** No odd integer can be expressed as the sum of three even integers.

**Proof** Assume, to the contrary, that there is an integer \( x \), such that \( x = a + b + c \), where \( a, b, \) and \( c \) are even integers. Then \( a = 2k, b = 2\ell, \) and \( c = 2p, \) for some \( k, \ell, p \) integers. Thus \( x = a + b + c = 2k + 2\ell + 2p = 2(k + \ell + p) \). Since \( k + \ell + p \) is an integer, it follows that \( x \) is even, which is a contradiction.

**Result:** If \( a \) is an even integer and \( b \) is an odd integer, then \( 4 \nmid (a^2 + 2b^2) \).

**Proof:** Let \( a \) be an even integer, and \( b \) be an odd integer, and assume to the contrary that \( 4 \mid (a^2 + 2b^2) \). Then \( a = 2k, b = 2\ell + 1, \) and \( a^2 + 2b^2 = 4p \) for some integers \( k, \ell, \) and \( p \). Then

\[
\begin{align*}
a^2 + 2b^2 &= 4p \\
(2k)^2 + 2(2\ell + 1)^2 &= 4p \\
4k^2 + 8\ell^2 + 8\ell + 2 &= 4p \\
2 &= 4p - 4k^2 - 8\ell^2 - 8\ell \\
2 &= 4(p - k^2 - 2\ell^2 - 2\ell),
\end{align*}
\]

which contradiction since 4 divides the right side but not the left side.

Note: You could dive through by a 2 above to get:

\[
\begin{align*}
2 &= 4(p - k^2 - 2\ell^2 - 2\ell) \\
1 &= 2(p - k^2 - 2\ell^2 - 2\ell).
\end{align*}
\]

Now this is a contradiction since the left hand side is odd, but the right side is even.

### 5.3 Review the proof techniques on page 116 – 118

Here is a result that is proved by three different proof techniques.

**Result:** If \( n \) is an even integer, then \( 3n + 1 \) is odd.

**Proof:** (direct proof) Assume that \( n \) is an even integer. Then \( n = 2k \) for some \( k \in \mathbb{Z} \). Thus \( 3n + 1 = 3(2k) + 1 = 2(3k) + 1 \). Since \( 3k \) is an integer, \( 3n + 1 \) is odd.

**Result:** If \( n \) is an even integer, then \( 3n + 1 \) is odd.

**Proof:** (contrapositive) Assume that \( 3n + 1 \) is even. Then \( 3n + 1 = 2k \). Thus \( n = (3n + 1) - (2n + 1) = 2k - 2n - 1 = 2(k - n - 1) + 1 \). Since \( k - n - 1 \) is an integer, it follows that \( 3n + 1 \) is odd.

**Result:** If \( n \) is an even integer, then \( 3n + 1 \) is odd.

**Proof:** (contradiction) Let \( n \) be even, and assume, to the contrary, that \( 3n + 1 \) is also even. So, \( n = 2k \) and \( 3n + 1 = 2\ell \), for some \( k, \ell \in \mathbb{Z} \). Then,

\[
\begin{align*}
3n + 1 &= 2\ell \\
3(2k) + 1 &= 2\ell \\
1 &= 2(\ell - 3k),
\end{align*}
\]
which is a contradiction since the left hand side is odd and the right hand side is even.

5.4 Existence proofs

To prove that $\exists x \in S, P(x)$, one needs to find some value $x$ that makes $P(x)$ true.

Result: There exist integers $x, y$, such that $(x - y)^2 = x^2 - y^2$.

Proof: Let $x = 2$ and $y = 0$. Then $(x - y)^2 = (2 - 0)^2 = 4$ and also $x^2 - y^2 = 2^2 - 0^2 = 4$. ■

Some statements are only true for one value, and it may be important to show that it is a unique value for which it is true. We say that there is only one value $x$ for which $P(x)$ by using $!$:

$$\exists! x \in S, P(x)$$

To prove that $\exists! x \in S, P(x)$, one needs to find some value $x$ that makes $P(x)$ true, and then show that there is no other value that makes it true.

Theorem: Let $a, b, \in \mathbb{R}, a \neq 0$. There exists a unique real number $x$, such that $ax + b = 0$.

Proof: Note that $x = -\frac{b}{a}$ verifies the equation. We now show that no other different value will verify it. Let $y$ be another solution of $ax + b = 0$. Then $ay + b = 0$, and so $ay = -b$ which implies that $y = -\frac{b}{a}$. Therefore $y = x$, and so the solution is unique. ■

Note that now you can use the above theorem to show that any linear equation has a unique solution. For example:

Theorem: There exists a unique real number $x$, such that $3x = 4$.

Proof: Observe that the value $x = \frac{4}{3}$ verifies the equation $3x = 4$. Also, by the theorem above, this is the only real number that verifies the equation. ■

5.5 Disproving existence statements

Recall: $\sim (\exists x \in S, P(x)) \iff \forall x \in S, \sim P(x)$

That is, to disprove an existence statement, one has to prove that its negation is always true.

Result: There exists an integer $x$, such that $x^2 - 2x + 7 = 0$.

Solution: Observe that $x^2 - 2x + 7 > x^2 - 2x + 4 = (x - 2)^2 \geq 0$. Therefore $x^2 - 2x + 7 > 0$, for $\forall x \in \mathbb{R}$. And so the statement is false. ◦